Stability of non localized responses for damaging materials

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Abstract

This work is devoted to the analysis of the stability of the homogeneous states of a bar made of a brittle strain softening material submitted to a tensile loading. We distinguish two types of damage models: local damage models and gradient damage models. We show that a local damage model necessarily leads to the unstability of the homogeneous response once the first damage threshold is reached. On the contrary, in the case of a gradient damage model, viewed as a regularization of the underlying local model, the homogeneous damage states of "sufficiently small" bars are stable.

1 Introduction

Prior to their complete rupture, many engineering materials such as concrete, rocks, wood or various composites show a strain-softening behavior when they are deformed beyond a certain limit. The theory of damage is generally used to model this behavior at a continuum level. Limiting our analysis to rate independent behaviors, we can distinguish two types of damage models: (i) the so-called local models where the only variables characterizing the state of the material point are the strain and the damage variable; (ii) the so-called non local models where additional information on the neighborhood of the material point are involved.

From the theoretical viewpoint, the boundary-value problem associated with local models is mathematically ill posed (Benallal *et al.* 1989 [2], Lasry and Belytschko, 1988 [7]) and lead to multiple (and even an infinite number of) solutions. From the numerical viewpoint, the computations give rise to spurious mesh dependences: upon refinement of the meshsize, no convergence is observed or more precisely the deformation is localized into narrow bands whose thickness corresponds to the meshsize (Bažant, 1993 [1]). Origins of these pathological localizations are usually understood in terms of bifurcations analyses via wave propagation phenomena although the loading is quasistatic (Pijaudier and Benallal [15].) All these bad properties suggest that local models must be revised.

Two main regularization techniques exist to avoid pathological localization, namely the integral (Pijaudier-Cabot and Bažant, 1987 [14]) or the gradient (Triantafyllidis and Aifantis, 1986 [16]) damage approaches. Both of them rely on an *ad-hoc* incorporation of a material lengthscale in the constitutive behavior (Lorentz and Andrieux, 2003 [8], de Borst *et al.*, 1993 [5], Peerlings *et al.*, 1996 [13]). However, despite the introduction of the gradient of damage as a state variable into the model, the associated initial boundary value remains ill-posed and does not prevent localized bifurcations.

We revisit here the issues of well-posedness and stability for, first, local models and, then, gradient damage models by using energetic formulations. Indeed these approaches have gained popularity in the last decade since they offer a physical and rigorous framework relying on the tools of the calculus of variations. This global variational approach was first proposed by Nguyen [12] for a large class of rate independent behaviours, then extended by Francfort and Marigo [6] and Bourdin, Francfort and Marigo [4] to Damage and Fracture Mechanics . The power of this concept is that the evolution problem and the stability analysis are understood respectively as a first and second order conditions of optimality of the total energy (Benallal and Marigo, 2007 [3]). In particular the stable states naturally correspond to local minima of the potential energy in a quasi-static loading. Accordingly, the search of stable states seems to be a more relevant approach than proving the ill-posedness (i.e. non uniqueness) characterization, while bifurcations which can lead to multiple equilibrium configurations in a non linear material (e.g. softening material) is just the reflection of the complexity of our world and can not be a discriminated criterion for a model.

In this paper, we investigate under which conditions an homogeneous state of damage in a softening material is stable and therefore will be observable during an experiment. As a matter of fact a spatially homogeneous damage evolution is experimentally significant since it does not result in a brutal fracture of the specimen but instead offers the possibility to identify some of the damage material parameters. The paper is structured as follows. Section 2 describes the statement of the problem and how a local damage approach of a softening behavior is bound to fail. Sections 3 shows how the regularization of the local model brings size effects and improvements in the stability results of homogeneous states.

We use the following notation: the prime denotes the spatial derivative or the derivative with respect to the damage parameter, the dot the time derivative, e.g. $u' = \partial u/\partial x$, $E'(\alpha) = dE(\alpha)/d\alpha$, $\dot{\alpha} = \partial\alpha/\partial t$.

2 Case of a local damage model

2.1 The damage evolution problem

We consider a homogeneous bar of length L, made of a softening material. The end $x = 0$ of the bar is fixed while the displacement of the end $x = L$ is controlled by a hard device so that the displacement field u_t at time t satisfies

$$u_t(x = 0) = 0, \qquad u_t(x = L) = tL, \qquad t \geq 0 \tag{2.1}$$

where t denotes the loading parameter growing from 0 and identified with the time. The damage state of the bar at time t is characterized by the scalar field $x \mapsto \alpha_t(x)$. The evolution of the displacement and the damage in the bar with the time is obtained via an energetic formulation, see [9] and [10] for a justification of a such energetic approach by thermodynamical arguments. Specifically, let $E(\alpha)$ be the Young modulus of the material in the damage state α and $w(\alpha)$ the energy dissipated when the material is damaged up to α. The functions $\alpha \mapsto E(\alpha)$ and $\alpha \mapsto w(\alpha)$ are respectively positive decreasing and positive increasing. Because of the irreversibility of damage, α_t can only increase with t. The evolution of the damage of a point is governed by a local yield criterion formulated in terms of the concept of critical elastic energy release rate. That leads to the following system of (in)equalities (sometimes called Kuhn-Tucker conditions) which must hold at each time and each point:

