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# NONLINEAR OPTIMAL PERTURBATIONS IN SUBCRITICAL INSTABILITIES

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**Abstract** Nonlinear optimal perturbations are defined here as those of minimum energy leading to subcritical instability. We show that a necessary condition for an initial perturbation to be a nonlinear optimal is that the initial perturbation energy growth is zero. The fulfillment of this condition does not depend on the disturbance amplitude but only on the linearized operator as long as the nonlinearity conserves energy. Saddle point solutions and linear optimal perturbations leading to maximum transient growth both satisfy the nonlinear optimality condition. We discuss these issues on low dimensional models of subcritical transition for which nonlinear optimals and the minimum threshold energy are computed.

**Keywords:** Subcritical transition, nonlinear stability, optimal perturbations, non-normality.

## 1. Introduction

The transition to turbulence in wall bounded shear flows has fascinated scientists since Reynolds' experiments in 1883 and is still not completely understood. Some of these flows, such as plane Couette flow and pipe Poiseuille flow, experience a transition to turbulence despite the linear stability of the laminar basic flow for all Reynolds numbers  $R$ . The fact that a turbulent state is observed for sufficiently large  $R$  is understood as an example of conditional stability, where asymptotic stability is guaranteed only if the energy of the perturbations  $\mathcal{E}$  is lower than a threshold  $\delta(R)$  which depend on the Reynolds number. For low values of the Reynolds number  $R < R_G$ , the threshold  $\delta$  is infinite and the flow is said to be globally stable. However, for  $R > R_G$ , the threshold  $\delta$  is finite and is thought to decrease with increasing  $R$  making the observation of the laminar basic flow less and less likely at large  $R$ . The direct computation of the curve  $\delta(R)$ , which is the scope of nonlinear hydrodynamic stability theory, is currently out of reach. Different types of stability analyses have therefore provided, along the years, upper and lower bounds on  $R_G$  and/or on  $\delta(R)$  itself or on its scaling with  $R$  as  $R \rightarrow \infty$ .

Energy theory (see Joseph, 1976 for a throughout account) provides a lower bound on  $R_G$  by providing sufficient conditions for monotonic stability. The idea is to start from the evolution equation for the energy of perturbations and then to maximize  $d\mathcal{E}/dt$  over all admissible perturbations at given  $R$ . Through variational calculus it is possible to find a critical Reynolds number  $R_E$  such that if  $R < R_E$  then  $d\mathcal{E}/dt < 0 \forall \mathbf{u}$ . The lower bounds  $R_E$  given by this kind of theory can however be far below the observed values of  $R_G$ .

A 'global bifurcation' perspective, completely different from the classical nonlinear stability analysis has emerged (Nagata, 1990, Clever and Busse, 1992 and many others thereafter) by looking for non-trivial solutions of the Navier-Stokes equations instead of examining the stability of the basic flow itself. In this approach, one usually looks for saddle-node bifurcations. Even if this kind of approach provide useful informations on the complexity of the phase space, for the moment it is not very effective in helping finding  $\delta$ . The main problems are (a) that standard techniques only allow for the search of 'simple' limit sets such as fixed points or limit cycles and are therefore unable for instance to provide upper bounds for  $\delta$  if the non-trivial attractors are not steady or periodic solutions, (b) that one is never sure to have found all the nontrivial attractors and (c) that to compute  $\delta$  one should compute the envelope of all the basins of attraction of all the non-trivial attractors which may be more complicated than computing the basin of attraction of

the laminar flow itself. This approach, appealing for the understanding of the structure of the turbulent flow itself, may be therefore less adapted to the determination of  $\delta(R)$ .

A complementary approach has emerged recently (see Schmid and Henningson, 2001 for a review) and has concentrated on the ability of linearly stable laminar shear flows, such as the Couette or Poiseuille solutions, to sustain, in the linear approximation, transient energy growths  $\mathcal{E}_L(t)/\mathcal{E}(0)$  whose maximum value can attain values of order  $O(R^2)$  (Gustavsson, 1991, Butler and Farrell, 1992). This huge potential for transient growth has been related to the strongly non-normal nature of the evolution operator linearized about the laminar basic flow (see Trefethen et al., 1993 for a review). Low-energy upper bounds on  $\delta(R)$  have been obtained by direct numerical simulation using finite amplitude linear optimal perturbations plus noise as initial condition (Kreiss et al., 1994, Reddy et al., 1998).

The scaling of  $\delta$  with the Reynolds number has been the subject of debate. Arguments based on the balance between linear-non-normal energy growth and nonlinear energy feedback applied to model systems suggested a scaling of the type  $\delta \sim R^\gamma$  with  $\gamma < -1$  (Trefethen et al., 1993, Baggett and Trefethen, 1997), where the upper limit  $\gamma = -1$  corresponds to the balance between nonlinear terms and viscous diffusion (Waleffe, 1995). The scaling estimates of  $\gamma$  provided by direct numerical simulation (Kreiss et al., 1994, Reddy et al., 1998) and asymptotic analyses (Chapman, 2002) for the Navier-Stokes equations are all near the  $-1$  value when streamwise vortices or oblique waves are given as initial conditions in the plane Couette and the plane Poiseuille flow (for  $R < 5772$ ). Recent experimental results (Hof et al., 2003) estimate  $\gamma = -1$  for pipe Poiseuille flow. On the other hand, using estimates on the scaling of the resolvent norm with  $R$ , which is strongly related to the non-normal nature of the linear operator, lower bounds on the threshold  $\delta(R)$  have been provided (Kreiss et al., 1994) as reviewed by Henningson in these proceedings. These analyses provide a lower bound  $\gamma > -4$  for the plane Couette flow.

Despite the great progress performed in recent years, an important gap still remains between lower bounds and upper estimates of the scaling exponent  $\gamma$  and of  $\delta(R)$  itself. One therefore remains wondering if other initial conditions, not considered in the cited numerical simulations asymptotic analyses and experiments, would not be able to exhibit values of  $\gamma$  lower than its upper limit  $-1$ . Even if the computation of  $\delta(R)$  seems still to be out of reach for the Navier-Stokes equations, it is not for model systems that have been used to mimic the subcritical transition and to propose new ideas for the Navier-Stokes case. The scope of the present contribution is to compute the nonlinear stability threshold  $\delta(R)$  itself and the associated nonlinear optimal perturbations for some low-dimensional models of subcritical transition. It is expected that the comparison of these results with those issued by previously cited approaches could induce some progress in the computation of  $\delta(R)$  in the Navier-Stokes case.

