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# A Gradient Flow Approach to Quantization of Measures

Emanuele Caglioti [1] François Golse [2] Mikaela Iacobelli [3 4]

**Abstract**

In this paper we study a gradient flow approach to the problem of quantization of measures in one dimension. By embedding our problem in $L^2$, we find a continuous version of it that corresponds to the limit as the number of particles tends to infinity. Under some suitable regularity assumptions on the density, we prove uniform stability and quantitative convergence result for the discrete and continuous dynamics.

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# 1 Introduction

### The quantization problem in the static case

The problem of quantization of a $d$-dimension probability distribution by discrete probabilities with a given number of points can be stated as follows: Given a probability density $\rho$, approximate it by a convex combination of a finite number $N$ of Dirac masses. The quality of the approximation is usually measured in terms of the Monge-Kantorovich metric. Much of the early attention in the engineering and statistical literature was concentrated on the one-dimensional quantization problem. This problem arises in several contexts and has applications in information theory (signal compression), cluster analysis (quantization of empirical measures), pattern recognition, speech recognition, numerical integration, stochastic processes (sampling design), mathematical models in economics (optimal location of service centers), and kinetic theory. For a detailed exposition and a complete list of references, we refer to the monograph [9].

We now introduce the setup of the problem. Fixed $r \geq 1$, consider $\rho$ a probability density on an open set $\Omega \subset \mathbb{R}^d$ such that

$$\int_\Omega |y|^r \rho(y) dy < \infty.$$

Given $N$ points $x^1, \dots, x^N \in \Omega$, one wants to find the best approximation of $\rho$, in the sense of Monge-Kantorovich, by a convex combination of Dirac masses centered at $x^1, \dots, x^N$. Hence one minimizes

$$\inf \left\{ MK_r\Big( \sum_i m_i \delta_{x^i}, \rho(y)dy \Big) \,:\, m_1, \dots, m_N \geq 0,\ \sum_i m_i = 1 \right\},$$

with

$$MK_r(\mu, \nu) := \inf \left\{ \int_{\Omega\times\Omega} |x-y|^r d\gamma(x,y) \,:\, (\pi_1)_\# \gamma = \mu,\ (\pi_2)_\# \gamma = \nu \right\},$$

where $\gamma$ varies among all probability measures on $\Omega \times \Omega$, and $\pi_i : \Omega \times \Omega \to \Omega$ ($i = 1, 2$) denotes the canonical projection onto the $i$-th factor (see [2, 15] for more details on the Monge-Kantorovitch distance between probability measures).

As shown for instance in [9, Chapter 1, Lemmas 3.1 and 3.4], the following facts hold:

1. The best choice of the masses $m_i$ is given by

$$m_i := \int_{W(\{x^1,\dots,x^N\}|x^i)} \rho(y)dy,$$

where

$$W(\{x^1,\dots,x^N\}|x^i) := \{y \in \Omega \ : \ |y - x^i| \le |y - x^j|, \ j \in 1,\dots,N\}$$

is the so called *Voronoi cell* of $x^i$ in the set $x^1,\dots,x^N$.

2. The following identity holds:

$$\inf\left\{ MK_r\Big(\sum_i m_i\delta_{x^i}, \rho(y)dy\Big) \ : \ m_1,\dots,m_N \ge 0, \ \sum_i m_i = 1\right\} = F_{N,r}(x^1,\dots,x^N),$$

where

$$F_{N,r}(x^1,\dots,x^N) := \int_\Omega \min_{1\le i\le N} |x^i - y|^r \rho(y)dy.$$

If one chooses $x^1,\dots,x^N$ in an optimal way by minimizing the functional $F_{N,r} : (\mathbb{R}^d)^N \to \mathbb{R}^+$, in the limit as $N \to \infty$, these points distribute themselves accordingly to a probability density proportional to $\rho^{d/d+r}$. In other words, by [9, Chapter 2, Theorem 7.5] one has

$$\frac{1}{N}\sum_{i=1}^N \delta_{x^i} \rightharpoonup \frac{\rho^{d/d+r}}{\int_\Omega \rho^{d/d+r}(y)dy}\, dx. \tag{1.1}$$

These issues are relatively well understood from the point of view of the calculus of variations [9, Chapter 1, Chapter 2]. Our goal here is to consider instead a dynamic approach to this problem, as we shall describe now.

## A dynamical approach to the quantization problem

Given $N$ points $x_0^1,\dots,x_0^N$, we consider their evolution under the gradient flow generated by $F_{N,r}$, that is, we solve the system of ODEs in $(\mathbb{R}^d)^N$

$$\begin{cases} \big(\dot{x}^1(t),\dots,\dot{x}^N(t)\big) = -\nabla F_{N,r}\big(x^1(t),\dots,x^N(t)\big), \\ \big(x^1(0),\dots,x^N(0)\big) = (x_0^1,\dots,x_0^N) \end{cases} \tag{1.2}$$

As usual in gradient flow theory, as $t \to \infty$ one expects that the points $\big(x^1(t), \dots, x^N(t)\big)$ converge to a minimizer $(\bar{x}^1, \dots, \bar{x}^N)$ of $F_{N,r}$. Hence (in view of (1.1)) the empirical measure

$$\frac{1}{N} \sum_{i=1}^{N} \delta_{\bar{x}^i}$$

is expected to converge to

$$\frac{\rho^{d/d+r}}{\displaystyle\int_\Omega \rho^{d/d+r}(y)dy} dx$$

as $N \to \infty$.

