Diffraction from a subwavelength elliptic aperture: Analytic approximate aperture fields

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An analytical approximate solution of the electromagnetic field on a subwavelength elliptical hole in a thin perfectly conducting screen is presented. Illumination is a linear polarized, normally incident plane wave. A polynomial development method is used and allows one to obtain an easy-to-use analytical solution of the fields, which can be used to build analytical expressions of aperture fields for apertures in anisotropic structures.

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1. INTRODUCTION

Diffraction by a subwavelength aperture on a plane screen is a classical problem in electromagnetism [1–3]. More recently, the question of the transmission through subwavelength structures has become central in metamaterials and subwavelength arrays [4–6]. Even though numerical calculations are often used to simulate subwavelength structures, analytical developments of the electromagnetic field propagation through apertures of various shapes are still of great interest. Elliptic-shaped apertures have a special interest because interactions between the radiation and the hole are strongly driven by the anisotropic geometry [7], yet they have enough symmetry to allow analytical approximations. These analytical solutions may also be useful to simulate the complex electromagnetic transmission through metamaterials allowing fast preconditioning of the fields propagation in the different subwavelength parts. One of the main issues in these simulations is the time to converge as both electric and magnetic fields tend to diverge near every boundary. Elliptical shapes are especially useful as they can be a good approximation of more complicated shapes.

The problem of diffraction by subwavelength circular aperture was the most investigated. Rayleigh [1] introduced the idea of solving the problem with power series in k, and Bethe [2] found a scalar potential solution with a little error in the first-order approximation. Bouwkamp [3] and Eggimann [8] corrected Bethe's solutions and gave exact power series development of the electromagnetic field in the near-field and far-field zone. These two authors also gave vast bibliographies [8,9] of diffraction problems, highlighting the most important aspects of these problems. For other shapes, far-field approximations based on magnetic and electric dipolar moments have been developed [6,10–12], but no satisfactory analytical solutions such as the ones found for circular apertures have been found. Yet, for a subwavelength square aperture a semi-analytical expression for the aperture was found for a linear polarized, normally incident planar wave, in a unique direction of polarization [13]. Obviously, vast numbers of numerical strategies [14–18] can be used to evaluate the aperture fields, yet analytical expansions are useful to investigate these diffractions problems, in particular for preconditioning conditions.

The problem, as Eggimann [8] wrote it, can be expressed in the following way: (i) Maxwell's equation must be followed, (ii) the tangential magnetic field must vanish on the aperture, (iii) the electromagnetic field energy must remain finite inside the aperture, (iv) Sommerfeld's [19] radiation conditions must be fulfilled. The problem is solved by expanding every field in the power series of ka and kb, then every term is expanded in polynomial forms and finally all the fields are extracted by solving the linear systems linking all the developments coefficients together.

2. CALCULATION OF THE FIELDS IN THE ELLIPTICAL APERTURE

A perfectly reflecting screen S of vanishing thickness lies at $z = 0$ with an elliptic hole centered at $(x = 0, y = 0)$ with semimajor axis a and semiminor b (Fig. 1.). A monochromatic electromagnetic plane wave field $\vec{E}^i$ is incident to the screen from $z < 0$. The transmitted electric field in $z \geq 0$ is $\vec{E}^t$. Only the steady-state problem is discussed. It is tacitly understood that the time factor is $e^{-j\omega t}$, where j is the imaginary unit, ω the angular frequency, and t the time. The wavenumber is denoted by $k = 2\pi/\lambda$ with λ the wavelength.

Copson [20] showed that the transmitted fields should be written, assuming $\vec{r} = (x, y, z)$,

$$\begin{cases} \vec{E}^t(\vec{r}) = \frac{1}{\varepsilon_0} \nabla \times \vec{F}(\vec{r}) \\ \vec{H}^t(\vec{r}) = \frac{1}{j\mu_0 kc} \nabla \times \vec{E}^t(\vec{r}) \end{cases}, \tag{1}$$

where ε_0 is the permittivity of vacuum, μ_0 the permeability of vacuum, c the celerity of light in vacuum, $\vec{H}^t(\vec{r})$ the

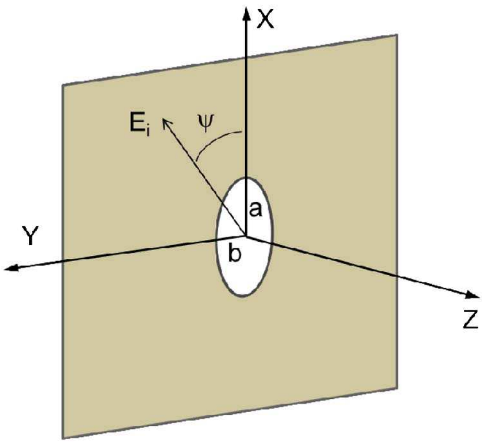

Fig. 1. (Color online) Infinitely thin, perfectly conducting screen with elliptical hole of semimajor axis a and semiminor axis b. Both $kb \ll 1$ and $ka \ll 1$. The electromagnetic plane wave is incident from $z < 0$, ψ is the angle between the incident electric field, and the x axis. The transmitted field $\vec{E}^t$ propagates in the $z > 0$ direction.

transmitted magnetic field, and $\vec{F}(\vec{r})$ a potential vector defined by

$$\vec{F}(\vec{r}) = \frac{\varepsilon_0}{2\pi} \iint_{\text{ellipse}} \vec{n} \times \vec{E}^t(x', y', 0) \frac{e^{jkR}}{R} \mathrm{d}x' \mathrm{d}y'$$
$$\text{with } R = \sqrt{(x - x')^2 + (y - y')^2 + z^2}, \tag{2}$$

and with $\vec{n}$ the unit vector normal to the surface of the screen in the $z > 0$ direction. In the aperture, the boundary conditions are given by

$$\begin{cases} H_x^t(\vec{r}) = -H_x^i(\vec{r}) \\ H_y^t(\vec{r}) = -H_y^i(\vec{r}) \\ E_z^t(\vec{r}) = -E_z^i(\vec{r}) \end{cases}. \tag{3}$$

Using Maxwell's equations and (2) lead to the following set of equations in the aperture:

$$\begin{cases} \nabla_{xy}^2 F_x + k^2 F_x = -\varepsilon_0 \frac{\partial E_y^i}{\partial z} \\ \nabla_{xy}^2 F_y + k^2 F_y = \varepsilon_0 \frac{\partial E_x^i}{\partial z} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \varepsilon_0 E_z^i \end{cases}, \tag{4}$$

with $\nabla_{xy}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Thus, in order to find the electromagnetic field inside the aperture, we must seek $\vec{F}$ inside the aperture and then solve (2). The last requirement is the fulfillment of Sommerfeld's divergence conditions [19] on the metal edge; namely, that the normal electric field component increases as $1/\sqrt{R}$.

The strategy to solve this problem is as follows: (i) the fields $\vec{n} \times \vec{E}^t$ and $\vec{F}$ are expanded in series of ka and kb, (ii) their components are developed with polynomials in x and y, (iii) Sommerfeld's boundary conditions are applied, (iv) the condition that the field remains finite at the rim of the disk imposes that the transmitted electric fields are to be found in the form $D(x,y)/\sqrt{1 - x^2/a^2 - y^2/b^2}$, where $D(x,y)$ is a polynomial whose degrees and coefficients are calculated.

A. Series Expansion

In the following, we attempt to find a power series expansion of the electric field in terms of ka and kb, which is expected to converge well for small elliptical apertures since $ka \ll 1$ and $kb \ll 1$. Let $\vec{J}(x', y') = \vec{n} \times \vec{E}^t(x', y', 0)$. $\vec{J}$ and $\vec{F}$ are developed in series of k

$$\begin{cases} \vec{J} = \vec{J}^0 + k\vec{J}^1 + k^2\vec{J}^2 + k^3\vec{J}^3 + \cdots \\ \vec{F} = \vec{F}^0 + k\vec{F}^1 + k^2\vec{F}^2 + k^3\vec{F}^3 + \cdots \end{cases}. \tag{5}$$

We then obtain

$$\vec{J}e^{jkr} = \vec{J}^0 + k(\vec{J}^1 + jr\vec{J}^0) + k^2\left(\vec{J}^2 + jr\vec{J}^1 - \frac{1}{2}r^2\vec{J}^0\right) + k^3\left(\vec{J}^3 + jr\vec{J}^2 - \frac{1}{2}r^2\vec{J}^1 - \frac{1}{6}jr^3\vec{J}^0\right) + \cdots. \tag{6}$$

Thus, including (1), and limiting to the third order:

$$\begin{cases} \vec{F}^0 = \frac{\varepsilon_0}{2\pi} \iint_{\text{ellipse}} \vec{J}^0 \frac{\mathrm{d}x'\mathrm{d}y'}{R} \\ \vec{F}^1 = \frac{\varepsilon_0}{2\pi} \left(\iint_{\text{ellipse}} \vec{J}^1 \frac{\mathrm{d}x'\mathrm{d}y'}{R} + j \iint_{\text{ellipse}} \vec{J}^0 \mathrm{d}x'\mathrm{d}y' \right) \\ \vec{F}^2 = \frac{\varepsilon_0}{2\pi} \left(\iint_{\text{ellipse}} \vec{J}^2 \frac{\mathrm{d}x'\mathrm{d}y'}{R} + j \iint_{\text{ellipse}} \vec{J}^1 \mathrm{d}x'\mathrm{d}y' - \frac{1}{2} \iint_{\text{ellipse}} J^0 R \mathrm{d}x'\mathrm{d}y' \right) \\ \vec{F}^3 = \frac{\varepsilon_0}{2\pi} \left(\iint_{\text{ellipse}} \vec{J}^3 \frac{\mathrm{d}x'\mathrm{d}y'}{R} + j \iint_{\text{ellipse}} \vec{J}^2 \mathrm{d}x'\mathrm{d}y' - \frac{1}{2} \iint_{\text{ellipse}} J^1 R \mathrm{d}x'\mathrm{d}y' - \frac{j}{6} \iint_{\text{ellipse}} J^0 R^2 \mathrm{d}x'\mathrm{d}y' \right) \end{cases}. \tag{7}$$