$$\text{Irreversibility condition} : \dot{\alpha}_t(x) \geq 0 \tag{2.2}$$

$$\text{Damage criterion} : -\frac{1}{2}E'(\alpha_t(x))\varepsilon_t(x)^2 \leq w'(\alpha_t(x)) \tag{2.3}$$

$$\text{Energy balance} : \dot{\alpha}_t(x)\left(\frac{1}{2}E'(\alpha_t(x))\varepsilon_t(x)^2 + w'(\alpha_t(x))\right) = 0 \tag{2.4}$$

where $\varepsilon_t(x)$ denotes the strain state of the point x at time t, $\varepsilon_t = u_t'$. Considering only quasi-static evolution, the bar must be at equilibrium at each time. That leads to

$$\sigma_t'(x) = 0, \qquad \sigma_t(x) = E(\alpha_t(x))\varepsilon_t(x). \tag{2.5}$$

The initial condition $\alpha_0(x) = 0$ saying that the bar is undamaged at $t = 0$ completes the formulation of the evolution problem.

2.2 Variational formulation of the evolution problem

Let us introduce the state function W_0 giving the strain work (or the total energy density) associated with an evolution of the state of a material point from $(0, 0)$ to (ε, α)

$$W_0(\varepsilon, \alpha) = \frac{1}{2}E(\alpha)\varepsilon^2 + w(\alpha). \tag{2.6}$$

Let $\mathcal{C}_t$ and $\mathcal{D}$ be respectively the kinematically admissible displacement fields at time t and the convex cone of admissible damage fields

$$\begin{aligned} \mathcal{C}_t &= \{v : v(0) = 0,\ v(L) = tL\} \\ \mathcal{C}_0 &= \{v : v(0) = 0,\ v(L) = 0\} \\ \mathcal{D} &= \{\beta : \beta \geq 0\} \end{aligned} \tag{2.7}$$

where $\mathcal{C}_0$ is the linear space associated with $\mathcal{C}_t$. Then with any admissible pair (u, α) in $\mathcal{C}_t \times \mathcal{D}$, we associate the total energy of the bar

$$\mathcal{P}_0(u, \alpha) = \int_0^L W_0(u'(x), \alpha(x))\, dx = \int_0^L \frac{1}{2} E(\alpha(x)) u'(x)^2 + w(\alpha(x))\, dx \tag{2.8}$$

The set of admissible displacement rates $\dot{u}$ can be identified with $\mathcal{C}_1$, while the set of admissible damage rates $\dot{\alpha}$ can be identified with $\mathcal{D}$. Under the assumption that the bar is sound at the initial time $t = 0$, the damage evolution problem is strictly equivalent to the following variational problem

$$\begin{gathered} \text{Find } (u_t, \alpha_t) \text{ in } \mathcal{C}_t \times \mathcal{D} \text{ such that} \\ \text{For all } (v, \beta) \in \mathcal{C}_1 \times \mathcal{D}, \quad \mathcal{P}_0'(u_t, \alpha_t)(v - \dot{u}_t, \beta - \dot{\alpha}_t) \geq 0 \end{gathered} \tag{2.9}$$

where $\mathcal{P}_0'(u, \alpha)(v, \beta)$ denotes the derivative of $\mathcal{P}_0$ at (u, α) in the direction (v, β) and is given by

$$\mathcal{P}_0'(u, \alpha)(v, \beta) = \int_0^L E(\alpha) u' v'\, dx + \int_0^L \left(\frac{1}{2} E'(\alpha) u'^2 + w'(\alpha) \right) \beta\, dx \tag{2.10}$$

Let us verify that (2.9) implies (2.2)–(2.5). Indeed, by inserting $\beta = \dot{\alpha}_t$ and $v = \dot{u}_t + w$ with $w \in \mathcal{C}_0$ in (2.9), we obtain the variational formulation of the equilibrium of the bar,

$$\int_0^L E(\alpha_t(x)) u'(x) w'(x)\, dx = 0, \qquad \forall w \in \mathcal{C}_0 \tag{2.11}$$

from which we deduce that the stress along the bar is homogeneous and depends only on time t

$$\sigma_t = E(\alpha_t(x)) u_t'(x), \quad \forall x \in (0, L) \tag{2.12}$$

and hence (2.5). Dividing (2.12) by $E(\alpha_t(x))$, integrating over $[0, L]$ and using boundary conditions (2.1), we find

$$\sigma_t \int_0^L \frac{dx}{E(\alpha_t(x))} = tL \tag{2.13}$$

The damage problem is obtained by inserting (2.11)–(2.13) into (2.9) which leads to the variational inequality governing the evolution of the damage

$$\int_0^L \frac{1}{2} E'(\alpha_t) u_t'^2 \beta\, dx + \int_0^L w'(\alpha_t) \beta\, dx \geq 0 \tag{2.14}$$

where the inequality (2.14) holds for all $\beta \in \mathcal{D}$ and becomes an equality when $\beta = \dot{\alpha}_t$. Again, by performing an integration by parts and using classical tools of the calculus of variations, we recover the local formulation of the damage problem (2.2)–(2.4).