We now seek to pass to the limit in the ODE above as $N \to \infty$. For this, we take a set of reference points $(\hat{x}^1, \dots, \hat{x}^N)$ and we parameterize a general family of $N$ points $x^i$ as the image of $\hat{x}^i$ via a smooth map $X : \mathbb{R}^d \to \mathbb{R}^d$, that is

$$x^i = X(\hat{x}^i).$$

In this way, the function $F_{N,r}(x^1, \dots, x^N)$ can be rewritten in terms of the map $X$ and (a suitable renormalization of it) should converge to a functional $\mathcal{F}[X]$. Hence, we expect that the evolution of $x^i(t)$ for $N$ large is well-approximated by the $L^2$-gradient flow of $\mathcal{F}$.

Although this formal argument may look convincing, already the 1 dimensional case is nontrivial, and will be studied in detail in the present paper. The higher dimensional case is much harder. For one thing, even in space dimension 2, there is no obvious analogue of the functional $\mathcal{F}[X]$ in the continuous limit. The present paper is focussed on the 1 dimensional setting, and the higher dimensional case is left for future work.

## The 1D case

With no loss of generality we take $\Omega$ to be the open interval $(0, 1)$ and we consider $\rho$ a smooth probability density on $\Omega$. In order to obtain a continuous version of the functional

$$F_{N,r}(x^1, \dots, x^N) = \int_0^1 \min_{1 \le i \le N} |x^i - y|^r \rho(y)\, dy,$$

with $0 \le x^1 \le \dots \le x^N \le 1$, assume that

$$x^i = X\left(\frac{i - 1/2}{N}\right), \qquad i = 1, \dots, N$$

with $X : [0, 1] \to [0, 1]$ a smooth non-decreasing map such that $X(0) = 0$ and $X(1) = 1$. Then, as explained in Appendix A,

$$N^r F_{N,r}(x^1, \dots, x^N) \longrightarrow C_r \int_0^1 \rho(X(\theta)) |\partial_\theta X(\theta)|^{r+1} d\theta := \mathcal{F}[X]$$

as $N \to \infty$, where $C_r := \frac{1}{2^r(r+1)}$.

By a standard computation [6] we obtain the gradient flow PDE for $\mathcal{F}$ for the $L^2$-metric,

$$\partial_t X(t,\theta) = C_r\Big((r+1)\partial_\theta\big(\rho(X(t,\theta))|\partial_\theta X(t,\theta)|^{r-1}\partial_\theta X(t,\theta)\big) - \rho'(X(t,\theta))|\partial_\theta X(t,\theta)|^{r+1}\Big), \quad (1.3)$$

completed with the Dirichlet boundary condition

$$X(t,0) = 0, \qquad X(t,1) = 1. \quad (1.4)$$

Let us notice that, in the particular case $\rho \equiv 1$, (1.3) becomes a $p$-Laplacian equation

$$\partial_t X = C_r(r+1)\partial_\theta\big(|\partial_\theta X|^{r-1}\partial_\theta X\big)$$

with $p-1 = r$ (see [5, 13] and references therein for a general treatment of this class of equations).

## From the Lagrangian to the Eulerian setting

Equation (1.3) provides a Lagrangian description of the evolution of our system of particles in the limit $N \to \infty$. We can also study the Eulerian picture for the gradient flow PDE. If we denote by $f(t,x)$ the image of the Lebesgue measure through the map $X$, i.e.

$$f(t,x)dx = X(t,\theta)_{\#}d\theta,$$

then the PDE satisfied by $f$ takes the form [1]

$$\partial_t f(t,x) = -rC_r\partial_x\left(f(t,x)\partial_x\left(\frac{\rho(x)}{f(t,x)^{r+1}}\right)\right), \quad (1.5)$$

with periodic boundary conditions, and we expect the following long time behavior

$$f(t,x) \longrightarrow \frac{\rho^{1/(r+1)}(x)}{\int_0^1 \rho(y)^{1/(r+1)}dy} \qquad \text{as } t \to \infty.$$

Notice that if $\rho \equiv 1$, (1.5) becomes

$$\partial_t f = -C_r(r+1)\partial_x^2\big(f^{-r}\big),$$

which is an equation of very fast diffusion type [4, 13, 14]. It is interesting to point out that the above equation set on the whole space $\mathbb{R}$ or with zero

[1] Indeed since $\partial_t X = b(t,X)$ with $b(t,y) := C_r r\left(\frac{\rho(y)}{f(t,y)^{r+1}}\right)$ (this follows by a direct computation starting from (1.3)), the function $f \equiv f(t,x)$ solves the continuity equation $\partial_t f(t,x) + \mathrm{div}(b(t,x)f(t,x)) = 0$, as shown for instance in [1].

Dirichlet boundary conditions has no solutions, since all the mass instantaneously disappear [12, Theorem 3.1]. It is therefore crucial that in our setting the equation has periodic boundary conditions. In particular, as we shall see, our equation satisfies a comparison principle (see Lemma 2.1).

### Assumptions on $\rho$ and convexity of the functionals

Notice that our heuristic arguments in the previous section were based on the assumption that both the gradient flows of $F_{N,2}$ and of $\mathcal{F}$ converge to a minimizer as $t \to \infty$. Of course this is true if $F_{N,2}$ and $\mathcal{F}$ are convex [2]. Actually, notice that we are trying to show that the limits as $N \to \infty$ and $t \to \infty$ commute, and for this we need to prove that the discrete and the continuous gradient flows remain close in the $L^2$ sense, uniformly with respect to $t$. Therefore, the convexity of $\mathcal{F}$ and $F_{N,2}$ seems to be a very natural issue for the validity of our gradient flow strategy.

As shown in Appendix B, for the hessian of $\mathcal{F}$ to be nonnegative at "points" $X$ which are Lipschitz and uniformly monotone, one has to assume $\rho$ to be sufficiently close to a constant in $C^2$. We shall therefore adopt this condition on $\rho$.