1. Zeroth-Order Development

We will demonstrate that, at zeroth order, both $\vec{J}^0$ and $\vec{F}^0$ are null, as well as all even orders in the series expansions of the fields. For a linearly polarized, normally incident plane wave, introducing (5) in (4) reduces to

$$\nabla_{xy}^2 F_x^0 = \nabla_{xy}^2 F_y^0 = \frac{\partial F_y^0}{\partial x} - \frac{\partial F_x^0}{\partial y} = 0. \tag{8}$$

Thus, both F_x^0 and F_y^0 are linear in variables x and y. In order to obtain the electric field at zeroth order, the first equation of (7) has to be solved. It is first kind Fredholm problem [21], and

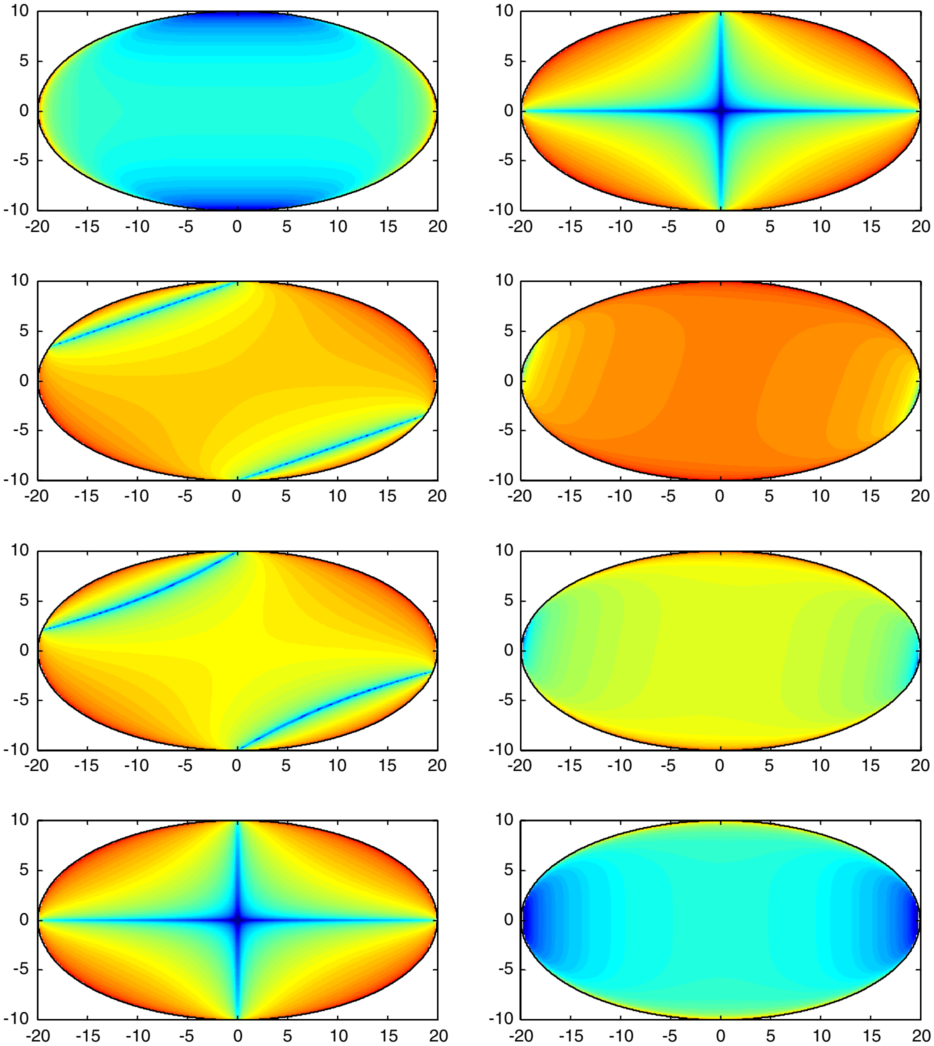

Fig. 2. (Color online) Evolution of the electric field in an elliptical hole ($a = 2b$) with varying incident polarization. On the left E_x and on the right E_y, from top to bottom, $\psi = (0, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2})$. The plotted quantity is $\log|E|$ in order to increase the contrast of the patterns.

its general solution is given in Appendix A. It follows that $\vec{J}^0$ may be written in the following form:

$$\vec{J}^0(x', y') = \frac{D_x^0(x', y')}{\sqrt{1 - \frac{x'^2}{a^2} - \frac{y'^2}{b^2}}}\vec{e}_x + \frac{D_y^0(x', y')}{\sqrt{1 - \frac{x'^2}{a^2} - \frac{y'^2}{b^2}}}\vec{e}_y, \tag{9}$$

where $D_x^0(x', y')$ and $D_y^0(x', y')$ are polynomials. In Appendix A, we show in (A.1.1) that the two polynomials have the same degree as F_x^0 and F_y^0. Furthermore, the supplementary condition of finiteness of the electromagnetic energy, corresponding to Sommerfeld's condition [8,19], is satisfied if

$$x' D_x^0(x', y') + y' D_y^0(x', y') = K^0(x', y')\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right), \tag{10}$$

where $K^0(x', y')$ is a polynomial. So, at zeroth-order approximation $\vec{J}^0$ and $\vec{F}^0$ must vanish. Bouwkamp found the same result for a circular aperture [22]. Note that if the electromagnetic field were not normal to the surface the two fields would not vanish (see section C). Furthermore, equations (4) with a normally incident plane wave lead to the nullity of all even orders development of the all the fields.

2. First-Order Development

Equations (4) reduce to

$$\begin{aligned} \nabla_{xy}^2 F_x^1 &= -j\varepsilon_0 E^i \sin(\psi), \\ \nabla_{xy}^2 F_y^1 &= j\varepsilon_0 E^i \cos(\psi), \\ \frac{\partial F_y^1}{\partial x} &= \frac{\partial F_x^1}{\partial y}, \end{aligned} \tag{11}$$

where ψ is the polarization angle of the incident plane wave, as seen in Fig. 1. Then Sommerfeld's conditions now become

$$x' D_x^1(x', y') + y' D_y^1(x', y') = K^1(x', y')\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right), \tag{12}$$

where K^1 is a polynomial. Since we showed that $\vec{J}^0 = 0$, (7) reduces at the first order to

$$\vec{F}^1 = \frac{\varepsilon_0}{2\pi}\iint_{\text{ellipse}} \vec{J}^1 \frac{\mathrm{d}x'\mathrm{d}y'}{R}. \tag{13}$$

We again write $\vec{F}$ as a polynomial in x and y as

$$\begin{aligned} F_x^1 &= \alpha_0^1 + \alpha_1^1 x + \alpha_2^1 y + \alpha_3^1 x^2 + \alpha_4^1 xy + \alpha_5^1 y^2, \\ F_y^1 &= \beta_0^1 + \beta_1^1 y + \beta_2^1 x + \beta_3^1 y^2 + \beta_4^1 yx + \beta_5^1 x^2, \end{aligned} \tag{14}$$

and using (11), this leads to

$$\begin{aligned} &2\alpha_3^1 + 2\alpha_5^1 = -j\varepsilon_0 E^i \sin(\psi), \qquad 2\beta_3^1 + 2\beta_5^1 = j\varepsilon_0 E^i \cos(\psi), \\ &\beta_2^1 = \alpha_2^1, \qquad 2\beta_5^1 = \alpha_4^1, \qquad 2\alpha_5^1 = \beta_4^1. \end{aligned} \tag{15}$$

Finally, we write

$$\begin{cases} J_x^1(x', y') = \dfrac{(\eta_0^1 + \eta_1^1 x' + \eta_2^1 y' + \eta_3^1 x'^2 + \eta_4^1 x' y' + \eta_5^1 y'^2)}{\sqrt{1 - \frac{x'^2}{a^2} - \frac{y'^2}{b^2}}} \\ J_y^1(x', y') = \dfrac{(\theta_0^1 + \theta_1^1 y' + \theta_2^1 x' + \theta_3^1 y'^2 + \theta_4^1 y' x' + \theta_5^1 x'^2)}{\sqrt{1 - \frac{x'^2}{a^2} - \frac{y'^2}{b^2}}} \end{cases}. \tag{16}$$

Solving η_i^1 and θ_i^1, where i ranges from 0 to 5, will provide the solution at the first-order development. Equation (12) leads to

$$\begin{aligned} &\eta_1^1 = \theta_1^1 = 0, \qquad \eta_2^1 = -\theta_2^1, \qquad \eta_3^1 = -\frac{\eta_0^1}{a^2}, \\ &\eta_4^1 + \theta_5^1 = -\frac{\theta_0^1}{a^2}, \qquad \eta_5^1 + \theta_4^1 = -\frac{\eta_0^1}{b^2}, \qquad \theta_3^1 = -\frac{\theta_0^1}{b^2}. \end{aligned} \tag{17}$$

Here are the equations linking the (α^1, β^1) to the (η^1, θ^1), using Table 1 and Eqs. (13), (14), and (16):

$$\begin{array}{lll} \alpha_0^1 = \pi^2 g_0 \eta_0^1 + \pi^2 C_3 \eta_3^1 + \pi^2 C_6 \eta_5^1 & \alpha_1^1 = 0 & \alpha_2^1 = \pi^2 C_0 \eta_2^1 \\ \alpha_3^1 = \pi^2 C_1 \eta_3^1 + \pi^2 C_5 \eta_5^1 & \alpha_4^1 = \pi^2 C_7 \eta_4^1 & \alpha_5^1 = \pi^2 C_2 \eta_3^1 + \pi^2 C_4 \eta_5^1 \\ \beta_0^1 = \pi^2 g_0 \theta_0^1 + \pi^2 C_6 \theta_3^1 + \pi^2 C_3 \theta_5^1 & \beta_1^1 = 0, & \beta_2^1 = \pi^2 C_{-1} \theta_2^1, \\ \beta_3^1 = \pi^2 C_4 \theta_3^1 + \pi^2 C_2 \theta_5^1, & \beta_4^1 = \pi^2 C_7 \theta_4^1 & \beta_5^1 = \pi^2 C_5 \theta_3^1 + \pi^2 C_1 \theta_5^1. \end{array} \tag{18}$$

We then obtain 12 linear independent equations for the (η^1, θ^1). Using (7) and Table 1, we find the solution presented in Table 3.

Solving (10), (12), and (13), and remembering that $\vec{J}^1 = \vec{n} \times \vec{E}^{t(1)}$, the electric field inside the elliptical aperture reads

$$\begin{cases} E_x^{t(1)} = \dfrac{\theta_0^1 + \theta_3^1 y^2 + \theta_4^1 yx + \theta_5^1 x^2}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} \\ E_y^{t(1)} = -\dfrac{\eta_0^1 + \eta_3^1 x^2 + \eta_4^1 xy + \eta_5^1 y^2}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}, \end{cases} \tag{19}$$

where the values of (η, θ) are given in Table 3.