The setting and framework of the problem is laid down in Section 2 where a necessary condition for nonlinear optimality is derived. Illustrative results, obtained for two representative low-dimensional model systems of subcritical transition are reported and commented in Section 3. Implications for the Navier-Stokes equations are briefly discussed and some conclusions are drawn in Section 4.

## 2. Mathematical formulation

**Framework.** We are interested in the (nonlinear) asymptotic stability of a linearly stable ‘laminar basic state’  $\mathbf{U}$  with respect to perturbations  $\mathbf{u}$ . The generic state being written as  $\mathbf{U} + \mathbf{u}$ , by substitution one obtains the evolution equation for the perturbations:

$$d\mathbf{u}/dt = \mathcal{L}_R \mathbf{u} + \mathcal{N}(\mathbf{u}), \quad (1)$$

where the ‘laminar basic state’ now corresponds to  $\mathbf{u} = \mathbf{0}$ .  $\mathcal{L}_R$  is a linear operator depending on the real parameter  $R$  and on the basic flow  $\mathbf{U}$  while  $\mathcal{N}$  is a nonlinear operator that we assume to be homogeneous (i.e. such that  $\mathcal{N}(\mathbf{0}) = \mathbf{0}$ ). The initial value problem for equation (1) is solved by giving an initial condition  $\mathbf{u}_0$  at  $t = t_0$  and considering its evolution  $\mathbf{u} = \mathbf{u}(t, \mathbf{u}_0, R)$  which depends on time, on the given initial condition and on the parameter  $R$ . The energy of the perturbations is defined in the standard way using a suitable inner product

$$\mathcal{E}[\mathbf{u}(t, \mathbf{u}_0, R)] = \frac{1}{2} \langle \mathbf{u}, \mathbf{u} \rangle. \quad (2)$$

We will denote the energy of the initial condition as  $\mathcal{E}_0(\mathbf{u}_0) = \langle \mathbf{u}_0, \mathbf{u}_0 \rangle / 2$ . The evolution equation for the perturbation energy is obtained by projection of equation (1) on  $\mathbf{u}$ :

$$\frac{d\mathcal{E}}{dt} = \langle \mathbf{u}, \mathcal{L}_R \mathbf{u} \rangle + \langle \mathbf{u}, \mathcal{N}(\mathbf{u}) \rangle. \quad (3)$$

We will call ‘energy preserving nonlinear terms’ nonlinear terms with the property that  $\langle \mathbf{u}, \mathcal{N}(\mathbf{u}) \rangle = 0 \forall \mathbf{u}$ . This is the case for Navier-Stokes equations and for most of low-dimensional models of subcritical transition. We will call ‘amplitude’ of  $\mathbf{u}$  the norm  $\|\mathbf{u}\|$  defined by the standard inner product (therefore  $\|\mathbf{u}\| = \sqrt{2\mathcal{E}(\mathbf{u})}$ ) and ‘shape’ of  $\mathbf{u}$  the direction of the corresponding vector in phase space i.e.  $\mathbf{u}/\|\mathbf{u}\|$ .

**Minimum threshold energy and nonlinear optimals.** The basin of attraction  $\mathcal{S}_R$  of the laminar basic flow  $\mathbf{u} = \mathbf{0}$  at fixed  $R$  is given by the set of initial perturbations  $\mathbf{u}_0$  such that  $\lim_{t \rightarrow \infty} \mathcal{E}[\mathbf{u}(t, \mathbf{u}_0, R)] / \mathcal{E}_0 = 0$ . As we assumed  $\mathbf{U}$  linearly (strictly) stable, its basin of attraction has non-zero measure. The complementary set  $\mathcal{U}_R$  is made of initial perturbations for which  $\lim_{t \rightarrow \infty} \mathcal{E}[\mathbf{u}(t, \mathbf{u}_0, R)] / \mathcal{E}_0 \neq 0$ ; this set has zero measure as long as the laminar basic flow is globally stable. In the case in which  $\mathcal{U}_R$  has non-zero measure, the *minimum threshold energy* can be defined as

$$\delta(R) = \min_{\mathbf{u}_0 \in \mathcal{U}_R} \mathcal{E}_0(\mathbf{u}_0). \quad (4)$$

The *nonlinear optimal perturbation* (from now on abbreviated into NLOP) is the initial perturbation for which the minimum  $\delta$  is attained.

**A necessary condition of nonlinear optimality.** We now show that not all possible initial perturbations are suitable candidates to be nonlinear optimals (NLOP) but only those that realize a local minimum for the perturbation energy along the trajectory selected by  $\mathbf{u}_0$ . In fact, in a neighborhood  $\Delta t \ll 1$  of  $t_0$ , we can develop the perturbation energy as  $\mathcal{E}[\mathbf{u}(t_0 + \Delta t, \mathbf{u}_0, R)] = \mathcal{E}_0 + \Delta t (d\mathcal{E}/dt)_0 + (\Delta t^2/2) (d^2\mathcal{E}/dt^2)_0 + O(\Delta t^3)$ . If  $\mathbf{u}_0 \in \mathcal{U}_R$  and  $(d\mathcal{E}/dt)_0 < 0$  then  $\mathbf{u}(t_0 + \Delta t, \mathbf{u}_0, R)$ , with  $\Delta t > 0$  would also belong to  $\mathcal{U}_R$  and have a perturbation energy  $\mathcal{E} < \mathcal{E}_0$ , and therefore  $\mathbf{u}_0$  cannot be the NLOP because it is not a minimum of  $\mathcal{E}$ . In the case  $(d\mathcal{E}/dt)_0 > 0$  the same argument with  $\Delta t < 0$  demonstrates that  $\mathbf{u}_0$  is also not optimal. This proves that *a necessary condition for  $\mathbf{u}_0$  to be a nonlinear optimal perturbation is that  $(d\mathcal{E}/dt)(t_0, \mathbf{u}_0, R) = 0$* . If this condition is satisfied an additional condition, by which we make sure to obtain a local minimum and not a maximum, is that  $(d^2\mathcal{E}/dt^2)(t_0, \mathbf{u}_0, R) \geq 0$ .