2.3 Homogeneous evolutions

As it is pointed out in [3], the evolution problem can admit an infinite number of solutions. We are interested here in particular solutions, the so-called homogeneous evolutions, for which the damage field and the strain field are uniform (ε_t and α_t depend on t but not on x). In such a case we have

$$\sigma_t = E(\alpha_t) t, \qquad u_t(x) = tx, \qquad \varepsilon_t = t \tag{2.15}$$

Then the damage criterion (2.3) and the energy balance condition (2.4) read as

$$\frac{1}{2}E'(\alpha_t)t^2 + w'(\alpha_t) \geq 0, \qquad \dot{\alpha}_t\left(\frac{1}{2}E'(\alpha_t)t^2 + w'(\alpha_t)\right) = 0 \tag{2.16}$$

Assuming that the whole bar is undamaged at the beginning of the loading ($\alpha_0 = 0$), we obtain that the first time t_e when the damage criterion becomes an equality is given by

$$t_e = \sqrt{-\frac{2w'(0)}{E'(0)}} \tag{2.17}$$

If t_e is strictly positive (i.e. if $w'(0) > 0$), then we observe an elastic phase in the time interval $[0, t_e)$: during this phase, the softening material is sound and its rigidity is E_0. For $t > t_e$, the damage criterion is an equality, and the damage grows homogeneously in the bar ($\dot{\alpha}_t > 0$). Finally, the value at any time t of the homogeneous damage is given by

$$\alpha_t = 0 \quad \text{if} \quad t \leq t_e, \qquad -\frac{2w'(\alpha_t)}{E'(\alpha_t)} = t^2 \quad \text{otherwise} \tag{2.18}$$

We remark that the homogeneous damage problem is well posed if the function $\alpha \to -2w'(\alpha)/E'(\alpha)$ is strictly monotone. If this monotonicity condition is satisfied then there exists a unique homogeneous damage state which verifies the evolution problem. However, this condition does not prevent bifurcation and appearance of non-homogeneous damage in the bar. To illustrate our study, we consider the following particular damage law for the softening material

$$E(\alpha) = E_0(1-\alpha)^2, \qquad w(\alpha) = \frac{\sigma_0^2}{E_0}\alpha, \quad \alpha \geq 0 \tag{2.19}$$

As $w'(0) > 0$ in (2.19) then there exists an elastic phase. For this law, the monotonicity condition above is satisfied and (2.18) admits a unique solution at each time t

$$\alpha_t = \max\left(0, 1 - \left(\frac{t_e}{t}\right)^2\right) \quad \text{with} \quad t_e = \frac{\sigma_0}{E_0} \tag{2.20}$$

Finally the relation between the stress of the homogeneous solution and the prescribed displacement at $x = L$ is given by

$$\sigma_t = \begin{cases} \sigma_0 \frac{t}{t_e} & \text{if } t \leq t_e \\ \sigma_0 \left(\frac{t_e}{t}\right)^3 & \text{otherwise} \end{cases} \tag{2.21}$$

We plot on (Fig.1) the stress in the bar versus the time (which is also the homogeneous strain) for the homogeneous damage solution. After the elastic phase the stress decreases, which is characteristic of the strain softening property of the material.

2.4 Unstability of homogeneous states

We see that the homogeneous damage evolution defined by (2.18) is always a solution of the evolution problem in the case of the local model. However, this response will be physically acceptable and experimentally observable if and only if, at each time, the corresponding state (u_t, α_t) is stable. Following [12] and [4], the stability of a state is defined in terms of local minimization of the total energy (2.8) at each time or more precisely, because of the irreversibility condition on the damage evolution, in terms of *unilateral* local minimization (see [6],[12]). We briefly recall this definition of stability hereafter.