3. Third-Order Development

We now consider the third-order development in (7). As we stated before, the second-order fields will vanish. Let us evaluate the third-order fields. Equation (4) now becomes

$$\nabla_{xy}^2 F_x^3 + F_x^1 = 0, \qquad \nabla_{xy}^2 F_y^3 + F_y^1 = 0, \qquad \frac{\partial F_y^3}{\partial x} = \frac{\partial F_x^3}{\partial y}. \tag{20}$$

Both (F_x^3, F_y^3) are fourth-order polynomials and so both (E_x^3, E_y^3) have a fourth-order numerator. The procedure to find the coefficients is similar to the one used before. Using (4), (7), and (10) with the help of Tables A1 and A2 leads to 30 linear equations for the coefficients of (E_x^3, E_y^3). We write

$$\begin{aligned} F_x^3(x, y) &= \sum_{(i+j \le 4)} \alpha_{f(i,j)}^3 x^i y^j \quad \text{and} \\ F_y^3(x, y) &= \sum_{(i+j \le 4)} \beta_{f(i,j)}^3 y^i x^j, \end{aligned} \tag{21}$$

$$\begin{aligned} J_x^3(x', y') &= \sum_{(i+j \le 4)} \frac{\eta_{f(i,j)}^3 x'^i y'^j}{\sqrt{1 - \frac{x'^2}{a^2} - \frac{y'^2}{b^2}}} \quad \text{and} \\ J_y^3(x', y') &= \sum_{(i+j \le 4)} \frac{\theta_{f(i,j)}^3 y'^i x'^j}{\sqrt{1 - \frac{x'^2}{a^2} - \frac{y'^2}{b^2}}}, \end{aligned} \tag{22}$$

with $f(i,j) = \frac{(i+j)(i+j+1)}{2} + j$. In a way similar as in first-order development, (4), (7), and (10) lead to 30 independent linear equations involving (η^3, θ^3). The system may be solved in a specific order to ease the solving procedure: $(\eta_7^3, \eta_9^3, \theta_7^3, \theta_9^3)$, $(\eta_6^3, \eta_8^3, \theta_6^3, \theta_8^3)$, (η_1^3, θ_1^3), (η_2^3, θ_2^3), $(\eta_{10}^3, \eta_{11}^3, \eta_{12}^3, \eta_{13}^3, \eta_{14}^3, \theta_{10}^3, \theta_{11}^3, \theta_{12}^3, \theta_{13}^3, \theta_{14}^3)$, and finally $(\eta_0^3, \eta_3^3, \eta_4^3, \eta_5^3, \theta_0^3, \theta_3^3, \theta_4^3, \theta_5^3)$.

The first four steps in the solving procedure lead to

$$\begin{aligned} &\eta_1^3 = \eta_2^3 = \eta_6^3 = \eta_7^3 = \eta_8^3 = \eta_9^3 = 0 \\ &\theta_1^3 = \theta_2^3 = \theta_6^3 = \theta_7^3 = \theta_8^3 = \theta_9^3 = 0. \end{aligned} \tag{23}$$

The next steps are found in Appendix A and lead to the transmitted electric field in the elliptical aperture at the third order:

$$\begin{cases} E_x^{t(3)} = \dfrac{\theta_0^3 + \theta_3^3 y^2 + \theta_4^3 yx + \theta_5^3 x^2 + \theta_{10}^3 y^4 + \theta_{11}^3 y^3 x + \theta_{12}^3 y^2 x^2 + \theta_{13}^3 yx^3 + \theta_{14}^3 y^4}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} \\ E_y^{t(3)} = -\dfrac{\eta_0^3 + \eta_3^3 x^2 + \eta_4^3 xy + \eta_5^3 y^2 + \eta_{10}^3 x^4 + \eta_{11}^3 x^3 y + \eta_{12}^3 x^2 y^2 + \eta_{13}^3 xy^3 + \eta_{14}^3 y^4}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}, \end{cases} \tag{24}$$

where the expressions of (η^3, θ^3) are given in Table 4. Finally, the total transmitted electric field in the elliptical aperture at a third-order development is given by

$$\begin{cases} E_x^t = kE_x^{t(1)} + k^3 E_x^{t(3)} \\ E_y^t = kE_y^{t(1)} + k^3 E_y^{t(3)} \end{cases}. \tag{25}$$

Electric field patterns can be found in Fig. 2.

B. Diffraction by an Elliptical Disk

The case of an elliptical disk can easily be solved with the generalized Babinet's principle [22,23]. We define the diffracted potential vector as

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \iint_{\text{ellipse}} \vec{\sigma}(x', y') \frac{e^{-jkR}}{R} \mathrm{d}x' \mathrm{d}y', \tag{26}$$

with $\vec{\sigma}(x', y')$ the electric surface current on the elliptic disk. The boundary conditions on the disk are

$$\begin{cases} E_x(x', y', 0) = -E_x^i(x', y', 0) \\ E_y(x', y', 0) = -E_y^i(x', y', 0) \\ H_z(x', y', 0) = -H_z^i(x', y', 0) \end{cases}, \tag{27}$$

leading to the following set of equations:

$$\begin{cases} \nabla_{xy}^2 A_x + k^2 A_x = \mu_0 \frac{\partial H_y^i}{\partial z} \\ \nabla_{xy}^2 A_y + k^2 A_y = -\mu_0 \frac{\partial H_x^i}{\partial z} \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = -\mu_0 H_z^i \end{cases}. \tag{28}$$

Thus, the disk and aperture problems are equivalent if the following substitution is made:

$$\begin{cases} \vec{F} \leftrightarrow \vec{A} \\ -\varepsilon_0 \vec{E}^i \leftrightarrow \mu_0 \vec{H}^i \\ \vec{\sigma} \leftrightarrow \vec{n} \times \vec{E}^t \end{cases}. \tag{29}$$

C. Nonplane Incident Electromagnetic Wave

We now briefly describe the case of a nonplane incident electromagnetic wave. We limit the polynomial development of the fields to the first order.

1. *Zeroth-Order developmentD*

Equation (5) becomes

$$\nabla_{xy}^2 F_x^0 = -\varepsilon_0 \frac{\partial E_y^i}{\partial z}, \qquad \nabla_{xy}^2 F_y^0 = \varepsilon_0 \frac{\partial E_x^i}{\partial z},$$
$$\frac{\partial F_y^0}{\partial x} - \frac{\partial F_x^0}{\partial y} = \varepsilon_0 E_z^i + \varepsilon_0 \frac{\partial E_z^i}{\partial x} x + \varepsilon_0 \frac{\partial E_z^i}{\partial y} y. \tag{30}$$

We use the same procedure, and then

$$\begin{cases} E_x^0 = \frac{\theta_0^0 + \theta_1^0 y + \theta_2^0 x + \theta_3^0 y^2 + \theta_4^0 yx + \theta_5^0 x^2}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} \\ E_y^0 = -\frac{\eta_0^0 + \eta_1^0 x + \eta_2^0 y + \eta_3^0 x^2 + \eta_4^0 xy + \eta_5^0 y^2}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} \end{cases}, \tag{31}$$

where the coefficients θ_i^0 and η_i^0 are found in Table 4.

2. *First-Order Development*

Equation (5) becomes

$$\nabla_{xy}^2 F_x^1 = -\varepsilon_0 \frac{\partial^2 E_y^i}{\partial k \partial z}, \qquad \nabla_{xy}^2 F_y^1 = \varepsilon_0 \frac{\partial^2 E_x^i}{\partial k \partial z},$$
$$\frac{\partial F_y^1}{\partial x} - \frac{\partial F_x^1}{\partial y} = \varepsilon_0 \frac{\partial E_z^i}{\partial k} + \varepsilon_0 \frac{\partial^2 E_z^i}{\partial k \partial x} x + \varepsilon_0 \frac{\partial^2 E_z^i}{\partial k \partial y} y. \tag{32}$$

With a notation similar to (16), it then follows that

$$\eta_0'^1 = \frac{\partial \eta_0^0}{\partial k}, \qquad \eta_1'^1 = 0, \qquad \eta_2'^1 = \frac{\partial \eta_2^0}{\partial k}, \qquad \eta_3'^1 = \frac{\partial \eta_3^0}{\partial k},$$
$$\eta_4'^1 = \frac{\partial \eta_4^0}{\partial k}, \qquad \eta_5'^1 = \frac{\partial \eta_5^0}{\partial k}, \qquad \theta_0'^1 = \frac{\partial \theta_0^0}{\partial k}, \qquad \theta_1'^1 = 0,$$
$$\theta_2'^1 = \frac{\partial \theta_2^0}{\partial k}, \qquad \theta_3'^1 = \frac{\partial \theta_3^0}{\partial k}, \qquad \theta_4'^1 = \frac{\partial \theta_4^0}{\partial k}, \qquad \theta_5'^1 = \frac{\partial \theta_5^0}{\partial k}, \tag{33}$$

and so

$$\begin{cases} E_x^1 = \frac{\theta_0'^1 + \theta_1'^1 y + \theta_2'^1 x + \theta_3'^1 y^2 + \theta_4'^1 yx + \theta_5'^1 x^2}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} \\ E_y^1 = -\frac{\eta_0'^1 + \eta_1'^1 x + \eta_2'^1 y + \eta_3'^1 x^2 + \eta_4'^1 xy + \eta_5'^1 y^2}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} \end{cases}. \tag{34}$$

Due to the structure of Eqs. (4), (7), and (10), we know that for all orders of development in k the coefficients (η', θ') will have the same structure, only the degree in k-differentiation will change. This property only holds if the degree of the polynomial development remains the same for all orders of k. Eggimann [8] was the first to point out this property for the circular disk diffraction.