**The  $\mathcal{Z}_R$  set and some of its properties.** The necessary condition of optimality allows to restrict the search of the nonlinear optimals to the set  $\mathcal{Z}_R$  of perturbations of zero energy growth  $d\mathcal{E}/dt = 0$ . By substitution in Eq. (3), it is easily verified that these perturbations must satisfy the condition  $\langle \mathbf{u}, \mathcal{L}_R \mathbf{u} \rangle + \langle \mathbf{u}, \mathcal{N}(\mathbf{u}) \rangle = 0$ . It is easy to verify<sup>1</sup> that all the nontrivial steady solutions  $\mathbf{u}_e$  such as saddles or nodes necessarily belong to  $\mathcal{Z}_R$ . For systems with energy-conserving nonlinear terms, the perturbations belonging to  $\mathcal{Z}_R$  satisfy the simpler condition  $\langle \mathbf{u}, \mathcal{L}_R \mathbf{u} \rangle = 0$ . For these systems the  $\mathcal{Z}_R$  set can therefore be determined by an analysis of the linear operator  $\mathcal{L}_R$  and belonging to that set is a property independent of the amplitude, i.e. if  $\mathbf{u}_0 \in \mathcal{Z}_R$  then  $c\mathbf{u}_0 \in \mathcal{Z}_R$ .

**Linear optimal perturbations.** Linear optimal perturbations are computed on the linear system  $d\mathbf{u}^{(L)}/dt = \mathcal{L}_R \mathbf{u}^{(L)}$  by a suitable optimization on the initial conditions. We will denote by LOP1 the linear optimals maximizing the linear transient growth  $\mathcal{E}^{(L)}/\mathcal{E}_0^{(L)}$  (e.g. Butler and Farrell, 1992, Trefethen et al., 1993). It can be shown<sup>2</sup> that these linear optimal perturbations satisfy the property  $\langle \mathbf{u}, \mathcal{L}_R \mathbf{u} \rangle = 0$ . Linear optimals LOP1 therefore also belong to  $\mathcal{Z}_R$  in the case of systems with energy preserving nonlinear terms.

Another type of linear optimal, denoted as LOP2 in the following, can be defined by maximizing the initial energy growth rate  $d\mathcal{E}^{(L)}/dt$ . This second type of optimal initial condition has for instance been used as initial condition to compute the scaling laws of the critical threshold for low-dimensional model systems (Baggett and Trefethen, 1997). Unless the maximum energy growth rate is zero, this second type of linear optimal initial condition does not belong to  $\mathcal{Z}_R$  in general.

### 3. Application to low dimensional models

Several low-dimensional models have been considered in order to discuss qualitative features of subcritical transitions. These models usually share properties of the Navier-Stokes equations that are thought to be relevant to the subcritical transition: (a) They admit the linearly stable laminar fixed point  $\mathbf{u} = 0 \forall R$ ; (b) the linear operator  $\mathcal{L}_R$  is non-normal and is able to induce energy growths of order  $R^2$  for sufficiently large  $R$ ; (c) nonlinear terms conserve energy. In the following we will compute the minimum threshold energy and the associated nonlinear optimal perturbations for two of these models. The first is a two-dimensional system proposed by Dauchot and Manneville, 1997 and is briefly reviewed by Manneville in these proceedings. It has also been independently studied by Baggett and Trefethen, 1997 as TTRD' model from a different perspective. We use this simple two-dimensional model to introduce, in a simple case, the technique by which the nonlinear optimals and threshold are computed. We then apply this technique to a four dimensional model, proposed by Waleffe, 1997, which is inspired by a low dimensional projection of the Navier-Stokes equations.

#### A two-dimensional model

**The model.** The model is in the form of system (1) with

$$\mathbf{u} = \begin{Bmatrix} u \\ v \end{Bmatrix}; \quad \mathcal{L}_R = \begin{bmatrix} -1/4R & 1 \\ 0 & -1/R \end{bmatrix}; \quad \mathcal{N}(\mathbf{u}) = \begin{Bmatrix} uv \\ -u^2 \end{Bmatrix}. \quad (5)$$

where all the variables are real. Using the standard inner product for real vectors the perturbation energy is defined as  $\mathcal{E} = (u^2 + v^2)/2$ . The equation governing the perturbation energy is readily obtained as  $d\mathcal{E}/dt = -u^2/4R + uv - v^2/R$ . The linear operator  $\mathcal{L}_R$  is non-normal and the nonlinear terms conserve  $\mathcal{E}$ .

**Global dynamics.** Many of the properties of system (5) can be computed by hand and its phase space dynamics has already been described (Dauchot and Manneville, 1997, Manneville in these proceedings). The laminar basic flow is represented by the trivial solution  $u = v = 0$ , coinciding with the origin  $O$  in the phase plane, and is stable for all  $R$ . For  $R < 1$  the laminar basic flow  $O$  is the sole attractor of the system and is therefore globally stable. At  $R = 1$  two additional solutions, a stable node  $A$  and a saddle  $S$  appear through a saddle-node bifurcation at finite distance from the laminar fixed point so that for  $R > 1$  the stability of laminar basic is only conditional because another attractor (the node  $A$ ) exists.<sup>3</sup>

**The  $\mathcal{Z}_R$  and the computation of  $\delta$  and the nonlinear optimals.** The states  $\{u_0, v_0\}$  belonging to  $\mathcal{Z}_R$  satisfy  $\langle \phi, \mathcal{L}_R \phi \rangle = -u_0^2/4R + u_0 v_0 - v_0^2/R = 0$  that gives the two solutions  $v_0 = u_0 (R \pm \sqrt{R^2 - 1})/2$ . The  $\mathcal{Z}_R$  is therefore defined only for  $R > R_E = 1$  and corresponds to two straight lines in the phase space (respectively  $\mathcal{Z}_R^+$  and  $\mathcal{Z}_R^-$  for the + or - sign of the root). In order to find the minimum threshold energy one should in principle test the stability of all possible initial conditions with energy  $(u_0^2 + v_0^2)/2 = \mathcal{E}_0$  starting from very low initial energy levels. These initial conditions corresponds to all the points on a circle in the phase space. When all the points of the circle corresponding to the given  $\mathcal{E}_0$  lead to stability, a new, slightly larger,  $\mathcal{E}_0$  can be considered and the computation repeated until the first initial condition leading to instability is encountered (see Figure 1). The necessary condition for optimality allows us instead to test the stability of only the four points on the circle which intersect the  $\mathcal{Z}_R$  set. To compute the threshold  $\delta$  and the corresponding nonlinear optimals we therefore start at sufficiently low  $\mathcal{E}_0$  and then we integrate up to  $t_{max} = 100 R$  the equations giving as initial condition the ‘‘potentially optimal’’ points of energy  $\mathcal{E}_0$ , on the set  $\mathcal{Z}_R$  and with  $(d^2\mathcal{E}/dt^2)_0 > 0$ , which are the two points on the  $\mathcal{Z}_R^+$  line. For low values of  $\mathcal{E}_0$  the two initial conditions lead to asymptotic stability. The initial energy is therefore increased by small steps  $\Delta\mathcal{E}_0$  and the computations repeated at each step. The minimum threshold  $\delta$  is found as the minimum value of  $\mathcal{E}_0$  for which no asymptotic decay of  $\mathcal{E}$  is observed and the corresponding initial condition represent the nonlinear optimal. This kind of computation has been repeated for values of  $R$  ranging from  $R_E = 1$  to 100. The nonlinear results are compared to the results that would have been found using linear optimization and saddle point tracking in the next paragraph.