Let α be an admissible damage field and u_t^α be the displacement field equilibrating the bar in this damage state at time t. Then the total energy of the bar can read as

$$\mathcal{E}_0^t(\alpha) \equiv \mathcal{P}_0(u_t^\alpha, \alpha) = \min_{v \in \mathcal{C}_t} \mathcal{P}_0(v, \alpha) \tag{2.22}$$

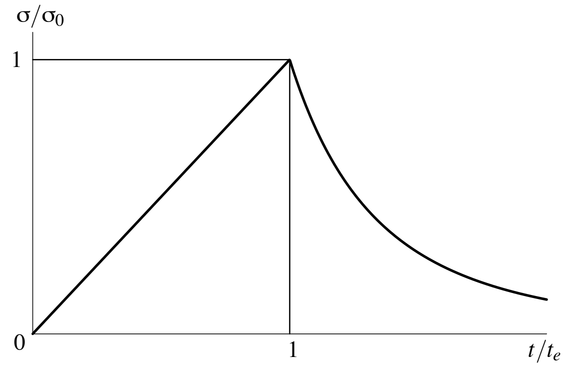

Figure 1: Stress vs time for the homogeneous damage response of the bar

A damage state α (not necessarily homogeneous) will be $\mathcal{E}_0^t$-*stable* (at time t) if and only if it exists a neighborhood of admissible states in which any accessible state has a greater potential energy. This condition can be written formally as follows

$$\mathcal{E}_0^t\textit{-stability:}\quad \exists r>0, \forall \beta \in \mathcal{D}: \|\beta\|=1, \forall h\in[0,r] \quad \mathcal{E}_0^t(\alpha) \le \mathcal{E}_0^t(\alpha+h\beta) \tag{2.23}$$

It is important to notice that the test directions are chosen in $\mathcal{D}$ and therefore must be positive as we cannot violate the irreversibility condition ($\alpha(x)+h\beta(x)\ge\alpha(x)$ for every x). The displacement u_t^α is obtained by solving the variational formulation of the equilibrium of the bar for the given damage α in $\mathcal{D}$ i.e.

$$\int_0^L E(\alpha(x)) {u_t^\alpha}'(x) w'(x)\,dx = 0 \qquad \forall w\in\mathcal{C}_0 \tag{2.24}$$

After integrating by parts we find

$$\sigma_t^\alpha = E(\alpha(x)){u_t^\alpha}'(x), \quad \forall x\in(0,L) \quad \text{with} \quad \sigma_t^\alpha = \frac{tL}{\int_0^L \frac{dx}{E(\alpha(x))}} \tag{2.25}$$

By inserting (2.25) in (2.8), we finally obtain the expression of $\mathcal{E}_0^t(\alpha)$

$$\mathcal{E}_0^t(\alpha) = \frac{t^2L^2}{2\int_0^L \frac{dx}{E(\alpha)}} + \int_0^L w(\alpha)\,dx \tag{2.26}$$

Now in practice, to check the stability inequality (2.23), we develop, for a given direction β, the functional $h\to\mathcal{E}_0^t(\alpha+h\beta)$ with respect to h around $h=0$ up to the second order

$$\mathcal{E}_0^t(\alpha+h\beta) = \mathcal{E}_0^t(\alpha) + h{\mathcal{E}_0^t}'(\alpha)(\beta) + \frac{h^2}{2}{\mathcal{E}_0^t}''(\alpha)(\beta) + o(h^2) \tag{2.27}$$

where the first derivative ${\mathcal{E}_0^t}'(\alpha)$ is given by

$${\mathcal{E}_0^t}'(\alpha)(\beta) = \int_0^L \left(w'(\alpha) + \frac{(\sigma_t^\alpha)^2}{2}\frac{E'(\alpha)}{E(\alpha)^2}\right)\beta\,dx \tag{2.28}$$

and the second derivative by

$${\mathcal{E}_0^t}''(\alpha)(\beta) \quad = \int_0^L \left(w''(\alpha) - \left(\frac{2E'(\alpha)^2}{E(\alpha)^3} - \frac{E''(\alpha)}{E(\alpha)^2}\right)\frac{(\sigma_t^\alpha)^2}{2}\right)\beta^2\,dx + \frac{(\sigma_t^\alpha)^3}{tL}\left(\int_0^L \frac{E'(\alpha)\beta dx}{E(\alpha)^2}\right)^2 \tag{2.29}$$

We see that the stability condition consists in finding the sign of each derivative of $\mathcal{E}_0^t$.

Let us study the stability of the homogeneous damage state α_t. Firstly, we focus on the first derivative and analyze under which condition it is positive or not. Using the homogeneous property of α_t, we obtain

$$\mathcal{E}_0^{t\prime}(\alpha_t)(\beta) = \int_0^L \left(w'(\alpha_t) + \frac{1}{2}E'(\alpha_t)t^2 \right) \beta \, dx = \left(w'(\alpha_t) + \frac{1}{2}E'(\alpha_t)t^2 \right) \int_0^L \beta \, dx \tag{2.30}$$