Whether this condition on $\rho$ ensures that $F_{N,2}$ is also convex is left undecided. Nevertheless we are able to prove that the discrete flow and the continuous one remain close by a combination of arguments including the maximum principle and $L^2$-stability (see Section 4).

### Statement of the results

In order to simplify our presentation, in the whole paper we shall focus only on the case $r = 2$. Indeed, this has the main advantage of simplifying some of the computations allowing us to highlight the main ideas. As will be clear from the sequel, this case already incorporates all the main features and difficulties of the problem, and this specific choice does not play any essential role.

As we mentioned in the previous section, the properties of $\rho$ are crucial in the proofs. Notice also that (1.3) is of $p$-laplacian type, which is a degenerate parabolic equation. In order to avoid degeneracy, it is necessary for the solution to be an increasing function of $\theta$. For this reason, we assume this on the initial datum and prove that this monotonicity is preserved along the flow.

It is worth noticing that the monotonicity estimate at the discrete level says that if $x^{i+1}(0) - x^i(0) \approx \frac{1}{N}$ for all $i$, this property is preserved in time (up to multiplicative constants). In particular the points $\{x^i(t)\}_{i=1,\dots,N}$ can never collide.

Our main result shows that, under the two above mentioned assumptions (that is, $\rho$ is close to a constant in $C^2$ and the initial datum is smooth and increasing) the discrete and the continuous gradient flows remain uniformly close in $L^2$ for *all* times. Notice however that the results in the case $\rho \equiv 1$ and $\rho \not\equiv 1$ are quite different. Indeed, when $\rho \equiv 1$ the equation (1.3) depends on $\partial_\theta X$ and $\partial_{\theta\theta} X$, but not on $X$ itself. This fact plays a role in several places, both for showing the monotonicity of solutions (in particular for the discrete case) and in the convergence estimate. In particular, while in the case $\rho \equiv 1$ we obtain convergence of the discrete flow to the continuous one for all initial data, the case $\rho \not\equiv 1$ requires an additional assumption at time 0 (see (1.6)).

One further comment concerns the time scaling: notice that, in order to obtain a nontrivial limit of our functional $F_{N,r}$, we needed to rescale them by $1/N^r$. In addition to this, since we want to compare gradient flows, we have to take into account that the Euclidean metric in $\mathbb{R}^N$ has to be rescaled by a factor $1/N$ to be compared with the $L^2$ norm.[2] Hence, to compare the discrete and the continuous gradient flows, we need to rescale the former in time by a factor $N^{r+1}$.

We now state our convergence results, first when $\rho \equiv 1$ and then for the general case. It is worth to point out that the best way to approximate the uniform measure on $[0,1]$ with the sum of $N$ Dirac masses it to put masses of size $1/N$ centered at points $(i-1/2)/N$ and

$$MK_1\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{(i-1/2)/N}, d\theta\right) = \frac{1}{4N}$$

(see the computation in the proof of Theorem 3.6). Hence the result in our next theorem shows that the gradient flow approach provides, for $N$ and $t$ large, the best approximation rate.

**Theorem 1.1.** *Let $\rho \equiv 1$, $\big(x^1(t), \dots, x^N(t)\big)$ the gradient flow of $F_{N,2}$, and $X(t)$ the gradient flow of $\mathcal{F}$ starting from $X_0$. Assume that $X_0 \in C^{4,\alpha}([0,1])$ and that there exist positive constants $c_0, C_0$ such that*

$$\frac{c_0}{N} \le \bar{x}^i(0) - \bar{x}^{i-1}(0) \le \frac{C_0}{N}, \text{ and } c_0 \le \partial_\theta X_0 \le C_0.$$

*Define $X^i(t) := X\left(t, \frac{i-1/2}{N}\right)$, $\bar{x}^i(t) := x^i(N^3 t)$, and $\mu_t^N := \frac{1}{N}\sum_i \delta_{x^i(t)}$ Then there exist two constants $c', C' > 0$, depending only on $c_0$, $C_0$, and*

[2] Let $\bar{x} := (x^1, \dots, x^N), \bar{y} := (y^1, \dots, , y^N) \in \mathbb{R}^N$, and embed these points into $L^2([0,1])$ by defining the functions

$$X(\theta) := x^i, \quad Y(\theta) := y^i, \qquad \forall\, \theta \in \left(\frac{i-1}{N}, \frac{i}{N}\right).$$

Then $|\bar{x} - \bar{y}|^2 = \sum_{i=1}^N |x^i - y^i|^2$ while $\|X - Y\|_{L^2}^2 = \frac{1}{N}\sum_{i=1}^N |x^i - y^i|^2$.

$\|X_0\|_{C^{4,\alpha}([0,1])}$, *such that, for all* $t \geq 0$,

$$\frac{1}{N}\sum_{i=1}^{N}\left(\bar{x}^i(t) - X^i(t)\right)^2 \leq e^{-c't}\frac{1}{N}\sum_{i=1}^{N}\left(\bar{x}^i(0) - X^i(0)\right)^2 + \frac{C'}{N^4}$$

*and*

$$MK_1(\mu_t^N, d\theta) \leq \frac{1}{4N} + C'\, e^{-c't/N^3} + \frac{C'}{N^2}.$$

*In particular*

$$MK_1(\mu_t^N, d\theta) \leq \frac{1}{4N} + \frac{2C'}{N^2} \qquad \forall\, t \geq \frac{2N^3 \log N}{c'}.$$

**Theorem 1.2.** *Let* $\left(x^1(t), \ldots, x^N(t)\right)$ *be the gradient flow of* $F_{N,2}$, *and* $X(t)$ *the gradient flow of* $\mathcal{F}$ *starting from* $X_0$. *Assume that* $X_0 \in C^{4,\alpha}([0,1])$ *for some* $\alpha > 0$ *and that there exist two positive constants* $c_0, C_0$ *such that*

$$\frac{c_0}{N} \leq x^i(0) - x^{i-1}(0) \leq \frac{C_0}{N}, \text{ and } c_0 \leq \partial_\theta X_0 \leq C_0.$$