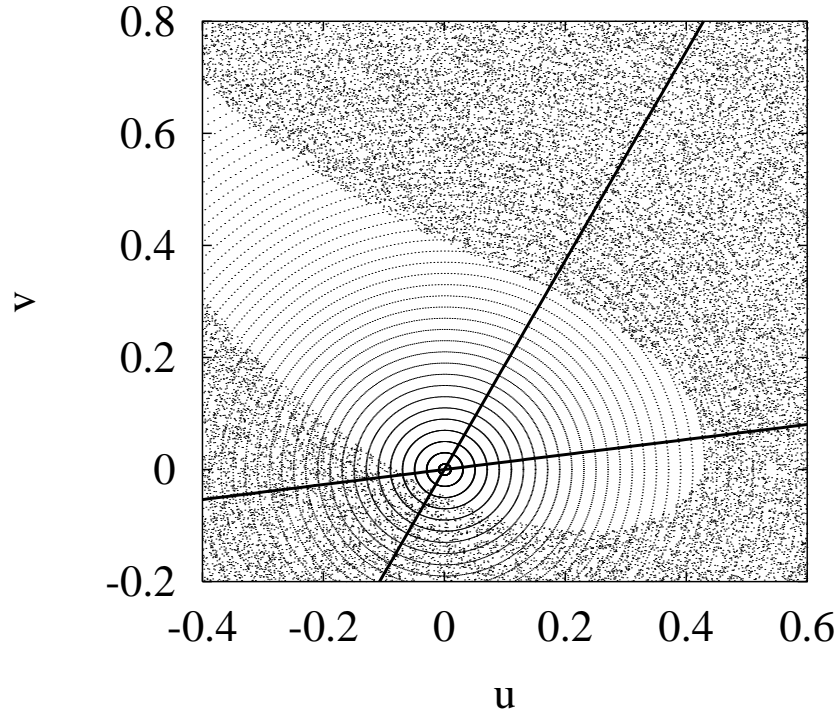


Figure 1. Phase space image of system (5) at  $R = 2$ . The two straight lines corresponding to the set  $\mathcal{Z}_R$  are reported as solid lines; they cross each other in origin corresponding to the laminar basic flow solution. The basin of attraction  $\mathcal{S}_R$  of the laminar basic flow is given by the white region surrounding the origin while its complementary  $\mathcal{U}_R$  is represented by the outside gray-shaded region. Iso-energy initial conditions lie on circles which are also reported in the figure.

**Comparison with other results.** The phase portrait of system (5) for  $R = 2$  is shown in Figs. 2 and 1. The linear optimal perturbations maximizing the energy growth (LOP1) lie on the  $\mathcal{Z}_R^+$  line (as already remarked by Dauchot and Manneville, 1997) while the corresponding linear optimal response coincides with the  $\mathcal{Z}_R^-$  line. The linear optimals maximizing the initial energy growth (LOP2) lie on the line reported in between  $\mathcal{Z}_R^+$  and  $\mathcal{Z}_R^-$  lines. The set of unstable initial conditions  $\mathcal{U}_R$  had been numerically computed and appears as the external gray region while the set  $\mathcal{S}_R$  is given by the inner white region surrounding the laminar fixed point  $O$ .  $\mathcal{U}_R$  and  $\mathcal{S}_R$  are separated by the stable manifold of the saddle  $S$  (Dauchot and Manneville, 1997). From Fig. 2 we therefore have a geometrical interpretations of the optimality condition: the minimum distance of  $\mathcal{U}_R$  from the laminar fixed point  $O$  is realized by the nonlinear optimal, denoted by  $N$ , which is situated on the  $\mathcal{Z}_R^+$  line that is orthogonal to the boundary of the basin of attraction and therefore at the minimum distance (see also Figure 1). Furthermore, in the present case, the nonlinear optimal and the linear optimal maximizing energy growth have the same shape. The same point  $N$  would have then been found by checking the stability threshold of linearly-optimal-shaped initial perturbations (LOP1) by increasing their amplitude (which is essentially the strategy followed by Reddy et al., 1998). The use of a strategy based on the linear optimals maximizing initial energy growth (Baggett and Trefethen, 1997), would have led to the critical point  $M$ , that, just like the saddle point  $S$ , is not only located farther from  $O$ , but also has a shape different from that of  $N$ . In Fig. 3 we show the critical amplitudes corresponding to the described “transition scenarios” as a function of  $R$ . The nonlinear minimum threshold amplitude  $\sqrt{2\delta(R)}$  and the critical amplitude found using the LOP1-strategy coincide, while the threshold amplitude found using the LOP2 strategy is slightly larger but they all scale as  $\sim R^{-3}$  for  $R > 10$ , a scaling predicted by Baggett and Trefethen, 1997 using non normal-nonlinear balance and by Dauchot and Manneville, 1997 using geometrical considerations. The ratio  $u_0/v_0$  scales as  $R^{-1}$  for the nonlinear optimals (and therefore also for the LOP1 linear optimals), while it tends to a constant for the LOP2. The saddle solution  $S$  has both