3. DISCUSSION

In the degenerate case of a circular aperture, (19) reduces to

$$\begin{cases} E_x^{t(1)} = -\frac{4j}{3\pi} \frac{2a^2 \cos\psi - x^2 \cos\psi + xy \sin\psi - 2y^2 \cos\psi}{\sqrt{a^2 - x^2 - y^2}} E^i \\ E_y^{t(1)} = -\frac{4j}{3\pi} \frac{2a^2 \sin\psi - 2x^2 \sin\psi + xy \cos\psi - y^2 \sin\psi}{\sqrt{a^2 - x^2 - y^2}} E^i \end{cases}, \tag{35}$$

which for $\psi = 0$ leads to

$$\begin{cases} E_x^{t(1)} = -\frac{4j}{3\pi} \frac{2a^2 - x^2 - 2y^2}{\sqrt{a^2 - x^2 - y^2}} E^i \\ E_y^{t(1)} = -\frac{4j}{3\pi} \frac{xy}{\sqrt{a^2 - x^2 - y^2}} E^i \end{cases}. \tag{36}$$

This is the same solution found by Bouwkamp [24] Eq. (35) [the factor k is absent here because it is in Eq. (25) of this paper].

In acoustics, mixed boundary conditions are common problems and recent development in subwavelength optics has spurred similar research in acoustics. Due to the numerous similarities between electromagnetic and acoustic propagation the results and calculi found here can find applications into the acoustic field.

4. CONCLUSION

We have presented an analytical approximate solution of the aperture fields for a subwavelength elliptical aperture in a thin perfectly conducting screen. We used Copson's formulation combined with Bouwkamp/Eggimann procedure and adapted it to elliptical geometry. Results are interesting because they lead to interesting insights into analytical expressions of electromagnetic interactions with anisotropic subwavelength structures. Results may be used to build analytical expressions of aperture fields for aperture with more anisotropy, and can also find some use in vibration theory. Finally, accessing higher-order terms could easily be done using symbolic programming.

APPENDIX A

A.1. Evaluation of the Integrals in Eq. (7)

The strategy is to use a change of variables to switch from elliptical to circular geometry. We define the following set of polar coordinates:

$$\begin{cases} x' = a\rho'\cos(\varphi') \\ y' = b\rho'\sin(\varphi') \end{cases} \quad \begin{cases} x = a\rho\cos(\varphi) \\ y = b\rho\sin(\varphi) \end{cases} \quad \rho' e^{i\varphi'} - \rho e^{i\varphi} = re^{i\theta}. \tag{A1}$$

Nonvanishing integrals encountered in Eq. (7) are of the forms

$$\vec{F} = \frac{\varepsilon_0}{2\pi}\iint_{\text{ellipse}} \vec{J}\frac{\mathrm{d}x'\mathrm{d}y'}{R} \tag{A2}$$

or

$$\vec{F} = \frac{\varepsilon_0}{2\pi}\iint_{\text{ellipse}} \vec{J}R\mathrm{d}x'\mathrm{d}y', \tag{A3}$$

where we recall that $R = \sqrt{(x'-x)^2 + (y'-y)^2}$ at $z = 0$.

A.1.1. Integral $\vec{F} = \frac{\varepsilon_0}{2\pi}\iint_{\text{ellipse}} \vec{J}\frac{\mathrm{d}x'\mathrm{d}y'}{R}$

The first integral (A1) reads in polar coordinates

$$G(x,y) = \frac{\varepsilon_0}{2\pi}\iint_{\text{ellipse}} \frac{J(x',y')\rho'\mathrm{d}\rho'\mathrm{d}\varphi'}{R}, \tag{A4}$$

which can be rewritten in a circular geometry as

$$G(x,y) = \frac{\varepsilon_0 b}{2\pi}\iint_{\text{circle}} \frac{J(x',y')\rho'\mathrm{d}\rho'\mathrm{d}\varphi'}{r\sqrt{1-p^2\sin^2(\theta)}}, \tag{A5}$$

with $p = \sqrt{1-\frac{b^2}{a^2}}$. The general solving was given by Boersma and Danicki [21], by expanding Wolfe's work [25]. In order to evaluate the integral (A4) we define $g(\theta) = \frac{\varepsilon_0 b}{2\pi}\frac{1}{\sqrt{1-p^2\sin^2(\theta)}}$ and decompose it on a Fourier basis $g(\theta) = \sum_{l,\text{even}} g_l e^{il\theta}$ with

$$\begin{aligned}
g_0 &= \frac{\varepsilon_0 b}{\pi^2}K(p) \\
g_{\pm 2} &= \frac{2\varepsilon_0 b}{\pi^2 p^2}\left[E(p) - \left(1-\frac{1}{2}p^2\right)K(p)\right] \\
g_{\pm 4} &= \frac{\varepsilon_0 b}{3\pi^2 p^4}[(3p^4 - 16p^2 + 16)K(p) + (8p^2 - 16)E(p)] \\
g_{\pm 6} &= \frac{\varepsilon_0 b}{15\pi^2 p^6}[(15p^6 - 158p^4 + 384p^2 - 256)K(p) \\
&\quad + (46p^4 - 256p^2 + 256)] \\
g_{\pm 8} &= \frac{\varepsilon_0 b}{105\pi^2 p^8}[(105p^8 - 1856p^6 + 8000p^4 - 12288p^2 + 6144)K(p) \\
&\quad + (352p^6 - 3776p^4 + 9216p^2 - 6144)E(p)],
\end{aligned} \tag{A6}$$

where $K(p)$ and $E(p)$ are the elliptic integrals of the first and second kind, respectively [26]. We only evaluate up the eighth order because the polynomial numerator of $J(x',y')$ won't exceed the fourth degree.

We evaluate the integrals of type (A3) and (A4) with $J(x',y') = \frac{x'^i y'^j}{\sqrt{1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}}}$, where i and j are integers. To do so, the polynomial term is expressed as series of Legendre functions multiplied by the complex exponential $e^{im\varphi'}$ and the following formula are used:

$$\begin{aligned}
&\int_0^{2\pi}\int_0^1 \frac{e^{il\theta}}{r}\frac{P_n^m\left(\sqrt{1-\rho'^2}\right)e^{im\varphi'}}{\sqrt{1-\rho'^2}}\rho'\mathrm{d}\rho'\mathrm{d}\varphi' = 0 \quad \text{if } |m+l| > n \\
&\int_0^{2\pi}\int_0^1 \frac{e^{il\theta}}{r}\frac{P_n^m\left(\sqrt{1-\rho'^2}\right)e^{im\varphi'}}{\sqrt{1-\rho'^2}}\rho'\mathrm{d}\rho'\mathrm{d}\varphi' \\
&\quad = L_{m,n,l}P_n^{m+l}\left(\sqrt{1-\rho^2}\right)e^{i(m+l)\varphi} \quad \text{if } |m+l| \le n
\end{aligned} \tag{A7}$$

with n, l, m integers, P_n^m Legendre functions, and

$$L_{m,n,l} = 2^{-l}\pi\frac{\Gamma\left(\frac{1}{2}n+\frac{1}{2}m+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}n-\frac{1}{2}m-\frac{1}{2}l+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}n-\frac{1}{2}m+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}n+\frac{1}{2}m+\frac{1}{2}l+1\right)}. \tag{A8}$$

Finally, the integral is expressed in (x,y) coordinates. Table 1 provides the value of $G(x,y)$ for various $J(x',y')$. Most interestingly, the degree of $G(x,y)$ is identical to the degree of the numerator of $J(x',y')$.

A.1.2. Integral $\vec{F} = \frac{\varepsilon_0}{2\pi}\iint_{\text{ellipse}} \vec{J}R\mathrm{d}x'\mathrm{d}y'$

The second integral (A2) reads

$$H(x,y) = \frac{\varepsilon_0}{2\pi}\iint_{\text{ellipse}} J(x',y')\sqrt{(x-x')^2+(y-y')^2}\mathrm{d}x'\mathrm{d}y'. \tag{A9}$$

They are evaluated in three steps:

$$H(0,0) = \frac{\varepsilon_0}{2\pi}\iint_{\text{ellipse}} J(x',y')\sqrt{x'^2+y'^2}\mathrm{d}x'\mathrm{d}y', \tag{A10}$$

which is expressed as elliptic integrals and

$$\begin{cases} \frac{\partial H(x,y)}{\partial x} = \frac{\varepsilon_0 x}{2\pi}\int\frac{J(x',y')}{\sqrt{(x-x')^2+(y-y')^2}}\mathrm{d}x'\mathrm{d}y - \frac{\varepsilon_0}{2\pi}\int\frac{J(x',y')x'}{\sqrt{(x-x')^2+(y-y')^2}}\mathrm{d}x'\mathrm{d}y' \\ \frac{\partial H(x,y)}{\partial y} = \frac{\varepsilon_0 y}{2\pi}\int\frac{J(x',y')}{\sqrt{(x-x')^2+(y-y')^2}}\mathrm{d}x'\mathrm{d}y - \frac{\varepsilon_0}{2\pi}\int\frac{J(x',y')y'}{\sqrt{(x-x')^2+(y-y')^2}}\mathrm{d}x'\mathrm{d}y' \end{cases}, \tag{A11}$$

Table 1. $G(x,y) = \frac{\varepsilon_0}{2\pi} \iint \frac{J(x',y')\rho' \mathrm{d}\rho' \mathrm{d}\varphi'}{\sqrt{(x-x')^2+(y-y')^2}}$