*Define* $X^i(t) := X\left(t, \frac{i-1/2}{N}\right)$, $\bar{x}^i(t) := x^i(N^3 t)$, *and* $\mu_t^N := \frac{1}{N}\sum_i \delta_{x^i(t)}$, *and assume that* $\rho : [0,1] \to (0,\infty)$ *is a periodic probability density of class* $C^{3,\alpha}$ *with* $\|\rho'\|_\infty + \|\rho''\|_\infty \leq \bar{\varepsilon}$ *and that*

$$|X^i(0) - x^i(0)| \leq \frac{\bar{C}}{N^2} \qquad \forall\, i = 1, \ldots, N. \tag{1.6}$$

*for some positive constants* $\bar{\varepsilon}, \bar{C}$. *Then there exist two constants* $c', C' > 0$, *depending only on* $c_0$, $C_0$, $\|\rho\|_{C^{3,\alpha}([0,1])}$ *and* $\|X_0\|_{C^{4,\alpha}([0,1])}$, *such that the following holds: if* $\bar{\varepsilon}$ *is small enough (in terms of* $c_0$, $C_0$, *and* $\bar{C}$*) we have*

$$\frac{1}{N}\sum_{i=1}^{N}\left(\bar{x}^i(t) - X^i(t)\right)^2 \leq \frac{C'}{N^4} \qquad \textit{for all } t \geq 0$$

*and*

$$MK_1(\mu_t^N, \gamma\rho^{1/3}\, d\theta) \leq C'\, e^{-c't/N^3} + \frac{C'}{N} \qquad \textit{for all } t \geq 0,$$

*where*

$$\frac{1}{\gamma} := \int_0^1 \rho(\theta)^{1/3}\, d\theta.$$

*In particular*

$$MK_1(\mu_t^N, \gamma\rho^{1/3}\, d\theta) \leq \frac{C'}{N} \qquad \textit{for all } t \geq \frac{N^3 \log N}{c'}.$$

As a consequence of our results, under the assumption that $\rho$ is $C^2$ close to 1 we obtain a quantitative version of the results in [9]:

**Corollary 1.3.** *There exist two constants $\bar{\varepsilon} > 0$ and $C > 0$ such that the following holds: assume that $\|\rho'\|_\infty + \|\rho''\|_\infty \leq \bar{\varepsilon}$, and let $(x^1, \dots, x^N)$ be a minimizer of $F_{N,2}$. Then*

$$MK_1(\mu^N, \gamma\rho^{1/3}\, d\theta) \leq \frac{C}{N}$$

*where*

$$\mu^N := \frac{1}{N}\sum_i \delta_{x^i}$$

*and*

$$\frac{1}{\gamma} := \int_0^1 \rho(\theta)^{1/3}\, d\theta.$$

The paper is structured as follows: in the next section we collect several preliminary results both on the discrete and the continuous gradient flow. Then, we prove the convergence result first in the case $\rho \equiv 1$, and finally in the case $\|\rho - 1\|_{C^2([0,1])} \ll 1$.

In the whole paper we assume that $0 < \lambda \leq \rho \leq 1/\lambda$.

# 2 Preliminary results

## 2.1 The discrete gradient flow

We begin by computing the discrete gradient flow: as shown in the appendix, given points $0 \leq x^1 \leq \ldots \leq x^N \leq 1$, one has

$$F_{N,r}(x^1, \dots, x^N) = \sum_{i=1}^N \int_{x^{i-1/2}}^{x^{i+1/2}} |y - x^i|^r \rho(y) dy$$

where

$$x^{i+1/2} := \frac{x^i + x^{i+1}}{2} \qquad \forall\, i = 2, \dots, N-1,$$

while we set $x^{1/2} := 0$ and $x^{N+1/2} := 1$. Then, a direct computation gives

$$\frac{\partial F_{N,2}}{\partial x^i}(x^1, \dots, x^N) = -2\int_{x^{i-1/2}}^{x^{i+1/2}} (y - x^i)\rho(y) dy. \tag{2.1}$$

Moreover, assuming that $\rho$ is at least of class $C^0$ it is easy to check that $\nabla F_{N,2}$ is bounded and continuously differentiable, hence $F_{N,2}$ is of class $C^2$. Thus the gradient flow of $F_{N,2}$ is unique and exists globally for all $t \geq 0$ by the Cauchy-Lipschitz Theorem for ODEs.

## 2.2 The continuous gradient flow

In order to construct a solution to the continuous gradient flow (2.3) we start from the Eulerian description that we look as a PDE on $[0, 1]$ with periodic boundary conditions.

### 2.2.1 The Eulerian flow

Recall that by assumption $\lambda \le \rho \le 1/\lambda$ for some $\lambda > 0$. Given $f(t,x)$ a solution of (1.5), we set

$$m(x) := \rho(x)^{1/3}, \qquad u(t,x) := \frac{f(t,x)}{m(x)}.$$

With these new unknowns (1.5) becomes

$$\partial_t u = -\frac{1}{4\,m(x)}\,\partial_x\Big(m(x)\partial_x\Big(\frac{1}{u^2}\Big)\Big) \qquad \text{on } [0,\infty)\times[0,1] \tag{2.2}$$

with periodic boundary conditions. The advantage of this form is double: first of all, the above PDE enjoys a comparison principle; secondly, constants are solutions. Since for our purposes, only comparison with constants is necessary, we will just show that.

**Lemma 2.1.** *Let $u$ be a nonnegative solution of* (2.2) *and $c$ be a positive constant. Then both*

$$t \mapsto \int_0^1 (u-c)_-\,dx \quad \textit{and} \quad t \mapsto \int_0^1 (u-c)_+\,dx$$

*are nonincreasing functions.*

*Proof.* We show just the first statement (the other being analogous).