If $t < t_e$, then $\alpha_t = 0$ and the damage criterion for the homogeneous damage state $(2.16)_1$ is a strict inequality. The first derivative (2.28) is then strictly positive if β is not equal to 0 and we conclude to the stability of the homogeneous state during the elastic phase. Now if $t \geq t_e$, then from (2.18), we deduce that the first derivative is equal to zero in any direction test $\beta \in \mathcal{D}$. Then the stability issue is given by the sign of the second order derivative (2.29). By using the non dependence of α_t on x, (2.29) can be simplified and read as

$$\mathcal{E}_0^{t\prime\prime}(\alpha_t)(\beta) \quad = \frac{t^2 E'(\alpha_t)^2}{LE(\alpha_t)} \left(\int_0^L \beta \, dx \right)^2 - \left(\left(\frac{2E'(\alpha_t)^2}{E(\alpha_t)} - E''(\alpha_t) \right) \frac{t^2}{2} - w''(\alpha_t) \right) \int_0^L \beta^2 \, dx \tag{2.31}$$

Inserting the damage law (2.19) into (2.31), we find for $t \geq t_e$

$$\mathcal{E}_0^{t\prime\prime}(\alpha_t)(\beta) \quad = \frac{4E_0 t^2}{L} \left(\int_0^L \beta \, dx \right)^2 - 3E_0 t^2 \int_0^L \beta^2 \, dx \tag{2.32}$$

Despite the presence of the first term in (2.32) which is positive, it is always possible to find a damage state for which (2.32) is negative and therefore leads to instabilities. Indeed, let (β_n) be the following sequence of admissible damage state defined by

$$\beta_n(x) = \begin{cases} 1 & \text{if} \quad 0 \leq x \leq \frac{L}{n} \\ 0 & \text{otherwise} \end{cases}$$

Then we obtain the following estimates when $n \to \infty$

$$\left(\int_0^L \beta_n \, dx \right)^2 \sim \frac{1}{n^2}, \qquad \int_0^L \beta_n^2 \, dx \sim \frac{1}{n} \tag{2.33}$$

We conclude that for n large enough $\mathcal{E}_0^{t\prime\prime}(\alpha_t)(\beta_n)$ will be strictly negative and the state $\alpha_t + h\beta_n$ will have a smaller energy than α_t. Therefore at time $t > t_e$, the homogeneous state is not stable.

3 Gradient damage model

3.1 Presentation

A way to prevent strong variations of the constitutive variables at the scale of the microstructure is to regularize the model by introducing some gradient effects. As the damage variable causes strain softening and localization in the evolution, we apply the regularization only to the damage variable. In the previous section, we see that the local formulation of the damage problem (2.2)–(2.4) and the equilibrium (2.5) were strictly equivalent to a global variational approach based on the strain work W_0 (2.6). This variational approach also permitted to define rigorously a stability criterion for a damage state in terms of *unilateral* local minimas of the potential energy. Now, (re-)starting from this variational framework, we introduce the non locality of our model directly in the postulated strain work W_ℓ as follows

$$W_\ell(\varepsilon, \alpha, \alpha') = \frac{1}{2}E_0 \ell^2 \alpha'^2 + W_0(\varepsilon, \alpha) \tag{3.1}$$

where α' is the space derivative of the damage field at the material point x, ℓ denotes the internal length of the material and E_0 is the Young modulus of the sound material ($E_0 = E(0)$). Then the total energy of the bar becomes

$$\mathcal{P}_\ell(u,\alpha) = \int_0^L W_\ell(u',\alpha,\alpha')\,dx = \int_0^L \frac{1}{2}E_0\ell^2\alpha'^2 + \frac{1}{2}E(\alpha)u'^2 + w(\alpha)\,dx \tag{3.2}$$

We remark that we obtain the energy of the local model by putting ℓ equals to 0 in (3.2). All the notions of "evolution" (2.9) and "stability" (2.23) which were introduced in the previous section are now being transposed to W_ℓ. In particular the evolution problem for the gradient damage model now reads as

$$\begin{gathered}\text{Find } (u_t,\alpha_t) \text{ in } \mathcal{C}_t \times \mathcal{D} \text{ such that}\\ \text{For all } (v,\beta) \in \mathcal{C}_1 \times \mathcal{D}, \quad \mathcal{P}'_\ell(u_t,\alpha_t)(v-\dot{u}_t,\beta-\dot{\alpha}_t) \geq 0\end{gathered} \tag{3.3}$$

where $\mathcal{P}'_\ell(u,\alpha)(v,\beta)$ denotes the derivative of $\mathcal{P}_\ell$ at (u,α) in the direction (v,β) and is given by

$$\mathcal{P}'_\ell(u,\alpha)(v,\beta) = \int_0^L E_0\ell^2\alpha'\beta'\,dx + \int_0^L E(\alpha)u'v'\,dx + \int_0^L \left(\frac{1}{2}E'(\alpha)u'^2 + w'(\alpha)\right)\beta\,dx \tag{3.4}$$