$J(x',y')$	$G(x,y)$
1	$\pi^2 g_0$
x'	$\frac{1}{2}\pi^2(g_0 - g_2)x = \pi^2 C_{-1} x$
y'	$\frac{1}{2}\pi^2(g_0 + g_2)y = \pi^2 C_0 y$
x'^2	$\pi^2[(\frac{5}{16}g_0 - \frac{1}{2}g_2 + \frac{3}{16}g_4)x^2 + (-\frac{1}{16}g_0 - \frac{1}{4}g_2 - \frac{3}{16}g_4)\frac{a^2}{b^2}y^2$ $+\frac{a^2}{4}(g_0 + g_2)] = \pi^2[C_1x^2 + C_2y^2 + C_3]$
y'^2	$\pi^2[(\frac{5}{16}g_0 + \frac{1}{2}g_2 + \frac{3}{16}g_4)y^2 + (-\frac{1}{16}g_0 + \frac{1}{4}g_2 - \frac{3}{16}g_4)\frac{b^2}{a^2}x^2$ $+\frac{b^2}{4}(g_0 - g_2)] = \pi^2[C_4y^2 + C_5x^2 + C_6]$
$x'y'$	$\frac{3}{8}\pi^2(g_0 - g_4)xy = \pi^2 C_7 xy$
x'^3	$\pi^2 x^3(\frac{7}{32}g_0 - \frac{27}{64}g_2 + \frac{9}{32}g_4 - \frac{5}{64}g_6)$ $+\frac{a^2}{b^2}\pi^2 xy^2(\frac{-3}{32}g_0 - \frac{15}{64}g_2 + \frac{3}{32}g_4 + \frac{15}{64}g_6)$ $+\pi^2 a^2 x(\frac{3}{16}g_0 - \frac{3}{16}g_4)$ $= \pi^2(C_8x^3 + C_9xy^2 + C_{10}x)$
x'^2y'	$\pi^2\frac{a^2}{b^2}y^3(-\frac{1}{32}g_0 - \frac{11}{64}g_2 - \frac{7}{32}g_4 - \frac{5}{64}g_6)$ $+\pi^2x^2y(\frac{9}{32}g_0 - \frac{15}{64}g_2 - \frac{9}{32}g_4 + \frac{15}{64}g_6)$ $+\pi^2a^2y(\frac{1}{16}g_0 + \frac{1}{4}g_2 + \frac{3}{16}g_4)$ $= \pi^2(C_{11}y^3 + C_{12}x^2y + C_{13}y)$
$x'y'^2$	$\pi^2\frac{b^2}{a^2}x^3(-\frac{1}{32}g_0 + \frac{11}{64}g_2 - \frac{7}{32}g_4 + \frac{5}{64}g_6)$ $+\pi^2xy^2(\frac{9}{32}g_0 + \frac{15}{64}g_2 - \frac{9}{32}g_4 - \frac{15}{64}g_6)$ $+\pi^2b^2x(\frac{1}{16}g_0 - \frac{1}{4}g_2 + \frac{3}{16}g_4)$ $= \pi^2(C_{14}x^3 + C_{15}xy^2 + C_{16}x)$
y'^3	$\pi^2y^3(\frac{7}{32}g_0 + \frac{27}{64}g_2 + \frac{9}{32}g_4 + \frac{5}{64}g_6)$ $+\pi^2\frac{b^2}{a^2}x^2y(-\frac{3}{32}g_0 + \frac{15}{64}g_2 + \frac{3}{32}g_4 - \frac{15}{64}g_6)$ $+\pi^2b^2y(\frac{3}{16}g_0 - \frac{3}{16}g_4)$ $= \pi^2(C_{17}y^3 + C_{18}x^2y + C_{19}y)$
x'^4	$\pi^2x^4(\frac{169}{1024}g_0 - \frac{11}{32}g_2 + \frac{77}{256}g_4 - \frac{5}{32}g_6 + \frac{35}{1024}g_8) +$ $\pi^2\frac{a^2}{b^2}x^2y^2(-\frac{51}{512}g_0 - \frac{15}{128}g_2 + \frac{39}{128}g_4 + \frac{15}{128}g_6 - \frac{105}{512}g_8) +$ $\pi^2a^2x^2(\frac{9}{64}g_0 - \frac{15}{128}g_2 - \frac{9}{64}g_4 + \frac{15}{128}g_6) +$ $\pi^2\frac{a^4}{b^2}y^2(-\frac{3}{64}g_0 - \frac{33}{128}g_2 - \frac{21}{64}g_4 - \frac{15}{128}g_6) +$ $\pi^2\frac{a^4}{b^4}y^4(\frac{9}{1024}g_0 + \frac{9}{128}g_2 + \frac{37}{256}g_4 + \frac{15}{128}g_6 + \frac{35}{1024}g_8) +$ $\pi^2a^4(\frac{9}{64}g_0 + \frac{3}{16}g_2 + \frac{3}{64}g_4) = \pi^2(C_{20}x^4 + C_{21}x^2y^2 + C_{22}x^2$ $+C_{23}y^2 + C_{24}y^4 + C_{25})$
x'^3y'	$\pi^2x^3y(\frac{55}{256}g_0 - \frac{5}{16}g_2 - \frac{5}{64}g_4 + \frac{5}{16}g_6 - \frac{35}{256}g_8) +$ $\pi^2\frac{a^2}{b^2}xy^3(-\frac{15}{256}g_0 - \frac{15}{64}g_2 - \frac{5}{64}g_4 + \frac{15}{64}g_6 + \frac{35}{256}g_8) +$ $\pi^2a^2xy(\frac{3}{32}g_0 + \frac{15}{64}g_2 - \frac{3}{32}g_4 - \frac{15}{64}g_6)$ $= \pi^2(C_{26}x^3y + C_{27}xy^3 + C_{28}xy)$
$x'^2y'^2$	$\pi^2\frac{b^2}{a^2}x^4(-\frac{35}{1024}g_0 + \frac{29}{256}g_2 - \frac{207}{512}g_4 + \frac{35}{256}g_6 - \frac{35}{1024}g_8) +$ $\pi^2\frac{a^2}{b^2}y^4(-\frac{35}{1024}g_0 - \frac{29}{256}g_2 - \frac{207}{512}g_4 - \frac{35}{256}g_6 - \frac{35}{1024}g_8) +$ $\pi^2x^2y^2(\frac{105}{512}g_0 - \frac{219}{256}g_4 + \frac{105}{512}g_8)$ $+\pi^2b^2x^2(\frac{1}{28}g_0 - \frac{17}{128}g_2 + \frac{15}{32}g_4 - \frac{15}{128}g_6) +$ $\pi^2a^2y^2(\frac{1}{28}g_0 + \frac{17}{128}g_2 + \frac{15}{32}g_4 + \frac{15}{128}g_6)$ $+\pi^2a^2b^2(\frac{3}{70}g_0 - \frac{3}{32}g_4) = \pi^2(C_{29}x^4 + C_{30}y^4 + C_{31}x^2y^2 +$ $C_{32}x^2 + C_{33}y^2 + C_{34})$
$x'y'^3$	$\pi^2\frac{b^2}{a^2}x^3y(-\frac{15}{256}g_0 + \frac{15}{64}g_2 - \frac{5}{64}g_4 - \frac{15}{64}g_6 + \frac{35}{256}g_8) +$ $\pi^2xy^3(\frac{55}{256}g_0 + \frac{5}{16}g_2 - \frac{5}{64}g_4 - \frac{5}{16}g_6 - \frac{35}{256}g_8) +$ $\pi^2b^2xy(\frac{3}{32}g_0 - \frac{15}{64}g_2 - \frac{3}{32}g_4 + \frac{15}{64}g_6)$ $= \pi^2(C_{35}x^3y + C_{36}xy^3 + C_{37}xy)$
y'^4	$\pi^2\frac{b^4}{a^4}x^4(\frac{9}{1024}g_0 - \frac{9}{128}g_2 + \frac{37}{256}g_4 - \frac{15}{128}g_6 + \frac{35}{1024}g_8) +$ $\pi^2y^4(\frac{169}{1024}g_0 + \frac{11}{32}g_2 + \frac{77}{256}g_4 + \frac{5}{32}g_6 + \frac{35}{1024}g_8) +$ $\pi^2\frac{b^2}{a^2}x^2y^2(-\frac{51}{512}g_0 + \frac{15}{128}g_2 + \frac{39}{128}g_4 - \frac{15}{128}g_6 - \frac{105}{512}g_8) +$ $\pi^2\frac{b^4}{a^2}x^2(-\frac{3}{64}g_0 + \frac{33}{128}g_2 - \frac{21}{64}g_4 + \frac{15}{128}g_6) +$ $\pi^2b^2y^2(\frac{9}{64}g_0 + \frac{15}{128}g_2 - \frac{9}{64}g_4 - \frac{15}{128}g_6)$ $+\pi^2b^4(\frac{9}{64}g_0 - \frac{3}{16}g_2 + \frac{3}{64}g_4)$ $= \pi^2(C_{38}x^4 + C_{39}y^4 + C_{40}x^2y^2$ $+C_{41}x^2 + C_{42}y^2 + C_{43})$

Table 2. $H(x,y) = \frac{\varepsilon_0}{2\pi}\iint J(x',y')\sqrt{(x-x')^2 + (y-y')^2}\mathrm{d}x'\mathrm{d}y'$

$J(x',y')$	$H(x,y)$
1	$h_0 + \frac{\pi^2}{2}(g_0 - C_{-1})x^2 + \frac{\pi^2}{2}(g_0 - C_0)y^2$
x'	$-\pi^2C_3x + \frac{\pi^2}{3}(C_{-1} - C_1)x^3 - \pi^2C_2xy^2$
y'	$-\pi^2C_6y - \pi^2C_5x^2y + \frac{\pi^2}{3}(C_0 - C_4)y^3$
x'^2	$h_1 + \frac{\pi^2}{2}(C_3 - C_{10})x^2 + \frac{\pi^2}{2}(C_3 - C_{13})y^2$ $+\frac{\pi^2}{4}(C_1 - C_8)x^4 + \frac{\pi^2}{2}(C_2 - C_9)x^2y^2 + \frac{\pi^2}{4}(C_2 - C_{11})y^4$
$x'y'$	$-\pi^2C_{13}xy + \frac{\pi^2}{3}(C_7 - C_{12})x^3y - \pi^2C_{11}xy^3$
y'^2	$h_2 + \frac{\pi^2}{2}(C_6 - C_{16})x^2 + \frac{\pi^2}{2}(C_6 - C_{19})y^2$ $+\frac{\pi^2}{4}(C_5 - C_{14})x^4 + \frac{\pi^2}{2}(C_4 - C_{15})x^2y^2 + \frac{\pi^2}{4}(C_4 - C_{17})y^4$

which can be evaluated using Table 1. Results are found in Table 2 with

$$\begin{cases} h_0 = \frac{1}{4}\mu_0 a^2 b E(p) \\ h_1 = \frac{\mu_0 a^4 b}{16p^2}[(p^2 - 1)K(p) + (p^2 + 1)E(p)] \\ h_2 = \frac{\mu_0 b^3 a^2}{16p^2}[(1 - p^2)K(p) + (2p^2 - 1)E(p)] \end{cases},$$

where K and E are the elliptic integrals previously defined. Note that Bouwkamp formula [22,24,27], modified to be used in elliptical geometry, could also have been used to evaluate these integrals.