Since constants are solutions of (2.2), it holds

$$\partial_t(u-c) = -\frac{1}{4\,m}\,\partial_x\Big(m\,\partial_x\Big(\frac{1}{u^2}-\frac{1}{c^2}\Big)\Big).$$

We now multiply the above equation by $-m\,\phi_\varepsilon\left(\frac{1}{u^2}-\frac{1}{c^2}\right)$, with $\phi_\varepsilon$ a smooth approximation the indicator function of $\mathbb{R}_+$ satisfying $\phi_\varepsilon' \ge 0$. Integrating by parts we get

$$\begin{aligned}\frac{d}{dt}\int_0^1 \Psi_\varepsilon(u-c)\,dx &= -\int_0^1 \phi_\varepsilon\left(\frac{1}{u^2}-\frac{1}{c^2}\right)\,\partial_t(u-c)\,m\,dx\\ &= -\frac{1}{4}\int_0^1 \left|\partial_x\Big(\frac{1}{u^2}-\frac{1}{c^2}\Big)\right|^2 \phi_\varepsilon'\Big(\frac{1}{u^2}-\frac{1}{c^2}\Big)\,m\,dx \le 0,\end{aligned}$$

where we have set

$$\Psi_\varepsilon(s) := -\int_0^s \phi_\varepsilon\Big(\frac{1}{(\sigma+c)^2}-\frac{1}{c^2}\Big)\,d\sigma.$$

Letting $\varepsilon \to 0$ we see that $\Psi_\varepsilon(s) \to s_-$, hence

$$\frac{d}{dt}\int_0^1 (u-c)_-\,dx \le 0,$$

proving the result. □

Thus, if $a_0 \le u(0,x) \le A_0$, then $a_0 \le u(t,x) \le A_0$ for all $t \ge 0$. We now apply this fact to show that if $f$ is bounded away from zero and infinity at the initial time, then so it is for all positive times. More precisely, recalling that by assumption $\lambda \le \rho \le \frac{1}{\lambda}$, we have

$$\begin{aligned} a_1 \le f(0,x) \le A_1 \quad &\Rightarrow \quad \lambda^{1/3} a_1 \le u(0,x) \le \frac{A_1}{\lambda^{1/3}} \\ &\Rightarrow \quad \lambda^{1/3} a_1 \le u(t,x) \le \frac{A_1}{\lambda^{1/3}} \\ &\Rightarrow \quad \lambda^{2/3} a_1 \le f(t,x) \le \frac{A_1}{\lambda^{2/3}} \qquad \forall\, t \ge 0. \end{aligned}$$

These a priori bounds show that (1.5) is a uniformly parabolic equation. In particular, since $f$ is uniformly bounded for all times, by parabolic regularity theory (see for instance [10, Theorem 8.12.1], [8, Chapter 3, Section 3, Theorem 7], and [11, Chapters 5, 6]) we conclude that:

**Proposition 2.2.** *Let $\lambda \in (0,1]$, and assume that $\rho : [0,1] \to [\lambda, 1/\lambda]$ is periodic and of class $C^{k,\alpha}$ for some $k \ge 0$ and $\alpha \in (0,1)$. Let $f(0,\cdot) : [0,1] \to \mathbb{R}$ be a periodic function of class $C^{k,\alpha}$ satisfying $0 < a_1 \le f(0,\cdot) \le A_1$, and let $f$ solve (1.5) with periodic boundary conditions. Then*

$$\lambda^{2/3} a_1 \le f(t,x) \le \frac{A_1}{\lambda^{2/3}} \qquad \text{for all } t \ge 0,$$

*$f(0,\cdot)$ is of class $C^{k,\alpha}$ for all $t \ge 0$, and there exists a constant $C$, depending only on $\lambda$, $\|\rho\|_{C^{k,\alpha}}$, $k$, $\alpha$, $a_1$, and $A_1$, such that $\|f(t,\cdot)\|_{C^{k,\alpha}([0,1])} \le C$ for all $t \ge 0$.*

### 2.2.2 The Lagrangian flow

To obtain now existence and uniqueness for the gradient flow of $\mathcal{F}$, we simply define $X(t)$ for any $t \ge 0$ as the solution of the ODE (in $\theta$)

$$\begin{cases} \partial_\theta X(t,\theta) = \frac{1}{f(t,X(t,\theta))} & \text{on } [0,1], \\ X(t,0) = 0, \end{cases} \qquad \forall\, t \ge 0. \tag{2.3}$$

Notice that the boundary conditions $X(t,1) = 1$ is automatically satisfied since

$$\int_0^{X(t,1)} f(t,x)\, dx = 1$$

and $f(t) > 0$ is a probability on $[0,1]$. Also, notice that $X(t)$ has exactly one derivative more than $f(t)$. Hence, by Proposition 2.2 we obtain:

**Proposition 2.3.** *Let $\lambda \in (0,1]$, and assume that $\rho : [0,1] \to [\lambda, 1/\lambda]$ is periodic and of class $C^{k,\alpha}$ for some $k \ge 0$ and $\alpha \in (0,1)$. Let $X(0,\cdot)$ satisfy $0 <$*

$a_1 \le \partial_\theta X(0,\cdot) \le A_1$, $X(0,0) = 1$, $X(0,1) = 1$, *and* $\|X(0,\cdot)\|_{C^{k+1,\alpha}([0,1])} < \infty$, *and let* $X(t,\cdot)$ *solve* (1.3)-(1.4). *Then*

$$\lambda^{2/3} a_1 \le \partial_\theta X(t,\theta) \le \frac{A_1}{\lambda^{2/3}} \qquad \textit{for all } t \ge 0,$$

*and there exists a constant* $C$, *depending only on* $\lambda$, $\|\rho\|_{C^{k,\alpha}}$, $k$, $\alpha$, $a_1$, *and* $A_1$, *such that* $\|X(t,\cdot)\|_{C^{k+1,\alpha}([0,1])} \le C$ *for all* $t \ge 0$.