On the one hand, as the regularization does not affect the kinematic variable, the equilibrium of the bar is not changed and is still given by (2.12) and (2.13). On the other hand because of the regularizing damage term, the obtention of the strong formulation for the damage evolution is slightly different than for the underlying local model though it is still deduced from the variational inequality (3.3). By inserting $v = \dot{u}$ and $\beta = \dot{\alpha} + \gamma$, with γ in $\mathcal{D}$ into (3.3), we obtain the variational inequality governing the damage field evolution

$$\int_0^L \left(E'(\alpha_t)\frac{\varepsilon_t^2}{2} + w'(\alpha_t)\right)\gamma + E_0\ell^2\alpha'_t\gamma'\,dx \geq 0 \tag{3.5}$$

where the inequality must hold for all $\gamma \in \mathcal{D}$ and becomes an equality when $\gamma = \dot{\alpha}$. By integrating by part (3.5), we obtain

$$\int_0^L \left(E'(\alpha_t)\frac{\varepsilon_t^2}{2} + w'(\alpha_t) - E_0\ell^2\alpha''_t\right)\gamma\,dx + [\ell^2\alpha'_t\gamma]_0^L \geq 0 \tag{3.6}$$

Using classical arguments of calculus of variations, the strong formulation of the evolution of the damage now reads as

$$\text{Irreversibility condition} : \dot{\alpha}_t \geq 0 \tag{3.7}$$

$$\text{Damage criterion} : -E_0\ell^2\alpha''_t + \frac{1}{2}E'(\alpha_t)\varepsilon_t^2 + w'(\alpha_t) \geq 0 \tag{3.8}$$

$$\text{Energy balance} : \dot{\alpha}_t\left(-E_0\ell^2\alpha''_t + \frac{1}{2}E'(\alpha_t)\varepsilon_t^2 + w'(\alpha_t)\right) = 0 \tag{3.9}$$

$$\text{Boundaries conditions} : \alpha'_t(0) \geq 0, \quad \alpha'_t(L) \geq 0, \quad \dot{\alpha}_t(0)\alpha'_t(0) = 0, \quad \dot{\alpha}_t(L)\alpha'_t(L) = 0 \tag{3.10}$$

Following the same variational approach than for the local model, we investigate for the gradient damage model the behavior and the stability of homogeneous states of damage. From (3.7)–(3.10), we deduce immediately that the evolution in time of homogeneous states is exactly governed by the same set of equalities and inequalities as for the local damage model. Indeed in the case of homogeneous states, the spatial derivatives are equal to 0 ($\alpha'_t = \alpha''_t = 0$). Therefore, the value of the homogeneous damage (2.20) at time t remains the same for the enhanced gradient damage model. However significant differences happen when we focus on the stability analysis. Indeed in the case of the local model, the instability of any solution is the consequence of the possibility to localize the damage on a vanishing length. As a result, the presence of gradient damage terms in the energy (3.2) should prevent from this kind of localization of zero energy and thereby we expect some improvements in the stability results.

3.2 Stability issue for homogeneous states

As the stability of a damaged state relies on its energy and because we modified the energy itself by incorporating gradient terms, we define for the non-local damage model the $\mathcal{E}_\ell^t$-stability where $\mathcal{E}_\ell^t$ is given by

$$\mathcal{E}_\ell^t(\alpha) \equiv \mathcal{P}_\ell(u_t^\alpha, \alpha) = \min_{v \in \mathcal{C}_t} \mathcal{P}_\ell(v, \alpha) \tag{3.11}$$

as follows

$$\mathcal{E}_\ell^t\textit{-stability:} \quad \exists r > 0, \forall \beta \in \mathcal{D} : \|\beta\| = 1, \forall h \in [0, r] \quad \mathcal{E}_\ell^t(\alpha) \le \mathcal{E}_\ell^t(\alpha + h\beta) \tag{3.12}$$

From (2.24)-(2.25) which remain true, we deduce

$$\mathcal{E}_\ell^t(\alpha) = \int_0^L \frac{1}{2} E_0 \ell^2 \alpha'^2 + \frac{t^2 L^2}{2 \int_0^L \frac{dx}{E(\alpha)}} + \int_0^L w(\alpha)\, dx \tag{3.13}$$

Using the same approach as for the underlying local model (see Section 2.4), we calculate the successive derivatives of $\mathcal{E}_\ell^t$ at state α_t. The first derivative is given by

$$\begin{aligned} \mathcal{E}_\ell^{t\prime}(\alpha_t)(\beta) &= \int_0^L E_0 \ell^2 \alpha_t'(x) \beta'(x)\, dx + \int_0^L \left(w'(\alpha_t) + \frac{1}{2} E'(\alpha_t) t^2 \right) \beta\, dx \\ &= \left(w'(\alpha_t) + \frac{1}{2} E'(\alpha_t) t^2 \right) \int_0^L \beta\, dx \end{aligned} \tag{3.14}$$

where we use the fact that $\alpha_t'(x) = 0$ over the whole bar. Then the first derivative is the same as for the underlying local model and we deduce with the same arguments that the elastic phase is stable. For $t > t_e$, $\mathcal{E}_\ell^{t\prime}(\alpha_t)(\beta) = 0$ for any $\beta \in \mathcal{D}$ and the stability is given by the sign of the second derivative. Here the second derivative reads as