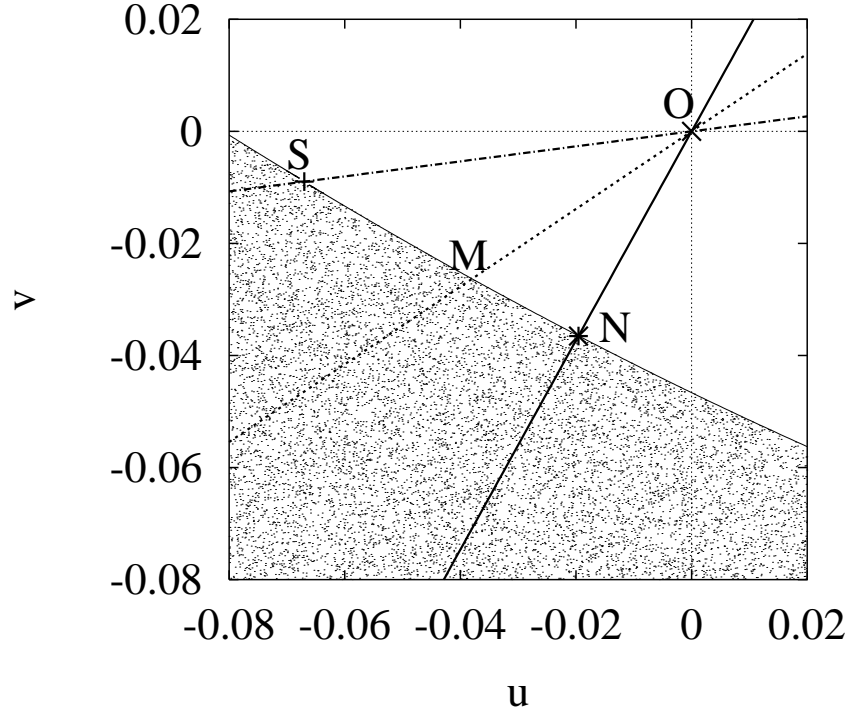


Figure 2. Phase space portrait of system (5) at  $R = 2$ . The laminar fixed point is denoted by  $O$ , the saddle by  $S$  and the nonlinear optimal perturbation by  $N$ . The basin of attraction  $\mathcal{S}_R$  is given by the white region surrounding  $O$ . The two straight lines corresponding to  $\mathcal{Z}_R^+$  and  $\mathcal{Z}_R^-$  are reported respectively as the solid line and the dashed-dotted lines. The LOP2 linear optimal set is reported as a dotted line.

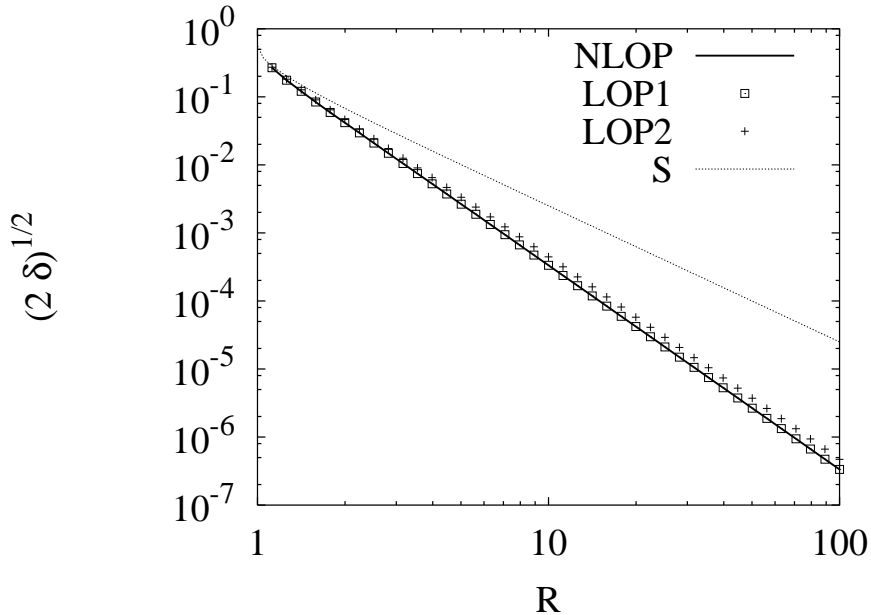


Figure 3. Dependence on the  $R$  of the (nonlinear) minimum threshold amplitude  $(2\delta(R))^{1/2}$  and of the critical amplitudes found using the other (LOP1, LOP2 and saddle point) strategies for system (5).

an amplitude scaling  $\sim R^{-2}$  and a shape  $u_0/v_0 \sim R$  which are completely different from the other critical perturbations therefore providing a less useful upper bound for critical thresholds.

**Discussion.** The coincidence of the shape of linear and non-linear optimal conditions is a general result for two-dimensional systems under the assumption of sufficiently low threshold energy.<sup>4</sup> Some of the conclusions that we may infer from the analysis of the two-dimensional model, however, may not extend to models of higher dimensions. In three dimensions for instance the set  $\mathcal{Z}_R$  would not be given by two lines

but by a cone and for instance the nonlinear optimals and the linear LOP1 optimals could be on the same cone without necessarily being on the same line and therefore have the same ‘shape’. This is why we now repeat our analysis on a four-dimensional model proposed by Waleffe, 1997 which has also the advantage of including key physical ingredients of the subcritical transition in shear flows.

### Waleffe’s model

**The model.** The four-dimensional model of Waleffe, 1997 is designed to mimic the subcritical instability of a Couette-like shear flow by a nonlinear self-sustained process involving the amplitude of streamwise vortices  $v$ , of streamwise streaks  $u$ , of sinuous perturbations of the streaks  $w$ , and of the mean shear  $m$  induced by perturbations. The model can be recast in the form of system (1) with

$$\mathbf{u} = \begin{Bmatrix} m \\ u \\ v \\ w \end{Bmatrix}; \quad \mathcal{N}(\mathbf{u}) = \begin{Bmatrix} \sigma_m w^2 - \sigma_u u v \\ -\sigma_w w^2 + \sigma_u m v \\ \sigma_v w^2 \\ (\sigma_w u - \sigma_m m - \sigma_v v) w \end{Bmatrix}. \quad (6)$$

$$\mathcal{L}_R = \begin{bmatrix} -k_m^2/R & 0 & 0 & 0 \\ 0 & -k_u^2/R & \sigma_u & 0 \\ 0 & 0 & -k_v^2/R & 0 \\ 0 & 0 & 0 & -k_w^2/R - \sigma_m \end{bmatrix}; \quad (7)$$

The same coefficients as those considered in Waleffe, 1997 have been selected.<sup>5</sup> The linear operator becomes strongly non-normal when  $R$  is increased and the nonlinear terms conserve energy.

**Global bifurcations.** The ‘laminar basic flow’  $m = u = v = w = 0$  is linearly stable for all  $R$ . For the parameters considered here, the phase space dynamics of the system has already been investigated (Waleffe, 1997): the laminar basic flow is the only steady solution of the system up to  $R = 104.84$  where a saddle node bifurcation takes place and two additional steady solutions appear. The ‘lower branch’ solution  $S$  is a saddle while the upper branch  $A$  is an unstable fixed point which become stable at  $R_G = 138.06$  thus rendering the stability of the laminar basic flow only conditional for  $R > R_G$ .