A.2. Third-Order Development

At third-order development, (4), (7), and (30) lead to 30 independent linear equations involving (η^3, θ^3). The system may be solved in a specific order to ease the solving procedure: $(\eta_7^3, \eta_9^3, \theta_7^3, \theta_9^3)$, $(\eta_6^3, \eta_8^3, \theta_6^3, \theta_8^3)$, (η_1^3, θ_1^3), (η_2^3, θ_2^3), $(\eta_{10}^3, \eta_{11}^3, \eta_{12}^3, \eta_{13}^3, \eta_{14}^3, \theta_{10}^3, \theta_{11}^3, \theta_{12}^3, \theta_{13}^3, \theta_{14}^3)$ and finally $(\eta_0^3, \eta_3^3, \eta_4^3, \eta_5^3, \theta_0^3, \theta_3^3, \theta_4^3, \theta_5^3)$. The first four steps in the solving procedure lead to

$$\eta_1^3 = \eta_2^3 = \eta_6^3 = \eta_7^3 = \eta_8^3 = \eta_9^3 = 0$$
$$\theta_1^3 = \theta_2^3 = \theta_6^3 = \theta_7^3 = \theta_8^3 = \theta_9^3 = 0. \quad \text{(A12)}$$

It follows

Table 3. First-Order Development

	Value
η_0^1	$\frac{1/2 j\varepsilon_0 E^i \sin(\psi) a^2 b^2 (2C_4+C_7)}{\pi^2(-2b^2C_5C_2+a^2C_5C_7+a^2C_4C_7+2b^2C_1C_4+b^2C_1C_7+b^2C_2C_7)}$
η_1^1	0
η_2^1	0
η_3^1	$-\frac{1/2 j\varepsilon_0 E^i \sin(\psi) b^2 (2C_4+C_7)}{\pi^2(-2b^2C_5C_2+a^2C_5C_7+a^2C_4C_7+2b^2C_1C_4+b^2C_1C_7+b^2C_2C_7)}$
η_4^1	$\frac{j\varepsilon_0 E^i \cos(\psi)(C_5a^2+C_1b^2)}{\pi^2(b^2C_1C_7-2a^2C_2C_5+b^2C_2C_7+a^2C_5C_7+2a^2C_1C_4+a^2C_4C_7)}$
η_5^1	$-\frac{1/2 j\varepsilon_0 E^i \sin(\psi)(-2C_2b^2+C_7a^2)}{\pi^2(-2b^2C_5C_2+a^2C_5C_7+a^2C_4C_7+2b^2C_1C_4+b^2C_1C_7+b^2C_2C_7)}$
θ_0^1	$-\frac{1/2 j\varepsilon_0 E^i \cos(\psi) a^2 b^2 (2C_1+C_7)}{\pi^2(b^2C_1C_7-2a^2C_2C_5+b^2C_2C_7+a^2C_5C_7+2a^2C_1C_4+a^2C_4C_7)}$
θ_1^1	0
θ_2^1	0
θ_3^1	$\frac{1/2 j\varepsilon_0 E^i \cos(\psi) a^2 (2C_1+C_7)}{\pi^2(b^2C_1C_7-2a^2C_2C_5+b^2C_2C_7+a^2C_5C_7+2a^2C_1C_4+a^2C_4C_7)}$
θ_4^1	$-\frac{j\varepsilon_0 E^i \sin(\psi)(C_2b^2+C_4a^2)}{\pi^2(-2b^2C_5C_2+a^2C_5C_7+a^2C_4C_7+2b^2C_1C_4+b^2C_1C_7+b^2C_2C_7)}$
θ_5^1	$-\frac{1/2 j\varepsilon_0 E^i \cos(\psi)(2C_5a^2-C_7b^2)}{\pi^2(b^2C_1C_7-2a^2C_2C_5+b^2C_2C_7+a^2C_5C_7+2a^2C_1C_4+a^2C_4C_7)}$

$$\begin{aligned}
\eta_{10}^{3} &= |\,S_1 \quad D_2 \quad D_3 \quad D_4 \quad D_5 \quad D_6 \quad D_7 \quad D_8 \quad D_9 \quad D_{10}\,|/\psi \\
\eta_{11}^{3} &= |\,D_1 \quad S_1 \quad D_3 \quad D_4 \quad D_5 \quad D_6 \quad D_7 \quad D_8 \quad D_9 \quad D_{10}\,|/\psi \\
\eta_{12}^{3} &= |\,D_1 \quad D_2 \quad S_3 \quad D_4 \quad D_5 \quad D_6 \quad D_7 \quad D_8 \quad D_9 \quad D_{10}\,|/\psi \\
\eta_{13}^{3} &= |\,D_1 \quad D_2 \quad D_3 \quad S_1 \quad D_5 \quad D_6 \quad D_7 \quad D_8 \quad D_9 \quad D_{10}\,|/\psi \\
\eta_{14}^{3} &= |\,D_1 \quad D_2 \quad D_3 \quad D_4 \quad S_1 \quad D_6 \quad D_7 \quad D_8 \quad D_9 \quad D_{10}\,|/\psi \\
\theta_{10}^{3} &= |\,D_1 \quad D_2 \quad D_3 \quad D_4 \quad D_5 \quad S_1 \quad D_7 \quad D_8 \quad D_9 \quad D_{10}\,|/\psi \\
\theta_{11}^{3} &= |\,D_1 \quad D_2 \quad D_3 \quad D_4 \quad D_5 \quad D_6 \quad S_1 \quad D_8 \quad D_9 \quad D_{10}\,|/\psi \\
\theta_{12}^{3} &= |\,D_1 \quad D_2 \quad D_3 \quad D_4 \quad D_5 \quad D_6 \quad D_7 \quad S_1 \quad D_9 \quad D_{10}\,|/\psi \\
\theta_{13}^{3} &= |\,D_1 \quad D_2 \quad D_3 \quad D_4 \quad D_5 \quad D_6 \quad D_7 \quad D_8 \quad S_1 \quad D_{10}\,|/\psi \\
\theta_{14}^{3} &= |\,D_1 \quad D_2 \quad D_3 \quad D_4 \quad D_5 \quad D_6 \quad D_7 \quad D_8 \quad D_9 \quad S_1\,|/\psi,
\end{aligned} \tag{A13}$$

with $|\,|$ being the determinant, and

$$\begin{aligned}
D_1 &= [\,\pi^2 C_{21}^{20} \quad 0 \quad \pi^2 C_{24}^{21} \quad 0 \quad 0 \quad -2\pi^2 C_{21} \quad 0 \quad -4\pi^2 C_{24} \quad -a^2/b^2 \quad 0\,]^T \\
D_2 &= [\,0 \quad \pi^2 C_{27}^{26} \quad 0 \quad 0 \quad -\pi^2 C_{26} \quad 0 \quad -3\pi^2 C_{27} \quad 0 \quad 0 \quad -a^2/b^2\,]^T \\
D_3 &= [\,\pi^2 C_{31}^{29} \quad 0 \quad \pi^2 C_{30}^{31} \quad 0 \quad 0 \quad -2\pi^2 C_{31} \quad 0 \quad -4\pi^2 C_{30} \quad 1 \quad 0\,]^T \\
D_4 &= [\,0 \quad \pi^2 C_{36}^{35} \quad 0 \quad 0 \quad \pi^2 C_{35} \quad 0 \quad -3\pi^2 C_{36} \quad 0 \quad 0 \quad 1\,]^T \\
D_5 &= [\,\pi^2 C_{40}^{38} \quad 0 \quad \pi^2 C_{39}^{40} \quad 0 \quad 0 \quad -2\pi^2 C_{40} \quad 0 \quad -4\pi^2 C_{39} \quad -b^2/a^2 \quad 0\,]^T \\
D_6 &= [\,0 \quad 0 \quad 0 \quad \pi^2 C_{39}^{40} \quad 4\pi^2 C_{38} \quad 0 \quad 2\pi^2 C_{40} \quad 0 \quad 0 \quad -b^2/a^2\,]^T \\
D_7 &= [\,0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 3\pi^2 C_{35} \quad 0 \quad \pi^2 C_{36} \quad -b^2/a^2 \quad 0\,]^T \\
D_8 &= [\,0 \quad 0 \quad 0 \quad \pi^2 C_{30}^{31} \quad 4\pi^2 C_{29} \quad 0 \quad 2\pi^2 C_{31} \quad 0 \quad 0 \quad 1\,]^T \\
D_9 &= [\,0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 3\pi^2 C_{26} \quad 0 \quad \pi^2 C_{27} \quad 1 \quad 0\,]^T \\
D_{10} &= [\,0 \quad 0 \quad 0 \quad \pi^2 C_{24}^{21} \quad 4\pi^2 C_{20} \quad 0 \quad 2\pi^2 C_{21} \quad 0 \quad 0 \quad -a^2/b^2\,]^T \\
S_1 &= [\,\xi_1^1 \quad \xi_2^1 \quad \xi_3^1 \quad \xi_4^1 \quad \xi_5^1 \quad \xi_6^1 \quad \xi_7^1 \quad \xi_7^1 \quad 0 \quad 0\,]^T \\
\psi &= |\,D_1 \quad D_2 \quad D_3 \quad D_4 \quad D_5 \quad D_6 \quad D_7 \quad D_8 \quad D_9 \quad D_{10}\,|,
\end{aligned} \tag{A14}$$

where T stands for transposition, with

$$\begin{array}{ll}
\delta_{10}^{1} = 1/8\pi^2(-C_1 + C_8)\eta_3^1 + 1/8\pi^2(-C_5 + C_{14})\eta_5^1 & \delta_{11}^{1} = 1/6\pi^2(-C_7 + C_{12})\eta_4^1 \\
\delta_{12}^{1} = 1/4\pi^2(-C_2 + C_9)\eta_3^1 + 1/4\pi^2(C_{15} - C_4)\eta_5^1 & \delta_{13}^{1} = 1/2\pi^2 C_{11}\eta_4^1 \\
\delta_{14}^{1} = 1/8\pi^2(C_{11} - C_2)\eta_3^1 + 1/8\pi^2(C_{17} - C_4)\eta_5^1 & \\
\lambda_{10}^{1} = 1/8\pi^2(C_{17} - C_4)\theta_3^1 + 1/8\pi^2(C_{11} - C_2)\theta_5^1 & \lambda_{11}^{1} = 1/2\pi^2 C_{11}\theta_4^1 \\
\lambda_{12}^{1} = 1/4\pi^2(C_{15} - C_4)\theta_3^1 + 1/4\pi^2(C_9 - C_2)\theta_5^1 & \lambda_{13}^{1} = 1/6\pi^2(C_{12} - C_7)\theta_4^1 \\
\lambda_{14}^{1} = 1/8\pi^2(C_{14} - C_5)\theta_3^1 + 1/8\pi^2(C_8 - C_1)\theta_5^1 &
\end{array} \tag{A15}$$