$$\begin{aligned} \mathcal{E}_\ell''(\alpha_t)(\beta) = \quad & E_0 \ell^2 \int_0^L \beta'^2\, dx + \frac{t^2}{L} \frac{E'(\alpha_t)^2}{E(\alpha_t)} \left(\int_0^L \beta dx \right)^2 \\ & - \left(\left(\frac{E'(\alpha_t)^2}{E(\alpha_t)} - \frac{E''(\alpha_t)}{2} \right) t^2 - w''(\alpha_t) \right) \int_0^L \beta^2\, dx \end{aligned} \tag{3.15}$$

By introducing the Rayleigh ratio $\mathcal{R}_\ell^t$ defined on $\mathcal{D} \backslash \{0\}$ by

$$\mathcal{R}_\ell^t(\beta) = \frac{E_0 \ell^2 \int_0^L \beta'^2\, dx + \frac{t^2}{L} \frac{E'(\alpha_t)^2}{E(\alpha_t)} \left(\int_0^L \beta dx \right)^2}{\left(\left(\frac{E'(\alpha_t)^2}{E(\alpha_t)} - \frac{E''(\alpha_t)}{2} \right) t^2 - w''(\alpha_t) \right) \int_0^L \beta^2\, dx} \tag{3.16}$$

we deduce that for $t > t_e$

$$\alpha_t \text{ is } \mathcal{E}_\ell^t\text{-stable at time } t \quad \Leftrightarrow \quad \lambda_\ell^t = \inf_{\beta \in \mathcal{D}} \mathcal{R}_\ell^t(\beta) > 1 \tag{3.17}$$

After some calculations which are not reproduced here, we find

$$\lambda_\ell^t = \frac{1}{\left(\frac{E'(\alpha_t)^2}{E(\alpha_t)} - \frac{E''(\alpha_t)}{2} \right) t^2 - w''(\alpha_t)} \cdot \min \left(\frac{E'(\alpha_t)^2}{E(\alpha_t)} t^2, \left(\frac{\sqrt{E_0} \pi t^2 E'(\alpha_t)^2}{E(\alpha_t)} \frac{\ell}{L} \right)^{2/3} \right) \tag{3.18}$$

By putting the internal length ℓ to 0 we recover the instability result of the underlying local model for $t > t_e$ since $\lambda_0^t = 0$. On the contrary, if $\ell \neq 0$ then we have $\lambda_\ell^t > \lambda_0^t$: the condition of localisation from the homogeneous state in the local damage model is always a lower bound of the criterion of localization in the non-local one. Moreover, the non-local damage model introduces a size effect which involves the length of the bar in the stability results. More precisely, depending on the value of the ratio $\eta = \ell/L$, we distinguish two different behaviors in the evolution of the bar after the end of the elastic phase. In particular, if we consider the damage law (2.19), then the stability criterion (3.18) reads as

$$\lambda_\ell^t = \min\left(\frac{4}{3}, \frac{4}{3}\left(\frac{2\pi E_0 t_e}{\sigma_0 t}\frac{\ell}{L}\right)^{2/3}\right) \tag{3.19}$$

Using (3.17) and (3.19), we can identify for any given ratios t/t_e and $\eta = L/\ell$ the zone of stability (i.e.

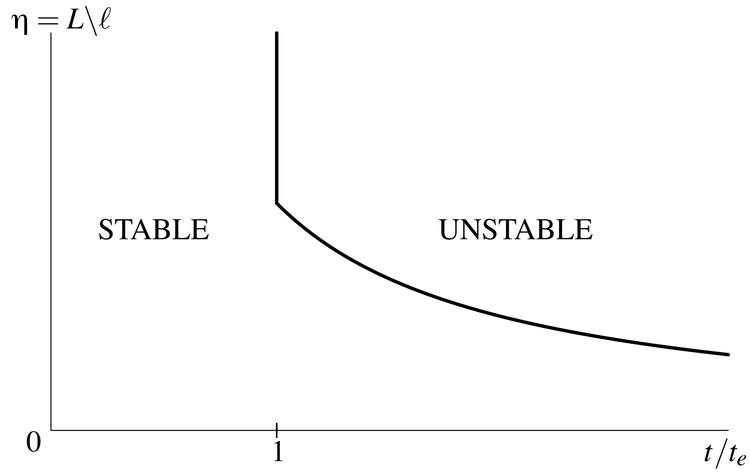

Figure 2: Stability diagram of the homogeneous states of the bar

$\lambda_\ell^t > 1$) and unstability (i.e. $\lambda_\ell^t < 1$) (Fig.2). While in the underlying local model any homogeneous state after the elastic phase is unstable for any length of bar, certain homogeneous states beyond the elastic phase are $\mathcal{E}_\ell^t$-stable if the bar is small enough. It means that it is physically possible to observe them during a tensile test provided that the length of the bar is sufficiently small. On the contrary, if the length of the bar exceeds a critical length, we see on Fig.2 that all homogeneous states remain unstable for $t > t_e$ and a non-homogeneous state necessarily appears at $t = t_e$.