**The set  $\mathcal{Z}_R$  and the computation of  $\delta$  and of the NLOPs.** The perturbation energy is defined through the standard inner product as  $\mathcal{E} = (m^2 + u^2 + v^2 + w^2)/2$  and it is easily found that the vectors belonging to the set  $\mathcal{Z}_R$  must satisfy  $-(k_m^2 m_0^2 + k_u^2 u_0^2 + k_v^2 v_0^2)/R + \sigma_u u_0 v_0 - (\sigma_m + k_w^2/R)w_0^2 = 0$ . Non-trivial solutions of the previous equation exist only when the monotonic stability is lost i.e. for  $R \geq R_E = 2k_u k_v / \sigma_u = 4.89$  and  $uv > 0$ . To compute  $\delta(R)$  only initial conditions in the  $\mathcal{Z}_R$  set were considered by randomly selecting  $m_0, u_0$  and  $w_0$  and by then solving in  $v_0$  the equation for  $\mathcal{Z}_R$ , which resulted in a second order equation. Real solutions for  $v_0$  are found when the discriminant of the equation is non-negative, which strongly reduces the number of initial conditions to be investigated because in a large part of the phase space no solution belonging to  $\mathcal{Z}_R$  exists. The minimum condition  $(d^2\mathcal{E}/dt^2)_0 > 0$  was also checked, reducing the number of computations even more. Then, the same  $\mathcal{E}_0$ -marching procedure described above for the two-dimensional model, was used for the four-dimensional system. The minimum threshold energies and the corresponding nonlinear optimals have then been computed for  $R$  ranging from  $R_G = 138$  up to  $R = 1000$ , where an asymptotic regime seems to be attained. The results are reported in figure 4 and in table 1 and are compared to the linear optimals and to the saddle solution in the next paragraph.

**Comparison with other results.** The linear optimal perturbations maximizing respectively the energy growth (LOP1) and the initial energy growth rate (LOP2) both systematically have  $m_0 = w_0 = 0$ . Like in the two-dimensional model previously considered, when  $R$  is sufficiently large,  $u_0/v_0$  scales as  $R^{-1}$  for the linear optimal perturbations maximizing energy growth (LOP1) while it tends to a constant for the linear optimals maximizing energy growth rate (LOP2). The minimum threshold amplitude  $\sqrt{2\delta(R)}$  and the critical amplitudes obtained using respectively the linear LOP1 and LOP2 optimals both with additional  $10^{-9}$  noise as initial condition, are reported in figure 4. The lower branch corresponding to the saddle node  $S$  amplitude



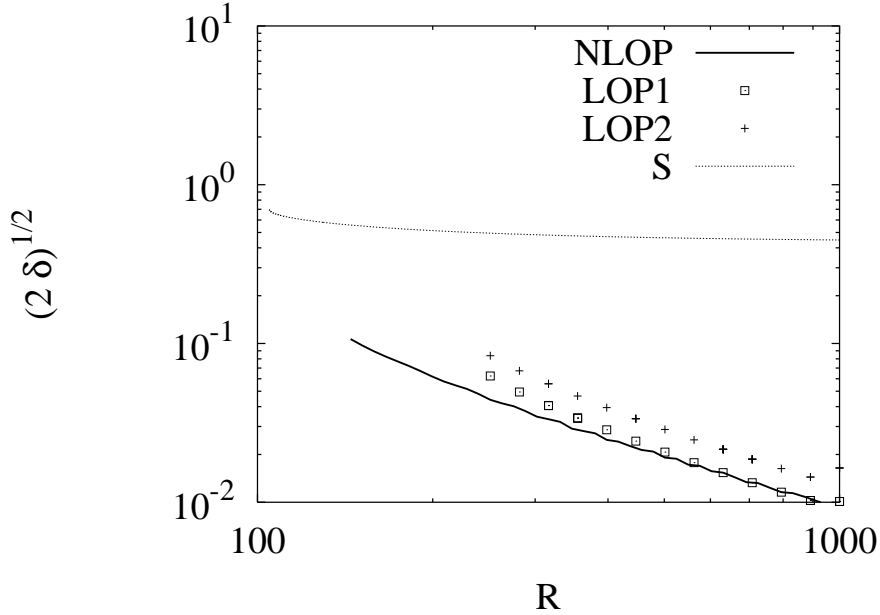


Figure 4. Dependence on the  $R$  of the (nonlinear) minimum threshold amplitude  $(2\delta(R))^{1/2}$  and of the critical amplitudes found using the other (LOP1, LOP2 and saddle point) strategies for system (7) with  $\sigma_m = 0.31$ .

Type	$v$	$u$	$w$	$m$	$(2\delta)^{1/2}$
NLOP	$15 R^{-1.07}$	$89 R^{-2.07}$	$0.00176 e^{-0.026 R}$	$35 R^{-1.99}$	$15 R^{-1.07}$
LOP1	$25 R^{-1.14}$	$189 R^{-2.18}$	$N$	$N$	$25 R^{-1.14}$
LOP2	$1.9 R^{-0.75}$	$8 R^{-0.97}$	$N$	$N$	$11 R^{-0.96}$

Table 1. Asymptotic ( $R > 500$ ) scaling of the nonlinear optimal perturbations NLOP and of the critical LOP1- and LOP2-shaped perturbations for system (7). The variables whose amplitude is at almost constant noise level have been denoted by N.

is also shown for comparison. The initial LOP1-shaped perturbations have critical amplitudes converging to the nonlinear minimum threshold for  $R$  sufficiently large, while the LOP2-shaped perturbations, even if leading to very low critical amplitudes, remain at finite distance from the minimum. The critical amplitudes all scale like  $\sim R^{-1}$  but only for  $R > 500$ . The saddle amplitude tends to a constant. This behaviour can be understood by examining the scalings observed for the critical perturbations for sufficiently large  $R$  ( $R > 500$ ) that are reported in table 1. The nonlinear optimal is essentially made of a linear LOP1 perturbation (remark that the ratio  $u_0/v_0$  is the same for the nonlinear and for the linear LOP1 optimal) plus an algebraically small mean shear perturbation  $m_0$  and an exponentially small  $w_0$  that, at sufficiently large  $R$ , is of the same order of magnitude of the noise ( $10^{-9}$ ) that is added to LOP1. The linear LOP2 initial condition is able to lead to so small critical amplitudes even if its shape is completely different from the NLOP shape.