# 3 The case $\rho \equiv 1$

As we already mentioned we shall focus only on increasing initial data, and as proved in the previous section this monotonicity is preserved in time, hence $\partial_\theta X \ge 0$.

We first observe that, in the case $\rho \equiv 1$, the equation (1.3) becomes

$$\partial_t X(t,\theta) = \frac{1}{4}\partial_\theta \left(\partial_\theta X(t,\theta)^2\right) = \frac{1}{2}\partial_\theta X(t,\theta)\partial^2_{\theta\theta} X(t,\theta) \tag{3.1}$$

with Dirichlet boundary conditions (1.4).

## 3.1 The $L^2$ estimate in the continuous case

The following result shows the exponential stability in $L^2$ of the continuous gradient flows.

**Proposition 3.1.** *Let* $X_1, X_2$ *be two solutions of* (3.1) *satisfying* (1.4) *and*

$$\partial_\theta X_i(0,\theta) \ge c > 0, \qquad i = 1,2. \tag{3.2}$$

*Then*

$$\int_0^1 |X_1(t,\theta) - X_2(t,\theta)|^2\, d\theta \le \left(\int_0^1 |X_1(0,\theta) - X_2(0,\theta)|^2\, d\theta\right) e^{-4ct}.$$

*Proof.* We first recall that the monotonicity condition (3.2) is preserved in time (apply Proposition 2.3 with $\lambda = 1$). Then, since $X_2 - X_1$ vanishes at the boundary, one has

$$\begin{aligned}
\frac{d}{dt}\int_0^1 |X_1 - X_2|^2\, d\theta &= \int_0^1 (X_1 - X_2)\left(\partial_\theta(\partial_\theta X_1^2) - \partial_\theta(\partial_\theta X_2^2)\right) d\theta \\
&= -\int_0^1 (\partial_\theta X_1 - \partial_\theta X_2)(\partial_\theta X_1^2 - \partial_\theta X_2^2)\, d\theta \\
&= -\int_0^1 (\partial_\theta X_1 - \partial_\theta X_2)^2(\partial_\theta X_1 + \partial_\theta X_2)\, d\theta.
\end{aligned}$$

Using the monotonicity condition $\partial_\theta X_i \geq c$ and the Poincaré inequality on $[0,1]$ (see for instance Lemma 3.5 and let $N \to \infty$), we get

$$\begin{aligned} -\int_0^1 (\partial_\theta X_1 - \partial_\theta X_2)^2 (\partial_\theta X_1 + \partial_\theta X_2)\, d\theta &\leq -2c \int_0^1 (\partial_\theta X_1 - \partial_\theta X_2)^2\, d\theta \\ &\leq -4c \int_0^1 (X_1 - X_2)^2\, d\theta \end{aligned}$$

so that

$$\frac{d}{dt}\left( e^{4ct} \int_0^1 |X_1 - X_2|^2(t,\theta)\, d\theta \right) \leq 0.$$

□

This argument shows that, if at time zero $X_1(0,\theta) = X_2(0,\theta)$ for a.e. $\theta \in (0,1)$, in particular $X_1(t,\theta) = X_2(t,\theta)$ for a.e. $\theta \in (0,1)$, for all $t \geq 0$. Moreover if $X_1(0,\theta) - X_2(0,\theta)$ is small in $L^2$ then it remains small in $L^2$ (continuity with respect to the initial datum), and actually converges to zero exponentially fast. In particular, noticing that $X(t,\theta) = \theta$ is a solution (corresponding to $f(t,x) = 1$), we deduce that all solutions converge exponentially to it: indeed, choosing $X_2(t,\theta) = \theta$ and assuming $c \leq 1$ we have

$$\int_0^1 |X(t,\theta) - \theta|^2\, d\theta \leq \left( \int_0^1 |X(0,\theta) - \theta|^2\, d\theta \right) e^{-4ct}.$$

### 3.2 Convergence of the gradient flows

The functional $F_{N,2}(x^1, \ldots, x^N)$ with $\rho \equiv 1$ is given by

$$F_{N,2}(x^1, \ldots, x^N) = \frac{|x^1|^3}{3} + \sum_{i=1}^{N-1} \frac{1}{12} |x^{i+1} - x^i|^3 + \frac{|1 - x^N|^3}{3}, \tag{3.3}$$

hence the defining equation for the gradient flow for $F_{N,2}$ is

$$\dot{x}^i = -\frac{\partial F_{N,2}}{\partial x^i} = \frac{1}{4}\Big( \big(x^{i+1} - x^i\big)^2 - \big(x^i - x^{i-1}\big)^2 \Big) \qquad \text{for all } i = 1, \ldots, N, \tag{3.4}$$

where by convention $x^0 := -x^1$ and $x^{N+1} := 2 - x^N$.

The former convention comes from the following observation: in order to avoid problems at the boundary, one could symmetrize the configuration of points $x^1, \ldots, x^N$ with respect to 0 to get $N$ points $y^1, \ldots, y^N \in [-1, 0]$ satisfying $y^i := -x^i$. By identifying $-1$ with 1, we then get a family of $2N$ points on the circle where the dynamics is completely equivalent to ours. This means that, by adding $x^0$ and $x^{N+1}$ defined as above, we can see $x^1$ and $x^N$ as interior points. In the next section we will apply the same observation

symmetrizing also the density $\rho$ in the way described above.

In order to prove convergence, we want to find an equation for $X$ evaluated on the grid $(i-1/2)N$.