4 Conclusion

A comparison between a local damage model and a gradient damage model has been carried out in the one-dimensional context of a bar made of a softening material and submitted to a tensile test. Even if for both models, the same homogeneous response is solution of the corresponding damage evolution problem, the properties of stability of such a response drastically depends on the model. In the case of a local model the homogeneous state is unstable once the first damage threshold is reached, while in the case of the enhanced gradient model the stability analysis is not so trivial and can be interpreted in terms of size effect. Specifically, for sufficiently "short bars" (i.e. for L/ℓ small enough), the homogeneous state is stable, at least in a certain time interval after the elastic phase. On the other hand, for long bars (L/ℓ large enough), all homogeneous states beyond the elastic phase are unstable (like for a local model) and the structure localizes its damage. A way to preserve the stability of the homogeneous damage in the bar in the full range of the test and for any bar length should be to modify the experimental procedure in the spirit of Mazars *et al.* work. Indeed, in [11], by sticking aluminium bars to a concrete specimen, Mazars *et al.* showed that a certain homogeneity of the strain is preserved during the tensile test. We could account for these added aluminium bars in our gradient damage model by introducing a residual elastic energy (the Young modulus will never fall to 0) and analyze their consequences on the stability of the homogeneous states. This quite appealing work is under investigation.

References

[1] Z. P. Bažant, T. Belytschko, and T. P. Chang. Continuum theory for strain-softening. *J. Engrg Mech.*, 110:1666–1692, 1984.

[2] A. Benallal, R. Billardon, and G. Geymonat. Bifurcation and localization in rate independent materials. In Q.S. Nguyen, editor, *C.S.I.M Lecture Notes on Bifurcation and Stability of Dissipative Systems*. Springer-Verlag, 1993.

[3] A. Benallal and J.-J. Marigo. Bifurcation and stability issues in gradient theories with softening. *Modelling Simul. Mater. Sci. Eng.*, 15:283–295, 2007.

[4] B. Bourdin, G.A. Francfort, and J.-J. Marigo. The variational approach to fracture. *J. Elasticity*, 91:5–148, 2008.

[5] R. de Borst, L.J. Sluys, Mühlhaus H.-B., and J. Pamin. Fundamental issues in finite element analysis of localization of deformation. *Eng. Comp.*, 10:99–121, 1993.

[6] G.A. Francfort and J.-J. Marigo. Stable damage evolution in a brittle continuous medium. *Eur. J. Mech. A/Solids*, 12:149–189, 1993.

[7] D. Lasry and Belytschko. Localization limiters in transient problems. *Int. J. Solids Structures*, 24:581–587, 1988.

[8] E. Lorentz and S. Andrieux. Analysis of non-local models through energetic formulations. *Int. J. Solids Struct.*, 40:2905–2936, 2003.

[9] J.-J. Marigo. Constitutive relations in Plasticity, Damage and Fracture Mechanics based on a Work Property. *Nuclear Engineering and Design*, 114:249–272, 1989.

[10] J.-J. Marigo. From Clausius-Duhem and Drucker-Ilyushin inequalities to standard materials. In G. A. Maugin, R. Drouot, and F. Sidoroff, editors, *Continuum Thermodynamics: the art and science of modelling material behaviour*, volume 76 of *Solids Mechanics and Its Applications: Paul Germain's anniversary*. Kluwer Acad. Publ., 2000.

[11] J. Mazars, S. Ramtani, and Y. Berthaud. An experimental procedure to delocalize tensile failure and to indentify the unilateral effect of tensile damage. In J. Mazars and Z. P. Bažant, editors, *Workshop on strain localization and size effect due to damage and cracking*, pages 581–587. Elsevier, Northwestern, 1988.

[12] Q. S. Nguyen. *Stability and Nonlinear Solid Mechanics*. Wiley & Son, London, 2000.

[13] R.H.J. Peerlings, R. de Borst, W.A.M. Brekelmans, J.H.P. de Vree, and I. Spee. Some observations on localisation in non-local and gradient damage models. *Eur. J. Mech. A/Solids*, 15:937–953, 1996.

[14] G. Pijaudier-Cabot and Z. P. Bažant. Non-local damage theory. *J. Eng. Mech.*, 113:1512–1533, 1987.

[15] G. Pijaudier-Cabot and A. Benallal. Strain localization and bifurcation in a non-local continuum. *Int. J. Solids Struct.*, 30:1761–1775, 1993.

[16] N. Triantafyllidis and E. C. Aifantis. A gradient approach to localization of deformation: I. hyperelastic materials. *J. Elasticity*, 16:225–237, 1986.