and

$$\begin{array}{llll}
C_{-1}^{0} = 1/2(g_0 - C_{-1}) & C_{0}^{0,-1} = g_0 - 1/2C_{-1} - 1/2C_0 & C_{0}^{0} = 1/2(C_0 - g_0) & C_{1}^{2} = 2C_2 + 2C \\
C_{3}^{10} = 1/2(C_3 - C_{10}) & C_{3}^{13} = 1/2(C_{13} - C_3) & C_{4}^{5} = 2C_5 + 2C_4 & C_{6}^{16} = 1/2(C_6 - C_{16}) \\
C_{6}^{19} = 1/2(C_{19} - C_6) & C_{10}^{13,3} = -1/2C_{13} + C_3 - 1/2C_{10} & C_{16}^{19,6} = -1/2C_{19} + C_6 - 1/2C_{16} & C_{21}^{20} = 12C_{20} + 2C_{21} \\
C_{24}^{21} = 2C_{21} + 12C_{24} & C_{22}^{23} = -2C_{23} - 2C_{22} & C_{27}^{26} = 6C_{26} + 6C_{27} & C_{31}^{29} = 12C_{29} + 2C_{31} \\
C_{30}^{31} = 2C_{31} + 12C_{30} & C_{32}^{33} = -2C_{33} - 2C_{32} & C_{36}^{35} = 6C_{35} + 6C_{36} & C_{40}^{38} = 12C_{38} + 2C_{40} \\
C_{39}^{40} = 2C_{40} + 12C_{39} & C_{41}^{42} = -2C_{42} - 2C_{41}. & &
\end{array} \tag{A16}$$

Finally, one obtains

$$\begin{array}{lll}
\xi_1 = -\alpha_3^1 - 12\delta_{10}^1 - 2\delta_{12}^1 & \xi_2 = -\alpha_4^1 - 6\delta_{11}^1 - 6\delta_{13}^1 & \xi_3 = -\alpha_5^1 - 2\delta_{12}^1 - 12\delta_{14}^1 \\
\xi_4 = -\beta_3^1 - 2\lambda_{12}^1 - 12\lambda_{10}^1 & \xi_5 = \delta_{11}^1 - 4\lambda_{14}^1 & \xi_6 = 2\delta_{12}^1 - 3\lambda_{13}^1 \\
\xi_7 = 3\delta_{13}^1 - 2\lambda_{12}^1 & \xi_8 = 4\delta_{14}^1 - \lambda_{11}^1 &
\end{array} \tag{A17}$$

with

$$\begin{aligned}
\alpha_0^3 &= \eta_0^3\pi^2 g_0 + \eta_3^3\pi^2 C_3 + \eta_5^3\pi^2 C_6 + \eta_{10}^3\pi^2 C_{25} + \eta_{12}^3\pi^2 C_{34} + \eta_{14}^3\pi^2 C_{43} - \frac{1}{2}\eta_3^1 h_1 - \frac{1}{2}\eta_0^1 h_0 - \frac{1}{2}\eta_5^1 h_2,\\
\alpha_1^3 &= \frac{1}{2}\eta_1^1\pi^2 C_3, \qquad \alpha_2^3 = \frac{1}{2}\eta_2^1\pi^2 C_6,\\
\alpha_3^3 &= \eta_3^3\pi^2 C_1 + \eta_5^3\pi^2 C_5 + \eta_{10}^3\pi^2 C_{22} + \eta_{12}^3\pi^2 C_{32} + \eta_{14}^3\pi^2 C_{41} - \frac{1}{2}C_{-1}^0\pi^2\eta_0^1 - \frac{1}{2}C_3^{10}\pi^2\eta_3^1 - \frac{1}{2}C_6^{16}\pi^2\eta_5^1,\\
\alpha_4^3 &= \eta_4^3\pi^2 C_7 + \eta_{11}^3\pi^2 C_{28} + \eta_{13}^3\pi^2 C_{37} + \frac{1}{2}C_{13}\pi^2\eta_4^1,\\
\alpha_5^3 &= \eta_3^3\pi^2 C_2 + \eta_5^3\pi^2 C_4 + \eta_{10}^3\pi^2 C_{23} + \eta_{12}^3\pi^2 C_{33} + \eta_{14}^3\pi^2 C_{42} - \frac{1}{2}C_0^0\pi^2\eta_0^1 - \frac{1}{2}C_3^{13}\pi^2\eta_3^1 - \frac{1}{2}C_6^{19}\pi^2\eta_5^1,\\
\alpha_6^3 &= \frac{1}{6}\pi^2(C_1 - C_{-1})\eta_1^1, \qquad \alpha_7^3 = \frac{1}{2}\eta_2^1\pi^2 C_5, \qquad \alpha_8^3 = \frac{1}{2}\eta_1^1\pi^2 C_2, \qquad \alpha_9^3 = \frac{1}{6}\pi^2\eta_2^1(C_4 - C_0),\\
\alpha_{10}^3 &= \eta_{10}^3\pi^2 C_{20} + \eta_{12}^3\pi^2 C_{29} + \eta_{14}^3\pi^2 C_{38} + \delta_{10}^1\\
\alpha_{11}^3 &= \eta_{11}^3\pi^2 C_{26} + \eta_{13}^3\pi^2 C_{35} + \delta_{11}^1\\
\alpha_{12}^3 &= \eta_{10}^3\pi^2 C_{21} + \eta_{12}^3\pi^2 C_{31} + \eta_{14}^3\pi^2 C_{40} + \delta_{12}^1\\
\alpha_{13}^3 &= \eta_{11}^3\pi^2 C_{27} + \eta_{13}^3\pi^2 C_{36} + \delta_{13}^1\\
\alpha_{14}^3 &= \eta_{10}^3\pi^2 C_{24} + \eta_{12}^3\pi^2 C_{30} + \eta_{14}^3\pi^2 C_{39} + \delta_{14}^1
\end{aligned} \tag{A18}$$

and

$$\begin{aligned}
\beta_0^3 &= \theta_0^3\pi^2 g_0 + \theta_3^3\pi^2 C_6 + \theta_5^3\pi^2 C_3 + \theta_{10}^3\pi^2 C_{43} + \theta_{12}^3\pi^2 C_{34} + \theta_{14}^3\pi^2 C_{25} - \frac{1}{2}\theta_0^1 h_0 - \frac{1}{2}\theta_3^1 h_2 - \frac{1}{2}\theta_5^1 h_1,\\
\beta_1^3 &= \frac{1}{2}\theta_1^1\pi^2 C_6, \qquad \beta_2^3 = \frac{1}{2}\theta_2^1\pi^2 C_3,\\
\beta_3^3 &= \theta_3^3\pi^2 C_4 + \theta_5^3\pi^2 C_2 + \theta_{10}^3\pi^2 C_{42} + \theta_{12}^3\pi^2 C_{33} + \theta_{14}^3\pi^2 C_{23} - \frac{1}{2}C_0^0\pi^2\theta_0^1 - \frac{1}{2}C_6^{19}\pi^2\theta_3^1 - \frac{1}{2}C_3^{13}\pi^2\theta_5^1,\\
\beta_4^3 &= \theta_4^3\pi^2 C_7 + \theta_{11}^3\pi^2 C_{37} + \theta_{13}^3\pi^2 C_{28} + \frac{1}{2}C_{13}\pi^2\theta_4^1,\\
\beta_5^3 &= \theta_3^3\pi^2 C_5 + \theta_5^3\pi^2 C_1 + \theta_{10}^3\pi^2 C_{41} + \theta_{12}^3\pi^2 C_{32} + \theta_{14}^3\pi^2 C_{22} - \frac{1}{2}C_{-1}^0\pi^2\theta_0^1 - \frac{1}{2}C_6^{16}\pi^2\theta_3^1 - \frac{1}{2}C_3^{10}\pi^2\theta_5^1,\\
\beta_6^3 &= \frac{1}{6}\pi^2(C_4 - C_0)\theta_1^1, \qquad \beta_7^3 = \frac{1}{2}\theta_2^1\pi^2 C_2, \qquad \beta_8^3 = \frac{1}{2}\theta_1^1\pi^2 C_5, \qquad \beta_9^3 = \frac{1}{6}\pi^2\theta_2^1(C_1 - C_{-1}),\\
\beta_{10}^3 &= \theta_{10}^3\pi^2 C_{39} + \theta_{12}^3\pi^2 C_{30} + \theta_{14}^3\pi^2 C_{24} + \lambda_{10}^1\\
\beta_{11}^3 &= \theta_{11}^3\pi^2 C_{36} + \theta_{13}^3\pi^2 C_{27} + \lambda_{11}^1\\
\beta_{12}^3 &= \theta_{10}^3\pi^2 C_{40} + \theta_{12}^3\pi^2 C_{31} + \theta_{14}^3\pi^2 C_{21} + \lambda_{12}^1\\
\beta_{13}^3 &= \theta_{11}^3\pi^2 C_{35} + \theta_{13}^3\pi^2 C_{26} + \lambda_{13}^1\\
v_{14}^3 &= \theta_{10}^3\pi^2 C_{38} + \theta_{12}^3\pi^2 C_{29} + \theta_{14}^3\pi^2 C_{20} + \lambda_{14}^1
\end{aligned} \tag{A19}$$

with

$$\begin{aligned}
\kappa_1 &= \pi^2 C_0^0\eta_0^1 + \pi^2 C_3^{13}\eta_3^1 + \pi^2 C_6^{19}\eta_5^1 - 1/2\pi^2 C_{13}\theta_4^1 + 2\pi^2 C_{23}\eta_{10}^3 + 2\pi^2 C_{33}\eta_{12}^3 + 2\pi^2 C_{42}\eta_{14}^3\\
&\quad - \pi^2 C_{37}\theta_{11}^3 - \pi^2 C_{28}\theta_{13}^3\\
\kappa_2 &= 1/2\pi^2 C_{13}\eta_4^1 + \pi^2 C_{-1}^0\theta_0^1 + \pi^2 C_6^{16}\theta_3^1 + \pi^2 C_3^{10}\theta_5^1 + \pi^2 C_{28}\eta_{11}^3 + \pi^2 C_{37}\eta_{13}^3 - 2\pi^2 C_{41}\theta_{10}^3 - 2\pi^2 C_{32}\theta_{12}^3\\
&\quad - 2\pi^2 C_{22}\theta_{14}^3\\
\kappa_3 &= -\alpha_0^1 + \pi^2 C_0^{0,-1}\eta_0^1 + \pi^2 C_{10}^{13,3}\eta_3^1 + \pi^2 C_{16}^{19,6}\eta_5^1 + \pi^2 C_{22}^{23}\eta_{10}^3 + \pi^2 C_{32}^{33}\eta_{12}^3 + \pi^2 C_{41}^{42}\eta_{14}^3\\
\kappa_4 &= -\beta_0^1 + \pi^2 C_0^{0,-1}\theta_0^1 + \pi^2 C_{16}^{19,6}\theta_3^1 + \pi^2 C_{10}^{13,3}\theta_5^1 + \pi^2 C_{41}^{42}\theta_{10}^3 + \pi^2 C_{32}^{33}\theta_{12}^3 + \pi^2 C_{22}^{23}\theta_{14}^3\\
\kappa_5 &= -\eta_{10}^3\\
\kappa_6 &= -\eta_{11}^3 - \theta_{14}^3\\
\kappa_7 &= -\eta_{14}^3 - \theta_{11}^3\\
\kappa_8 &= -\theta_{10}^3.
\end{aligned} \tag{A20}$$