**Lemma 3.2.** *Let $X(t,\theta)$ be a solution of* (3.1)-(1.4) *starting from an initial datum $X_0 \in C^{4,\alpha}([0,1])$ with $\partial_\theta X_0 \geq c_0 > 0$. Let $X^i$ be the discretized solution defined at the points $(\frac{i-1/2}{N}, t)$, that is*

$$X^i(t) := X\left(\frac{i-1/2}{N}, t\right) \qquad \textit{for all } i = 1, \dots, N. \tag{3.5}$$

*Then*

$$\partial_t X^i - N^3 \frac{\partial F_{N,2}}{\partial x^i}(X^1, \dots, X^N) = R^i.$$

*with*

$$|R^i(t)| \leq \frac{\hat{C}}{N^2} \qquad \textit{for all } t \geq 0, \quad \textit{for all } i = 1, \dots, N,$$

*where $\hat{C}$ depends only on $c_0$ and $\|X_0\|_{C^{4,\alpha}([0,1])}$.*

*Proof.* As we showed in Proposition 2.3 we have $\partial_\theta X(t,\theta) \geq c > 0$ for all $t$, so that the equation (3.1) remains uniformly parabolic and under our assumptions the solution $X(t)$ remains of class $C^4$ for all times, with

$$\|X(t)\|_{C^4} \leq C \qquad \forall\, t \geq 0.$$

By Taylor's expansion centered at $(\frac{i-1/2}{N}, t)$, one has

$$X^{i+1} = X^i + \frac{1}{N}\partial_\theta X^i + \frac{1}{2N^2}\partial_{\theta\theta} X^i + \frac{1}{6N^3}\partial_{\theta\theta\theta} X^i + O\left(\frac{\|X(t)\|_{C^4}}{N^4}\right),$$

$$X^{i-1} = X^i - \frac{1}{N}\partial_\theta X^i + \frac{1}{2N^2}\partial_{\theta\theta} X^i - \frac{1}{6N^3}\partial_{\theta\theta\theta} X^i + O\left(\frac{\|X(t)\|_{C^4}}{N^4}\right).$$

Thus, with the convention $X^0 := -X^1$ and $X^{N+1} := 2 - X^N$,

$$\begin{aligned}
&\partial_t X^i - \frac{N^3}{4}\Big( \big(X^{i+1} - X^i\big)^2 - \big(X^i - X^{i-1}\big)^2 \Big) = \\
&\partial_t X^i - \frac{N^3}{4}\left[\frac{1}{N}\partial_\theta X^i + \frac{1}{2N^2}\partial_{\theta\theta} X^i + \frac{1}{6N^3}\partial_{\theta\theta\theta} X^i + O\left(\frac{\|X(t)\|_{C^4}}{N^4}\right)\right]^2 \\
&\qquad + \frac{N^3}{4}\left[-\frac{1}{N}\partial_\theta X^i + \frac{1}{2N^2}\partial_{\theta\theta} X^i - \frac{1}{6N^3}\partial_{\theta\theta\theta} X^i + O\left(\frac{\|X(t)\|_{C^4}}{N^4}\right)\right]^2,
\end{aligned}$$

hence

$$\begin{aligned}
&\partial_t X^i - \frac{N^3}{4}\Big( \big(X^{i+1} - X^i\big)^2 - \big(X^i - X^{i-1}\big)^2 \Big) \\
&\qquad\qquad = \partial_t X^i - \frac{1}{2}\partial_\theta X^i \partial_{\theta\theta} X^i + R^i = R^i,
\end{aligned}$$

with

$$|R^i(t)| \le C\,\frac{\|X(t)\|_{C^4}}{N^2} \le \frac{\hat{C}}{N^2}.$$

with $\hat{C} := C \sup_{t\ge 0} \|X(t)\|_{C^4}$. □

In order to compare $X$ with $x^i$ we need to rescale times. More precisely, let us denote with $\bar{x}^i(t) := x^i(N^3 t)$. Then

$$\dot{\bar{x}}^i = \frac{N^3}{4}\Big( \big(\bar{x}^{i+1} - \bar{x}^i\big)^2 - \big(\bar{x}^i - \bar{x}^{i-1}\big)^2 \Big). \tag{3.6}$$

For simplicity of notation we set

$$W_X^i := N\left(X^{i+1} - X^i\right), \quad Y_X^i := \left(W_X^i\right)^2,$$

$$W_{\bar{x}}^i := N\left(\bar{x}^{i+1} - \bar{x}^i\right), \quad Y_{\bar{x}}^i := \left(W_{\bar{x}}^i\right)^2,$$

(recall the convention $X^0 := -X^1$ and $X^{N+1} := 2 - X^N$). The equation for $X^i$ can be written as

$$\partial_t X^i = \frac{N}{4}\big(Y_X^i - Y_X^{i-1}\big) + R^i,$$

while the equation for $W_{\bar{x}}^i$ (which follows easily from (3.6)) is given by

$$\partial_t W_{\bar{x}}^i = \frac{N^2}{4}\Big( (W_{\bar{x}}^{i+1})^2 - 2(W_{\bar{x}}^i)^2 + (W_{\bar{x}}^{i-1})^2 \Big). \tag{3.7}$$

We now prove a discrete monotonicity result:

**Lemma 3.3.** *Assume that $C \ge \partial_\theta X(0,\theta) \ge c$ and $C \ge W_{\bar{x}}^i(0) \ge c$ for all $i$ and $\theta \in (0,1)$. Then $C \ge W_{\bar{x}}^i(t), W_X^i(t) \ge c$ for all $i = 1,\dots,N$, and all $t \ge 0$.*

*Proof.* The inequality for $W_X^i$ follows from the fact that the bound $C \ge \partial_\theta X(0) \ge c$ is propagated in time (see Proposition 2.3 and recall that here $\lambda = 1$).