**The case  $\sigma_m = 0$ .** The results for the considered systems could lead to the belief that the nonlinear optimals (NLOP) always asymptotically approach the linear optimals maximizing transient growth (LOP1). A counter-example is found by setting  $\sigma_m = 0$  in system (7), reverting to an older model of subcritical self-sustained process (Waleffe, 1995). The thresholds obtained using initial conditions with the shape of both types of linear optimals are now slightly larger than the minimum nonlinear threshold and are reported in Fig. 5. This is explained by the fact that, in this case, the ratio  $w_0/v_0$  of the nonlinear optimal NLOP tends to a constant (see table 2); even if the ratios  $u_0/v_0$  are almost the same, the NLOP shape remains therefore asymptotically different from the shape of linear optimals for which  $w_0/v_0 = 0$ . Again, however, the nonlinear optimal and the linear optimal strategies lead to the same asymptotic scaling of the critical amplitude  $\sim R^{-2}$  for sufficiently large  $R$ , while the saddle  $S$  amplitude has very different scaling and shape.

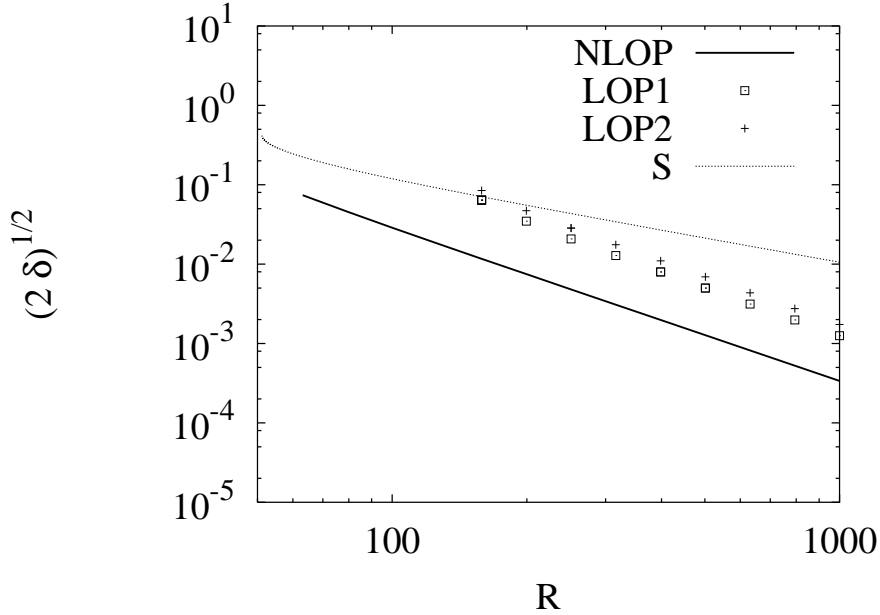


Figure 5. Same as in Fig. 4 but in the case  $\sigma_m = 0$ .

Type	$v$	$u$	$w$	$m$	$(2\delta)^{1/2}$
NLOP	$164 R^{-1.9}$	$1337 R^{-2.93}$	$115 R^{-1.97}$	$N$	$194 R^{-1.92}$
LOP1	$1309 R^{-2.00}$	$7831 R^{-3.00}$	$N$	$N$	$357 R^{-1.81}$
LOP2	$1174 R^{-1.99}$	$1200 R^{-1.99}$	$N$	$N$	$1679 R^{-1.99}$

Table 2. Asymptotic ( $R > 500$ ) scaling of the nonlinear optimal perturbations NLOP and of the critical LOP1- and LOP2-shaped perturbations for system (7) when  $\sigma_m = 0$ . The variables whose amplitude is at almost constant noise level have been denoted by N.

#### 4. Summary and conclusions

The scope of this study was to compute the minimum energy threshold  $\delta(R)$  and the associated nonlinear optimal perturbations for low-dimensional systems of subcritical transition. The main results may be summarized as follows:

- A necessary condition for a perturbation to be nonlinearly optimal, i.e., to be the one of minimum energy outside of the basin of attraction of the laminar basic flow, is that it realizes a local minimum for the perturbation energy i.e.  $d\mathcal{E}/dt = 0$  and  $d^2\mathcal{E}/dt^2 \geq 0$ .
- For systems with energy preserving nonlinear terms the first condition of optimality is equivalent to the condition  $\langle \mathbf{u}, \mathcal{L}_R \mathbf{u} \rangle = 0$  which can be determined from the analysis of the linear operator  $\mathcal{L}_R$  and does not depend on the amplitude of the perturbations.
- Linear optimal perturbations maximizing energy growth and steady nontrivial solutions such as saddles and nodes all necessarily belong to the set of perturbations satisfying the previous condition.
- Using these results it has been possible to demonstrate that in nonlinear systems with two degrees of freedom and nonlinear terms conserving energy, the nonlinear optimal perturbations coincide in general with the linear optimal perturbations maximizing energy growth provided that the threshold energy is sufficiently small.
- This coincidence of linear and nonlinear optimals is no longer necessary in systems with more than two degrees of freedom. However, the enforcement of the optimality condition allows one to reduce the extent of perturbations to be investigated for the determination of  $\delta$ .
- The analysis of Waleffe's models, reveals that in general nonlinear optimals do not have the same symmetries as the linear optimals. However, the projection of nonlinear optimals in the subspace

optimizing the linear growth has the same shape as the linear optimal. It is not clear if this result is general or peculiar to the considered systems.

- For the systems considered in this study, the minimum energy threshold asymptotically satisfy the  $R$  scalings predicted by dominant balance of non normal growth and nonlinear-feedback (Trefethen et al., 1993, Baggett and Trefethen, 1997). The thresholds computed using both nonlinear and linear optimal strategies are shown to reach the asymptotic scalings only for relatively large values of  $R$ .

The next extension of this study will probably be concerned with the analysis of the Navier-Stokes equations. In that case the necessary optimality condition still applies. If the perturbation energy is defined as  $\mathcal{E} = \int_V (u'_i u'_i / 2) dV$ , where  $u_i$  denote the velocity components of the perturbations of a basic flow  $U_i$  and  $V$  is a control volume, then the necessary condition  $\langle \mathbf{u}, \mathcal{L}_R \mathbf{u} \rangle = 0$  is easily derived by making use of the Reynolds-Orr equation<sup>6</sup>

$$\int_V \left( -\frac{\partial U_i}{\partial x_j} u_i u_j - \frac{1}{R} \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \right) dV = 0. \quad (8)$$

Only perturbation satisfying the previous property are eligible to be nonlinear optimals. The problem is that the number of perturbations to examine may be large. The previous condition should therefore be completed by an optimization algorithm and/or some supplementary condition. This is subject of current investigation.

## Acknowledgments

Stimulating discussions with Paul Manneville are kindly acknowledged.