Table 4. Arbitrary Incident Electromagnetic Field

	Value
η_0^3	$\frac{-a^2b^2C_4^5\kappa_1-a^2b^2(C_7+2C_4)\kappa_3-\pi^2a^2b^2(2a^2C_2C_4^5-a^2C_7C_1^2-2a^2C_4C_1^2)\kappa_5+\pi^2a^2b^4C_7C_4^5\kappa_7}{\pi^2(C_7C_4^5a^2+C_7C_1^2b^2-2C_2C_4^5b^2+2C_4C_1^2b^2)}$
η_3^3	$\frac{b^2C_4^5\kappa_1+b^2(C_7+2C_4)\kappa_3+\pi^2a^4C_7C_4^5\kappa_5-\pi^2b^4C_7C_4^5\kappa_7}{\pi^2(C_7C_4^5a^2+C_7C_1^2b^2-2C_2C_4^5b^2+2C_4C_1^2b^2)}$
η_4^3	$\frac{-(a^2C_4^5+b^2C_1^2)\kappa_2+2(a^2C_5+b^2C_1)\kappa_4-\pi^2a^4(-2C_1C_4^5+2C_5C_1^2)\kappa_6-2\pi^2b^4(C_1C_4^5-C_5C_1^2)\kappa_8}{\pi^2(C_7C_4^5a^2+C_7C_1^2b^2+2C_1C_4^5a^2-2C_5C_1^2a^2)}$
η_5^3	$\frac{-b^2C_1^2\kappa_1-(-C_7a^2+2C_2b^2)\kappa_3-\pi^2a^4C_7C_1^2\kappa_5+\pi^2b^4C_7C_1^2\kappa_7}{\pi^2(C_7C_4^5a^2+C_7C_1^2b^2-2C_2C_4^5b^2+2C_4C_1^2b^2)}$
θ_0^3	$\frac{a^2b^2C_1^2\kappa_2-a^2b^2(2C_1+C_7)\kappa_4+\pi^2a^4b^2C_7C_1^2\kappa_6+\pi^2a^2b^2(2b^2C_1C_4^5+b^2C_7C_4^5-2b^2C_5C_1^2)\kappa_8}{\pi^2(C_7C_4^5a^2+C_7C_1^2b^2+2C_1C_4^5a^2-2C_5C_1^2a^2)}$
θ_3^3	$\frac{-a^2C_1^2\kappa_2+a^2(C_7+2C_1)\kappa_4-\pi^2a^4C_7C_1^2\kappa_6+\pi^2b^4C_7C_1^2\kappa_8}{\pi^2(C_7C_4^5a^2+C_7C_1^2b^2+2C_1C_4^5a^2-2C_5C_1^2a^2)}$
θ_4^3	$\frac{(a^2C_4^5+b^2C_1^2)\kappa_1+(2C_4a^2+2C_2b^2)\kappa_3+2\pi^2a^4(C_2C_4^5-C_4C_1^2)\kappa_5+2\pi^2b^4(-C_2C_4^5+C_4C_1^2)\kappa_7}{\pi^2(C_7C_4^5a^2+C_7C_1^2b^2-2C_2C_4^5b^2+2C_4C_1^2b^2)}$
θ_5^3	$\frac{a^2C_4^5\kappa_2+(C_7b^2-2a^2C_5)\kappa_4+\pi^2a^4C_7C_4^5\kappa_6-\pi^2b^4C_7C_4^5\kappa_8}{\pi^2(C_7C_4^5a^2+C_7C_1^2b^2+2C_1C_4^5a^2-2C_5C_1^2a^2)}$

Table 5. Arbitrary Incident Electromagnetic Field

	Value
η_0^0	$-\frac{\varepsilon_0a^2b^2((2C_5+2C_4)\frac{\partial E_z^i}{\partial y}-(C_7+2C_4)\frac{\partial E_y^i}{\partial z})}{2\pi^2(-2C_5C_2b^2+C_7C_1b^2+2C_1C_4b^2+C_7C_2b^2+C_7C_5a^2+C_7C_4a^2)}$
η_1^0	0
η_2^0	$-\frac{\varepsilon_0E_z^i}{\pi^2(C_0+C_{-1})}$
η_3^0	$-\frac{\varepsilon_0b^2((2C_5+2C_4)\frac{\partial E_z^i}{\partial y}-(C_7+2C_4)\frac{\partial E_y^i}{\partial z})}{2\pi^2(-2C_5C_2b^2+C_7C_1b^2+2C_1C_4b^2+C_7C_2b^2+C_7C_5a^2+C_7C_4a^2)}$
η_4^0	$-\frac{\varepsilon_0((-C_5a^2-C_1b^2)\frac{\partial E_x^i}{\partial z}+(C_5a^2+C_4a^2+C_1b^2+C_2b^2)\frac{\partial E_z^i}{\partial x})}{\pi^2(-2C_2C_5a^2+C_7C_1b^2+C_7C_2b^2+2C_1C_4a^2+C_7C_5a^2+C_7C_4a^2)}$
η_5^0	$-\frac{\varepsilon_0((-2C_2b^2+C_7a^2)\frac{\partial E_y^i}{\partial z}+b^2(2C_1+2C_2)\frac{\partial E_z^i}{\partial y})}{2\pi^2(-2C_5C_2b^2+C_7C_1b^2+2C_1C_4b^2+C_7C_2b^2+C_7C_5a^2+C_7C_4a^2)}$
θ_0^0	$-\frac{\varepsilon_0a^2b^2(-(2C_1+2C_2)\frac{\partial E_z^i}{\partial x}+(2C_1+C_7)\frac{\partial E_x^i}{\partial z})}{2\pi^2(-2C_2C_5a^2+C_7C_1b^2+C_7C_2b^2+2C_1C_4a^2+C_7C_5a^2+C_7C_4a^2)}$
θ_1^0	0
θ_2^0	$\frac{\varepsilon_0E_z^i}{\pi^2(C_{-1}+C_0)}$
θ_3^0	$-\frac{\varepsilon_0a^2((2C_1+2C_2)\frac{\partial E_z^i}{\partial x}-(2C_1+C_7)\frac{\partial E_x^i}{\partial z})}{2\pi^2(-2C_2C_5a^2+C_7C_1b^2+C_7C_2b^2+2C_1C_4a^2+C_7C_5a^2+C_7C_4a^2)}$
θ_4^0	$\frac{\varepsilon_0(-(C_2b^2+C_4a^2)\frac{\partial E_y^i}{\partial z}+(C_1b^2+C_5a^2+C_2b^2+C_4a^2)\frac{\partial E_z^i}{\partial y})}{\pi^2(-2C_5C_2b^2+C_7C_1b^2+2C_1C_4b^2+C_7C_2b^2+C_7C_5a^2+C_7C_4a^2)}$
θ_5^0	$\frac{\varepsilon_0((-2C_5a^2+C_7b^2)\frac{\partial E_x^i}{\partial z}+a^2(2C_5+2C_4)\frac{\partial E_z^i}{\partial x})}{2\pi^2(-2C_2C_5a^2+C_7C_1b^2+C_7C_2b^2+2C_1C_4a^2+C_7C_5a^2+C_7C_4a^2)}$
θ_i^0	value
θ_0^0	$-\frac{2\pi}{\varepsilon_0}\frac{\varepsilon_0a^2b^2(-(2C_1+2C_2)\frac{\partial E_z^i}{\partial x}+(2C_1+C_7)\frac{\partial E_x^i}{\partial z})}{2\pi^2(-2C_2C_5a^2+C_7C_1b^2+C_7C_2b^2+2C_1C_4a^2+C_7C_5a^2+C_7C_4a^2)}$
θ_1^0	0
θ_2^0	$\frac{2\pi}{\varepsilon_0}\frac{\varepsilon_0E_z^i}{\pi^2(C_{-1}+C_0)}$
θ_3^0	$-\frac{2\pi}{\varepsilon_0}\frac{\varepsilon_0a^2((2C_1+2C_2)\frac{\partial E_z^i}{\partial x}-(2C_1+C_7)\frac{\partial E_x^i}{\partial z})}{2\pi^2(-2C_2C_5a^2+C_7C_1b^2+C_7C_2b^2+2C_1C_4a^2+C_7C_5a^2+C_7C_4a^2)}$
θ_4^0	$\frac{2\pi}{\varepsilon_0}\frac{\varepsilon_0(-(C_2b^2+C_4a^2)\frac{\partial E_y^i}{\partial z}+(C_1b^2+C_5a^2+C_2b^2+C_4a^2)\frac{\partial E_z^i}{\partial y})}{\pi^2(-2C_5C_2b^2+C_7C_1b^2+2C_1C_4b^2+C_7C_2b^2+C_7C_5a^2+C_7C_4a^2)}$
θ_5^0	$\frac{2\pi}{\varepsilon_0}\frac{\varepsilon_0((-2C_5a^2+C_7b^2)\frac{\partial E_x^i}{\partial z}+a^2(2C_5+2C_4)\frac{\partial E_z^i}{\partial x})}{2\pi^2(-2C_2C_5a^2+C_7C_1b^2+C_7C_2b^2+2C_1C_4a^2+C_7C_5a^2+C_7C_4a^2)}$

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