## Notes

1. A nontrivial steady solution  $\mathbf{u}_e$  must satisfy the equilibrium equation  $\mathcal{L}_R \mathbf{u}_e + \mathcal{N}(\mathbf{u}_e) = 0$ . By projecting this equation on  $\mathbf{u}_e$  it follows that  $\mathbf{u}_e \in \mathcal{Z}_R$ .

2. The linear optimal perturbations are the initial conditions for which the maximum transient growth of linear solutions  $\mathbf{u}^{(L)}$  is reached:  $G_{max} = \max_t \max_{\mathbf{u}_0^{(L)}} \mathcal{E}[\mathbf{u}^{(L)}(t, \mathbf{u}_0^{(L)}, R)] / \mathcal{E}_0^{(L)}(\mathbf{u}_0^{(L)})$ . The two maximizations (in time and over the initial condition) may be swapped and therefore one is reduced to a problem which is very similar to the nonlinear one for the finding of  $\delta$ , except that now we have to jointly maximize  $\mathcal{E}^{(L)}$  and minimize  $\mathcal{E}_0^{(L)}$  and the trajectories in phase space are given by the linear evolution operator. It is however clear that if  $\mathbf{u}_0^{(L)}$  is not located at a local minimum of  $\mathcal{E}^{(L)}$  along the trajectory then one could find another initial condition on the same trajectory with lowest  $\mathcal{E}_0^{(L)}$  which therefore could realize a better growth. A necessary condition therefore for  $\mathbf{u}_0^{(L)}$  to be a linear optimal is that at  $t = 0$   $(d\mathcal{E}^{(L)}/dt)(0, \mathbf{u}_0^{(L)}, R) = 0$  and that if this condition is satisfied, furthermore  $(d^2\mathcal{E}^{(L)}/dt^2)(0, \mathbf{u}_0^{(L)}, R) \geq 0$ . As the evolution equation here is linear, the first condition can be recasted in the form  $\langle \mathbf{u}_0^{(L)}, \mathcal{L}_R \mathbf{u}_0^{(L)} \rangle = 0$

3. The picture is essentially given by Figure 9 in the contribution by Manneville in these proceedings. In his notations,  $u$  is called  $X_1$ ,  $v$  is  $X_2$ , the saddle point is denoted by  $M^+$  and the stable node by  $M^-$ .

4. The nonlinear optimal has to belong to the  $\mathcal{Z}_R$  set which is formed of two lines. It therefore has essentially the choice between having the shape of the linear optimal maximizing energy growth (LOP1) or the shape of the linear optimal response. The linear optimal perturbation is distinguished from the optimal response by the sign of  $d^2\mathcal{E}^{(L)}/dt^2$  which is positive for the optimal perturbation and negative for the optimal response. The nonlinear optimal perturbation also has a positive  $d^2\mathcal{E}/dt^2$  but now computed with the full nonlinear equation. By deriving equation (3) with respect to  $t$  and then eliminating  $d\mathbf{u}/dt$  using equation (1), it is possible to show that  $d^2\mathcal{E}/dt^2 = d^2\mathcal{E}^{(L)}/dt^2 + O(\mathcal{E}^{1+\beta})$  where the order of the derivative computed with linear theory is proportional to  $\mathcal{E}$  while the additional terms, due to the nonlinear contributions, are of order  $\mathcal{E}^{1+\beta}$  with  $\beta > 0$  and therefore larger than  $\mathcal{E}$ . For sufficiently small  $\mathcal{E}$  therefore the second derivative computed by including nonlinear terms has the same sign as the second order derivative computed with the linear terms.

5. They correspond to a streamwise wavenumber  $\alpha = 1.3$  and a spanwise wavenumber  $\gamma = 2.28$  and are:  $[k_m, k_u, k_v, k_w] = [1.57, 2.28, 2.77, 2.67]$ , and  $[\sigma_m, \sigma_u, \sigma_v, \sigma_w] = [0.31, 1.29, 0.22, 0.68]$ .

6. Here one may want to assume *a priori* that the enstrophy of the nonlinear optimals is finite so as to ensure, at finite  $R$ , the integrability of the dissipation term  $\frac{1}{R} \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j}$  appearing in Eq. (8).

## References

- Baggett, J.S. and Trefethen, L.N. (1997). Low-dimensional models of subcritical transition to turbulence. *Phys. Fluids*, 9:1043–1053.
- Butler, K. M. and Farrell, B. F. (1992). Three-dimensional optimal perturbations in viscous shear flow. *Phys. Fluids A*, 4:1637–1650.
- Chapman, S.J. (2002). Subcritical transition in channel flows. *J. Fluid Mech.*, 451:35–97.
- Clever, R. M. and Busse, F. H. (1992). Three-dimensional convection in a horizontal fluid layer subjected to a constant shear. *J. Fluid Mech.*, 234:511–527.
- Dauchot, O. and Manneville, P. (1997). Local versus global concepts in hydrodynamic stability theory. *J. Phys. II France*, 7:371–389.

- Gustavsson, L. H. (1991). Energy growth of three-dimensional disturbances in plane Poiseuille flow. *J. Fluid Mech.*, 224:241–260.
- Hof, B., Juel, A., and Mullin, T. (2003). Scaling of the turbulence transition threshold in a pipe. *Phys. Rev. Lett.*, 91:244502–1–4.
- Joseph, D. D. (1976). *Stability of Fluid Motions*. Springer Tracts in Natural Philosophy, vol. 27. Springer, New York.
- Kreiss, G., Lundbladh, A., and Henningson, D. S. (1994). Bounds for threshold amplitudes in subcritical shear flows. *J. Fluid Mech.*, 270:175–198.
- Nagata, M. (1990). Three-dimensional finite-amplitude solutions in plane Couette flow: bifurcation from infinity. *J. Fluid Mech.*, 217:519–527.
- Reddy, S. C., Schmid, P. J., Baggett, J. S., and Henningson, D. S. (1998). On the stability of streamwise streaks and transition thresholds in plane channel flows. *J. Fluid Mech.*, 365:269.
- Schmid, P. J. and Henningson, D. S. (2001). *Stability and Transition in Shear Flows*. Springer, New York.
- Trefethen, L. N., Trefethen, A. E., Reddy, S. C., and Driscoll, T. A. (1993). A new direction in hydrodynamic stability: Beyond eigenvalues. *Science*, 261:578–584.
- Waleffe, F. (1995). Transition in shear flows. nonlinear normality versus non-normal linearity. *Phys. Fluids*, 7:3060.
- Waleffe, F. (1997). On a self-sustaining process in shear flows. *Phys. Fluids*, 9:883–900.