To prove that $W_{\bar{x}}^i(t) \ge c > 0$, it suffices to prove that, for any $\varepsilon > 0$ small,

$$W_{\bar{x}}^i(t) \ge c - \varepsilon(2 - e^{-t}) := f(t) \quad \forall i, \forall\, t \ge 0 \tag{3.8}$$

(the bound $W_{\bar{x}}^i(t) \le C$ being obtained in a is completely analogous manner). Notice that, with this choice, $f(0) < \min_i W_{\bar{x}}^i(0)$. Suppose by contradiction that

$$\min_i W_{\bar{x}}^i(t) \ngeq f(t) \text{ in } \mathbb{R}^+$$

Then there exist a first time $t_0$ such that $W_X^{i_0}(t_0) = f(t_0) \ge 0$ for some $i_0$, i.e., $f(t) < W_{\bar{x}}^i(t)$ for all $t \in [0, t_0)$ and all $i = 1,\dots,n$, and $f(t)$ touches

$W^i_{\bar{x}}(t)$ from below at $(i_0, t_0)$. From the equation (3.7) and the condition (3.8) we get a contradiction: indeed, since $t_0$ is the first contact time we get

$$\dot{W}^{i_0}_{\bar{x}}(t_0) \leq \dot{f}(t_0) = -\varepsilon e^{-t_0} < 0,$$

while since $\big(W^{i_0+1}_{\bar{x}}(t_0)\big)^2, \big(W^{i_0-1}_{\bar{x}}(t_0)\big)^2 \geq f(t_0)^2 = \big(W^{i_0}_{\bar{x}}(t_0)\big)^2$ (here we used that $f(t) \geq 0$ provided $\varepsilon$ is sufficiently small to deduce that $W^i \geq f$ implies $(W^i)^2 \geq f^2$)

$$\dot{W}^{i_0}_{\bar{x}}(t_0) = \frac{N^2}{4}\Big(\big(W^{i_0+1}_{\bar{x}}(t_0)\big)^2 - 2\big(W^{i_0}_{\bar{x}}(t_0)\big)^2 + \big(W^{i_0-1}_{\bar{x}}(t_0)\big)^2\Big) \geq 0.$$

This proves that $\min_i W^i_{\bar{x}}(t) \geq f(t)$ for all $t \geq 0$, and letting $\varepsilon \to 0$ we have the desired result. □

We can now prove our convergence theorem.

**Theorem 3.4.** *Let $\bar{x}^i$ be a solution of the ODE* (3.6)*, and let $X^i$ be as in* (3.5)*. Assume that $X_0 \in C^{4,\alpha}([0,1])$ and that there exist positive constants $c_0, C_0$ such that*

$$\frac{c_0}{N} \leq \bar{x}^i(0) - \bar{x}^{i-1}(0) \leq \frac{C_0}{N}, \quad c_0 \leq \partial_\theta X_0 \leq C_0.$$

*Then there exist two constants $\bar{c}, \bar{C} > 0$, depending only on $c_0$, such that*

$$\frac{1}{N}\sum_{i=1}^N \big(\bar{x}^i(t) - X^i(t)\big)^2 \leq e^{-\bar{c}t}\frac{1}{N}\sum_{i=1}^N \big(\bar{x}^i(0) - X^i(0)\big)^2 + \bar{C}\left(\frac{\hat{C}}{N^2}\right)^2$$

*for all $t \geq 0$, where $\hat{C}$ is as in Lemma 3.2.*

*Proof.* We begin by observing that, because of Lemma 3.3,

$$\frac{c_0}{N} \leq \bar{x}^i(t) - \bar{x}^{i-1}(t) \leq \frac{C_0}{N}, \quad \text{and} \quad \frac{c_0}{N} \leq X^i(t) - X^{i-1}(t) \leq \frac{C_0}{N},$$

for all $t \geq 0$. We now estimate the $L^2$ distance between $X^i$ and $\bar{x}^i$: recalling

Lemma 3.2 we have

$$
\begin{aligned}
&\frac{d}{dt}\frac{1}{N}\sum_{i=1}^{N}|X^i-\bar{x}^i|^2\\
&=\frac{1}{8N}\sum_{i=1}^{N}N\left(X^i-\bar{x}^i\right)\left[Y_X^i-Y_X^{i-1}-(Y_{\bar{x}}^i-Y_{\bar{x}}^{i-1})\right]+\frac{2}{N}\sum_{i=1}^{N}(X^i-\bar{x}^i)R^i\\
&=\frac{1}{8N}\sum_{i=1}^{N}N\left(X^i-\bar{x}^i\right)\left[Y_X^i-Y_{\bar{x}}^i\right]-\frac{1}{8N}\sum_{i=1}^{N}N\left(X^i-\bar{x}^i\right)\left[Y_X^{i-1}-Y_{\bar{x}}^{i-1}\right]\\
&\qquad+\frac{2}{N}\sum_{i=1}^{N}(X^i-\bar{x}^i)R^i\\
&=\frac{1}{8N}\sum_{i=0}^{N-1}N\left(X^i-\bar{x}^i\right)\left[Y_X^i-Y_{\bar{x}}^i\right]-\frac{1}{8N}\sum_{i=0}^{N-1}N\left(X^{i+1}-\bar{x}^{i+1}\right)\left[Y_X^i-Y_{\bar{x}}^i\right]\\
&\qquad+\frac{2}{N}\sum_{i=1}^{N}(X^i-\bar{x}^i)R^i\\
&=-\frac{1}{8N}\sum_{i=0}^{N-1}N\left((X^{i+1}-X^i)-(\bar{x}^{i+1}-\bar{x}^i)\right)\left[Y_X^i-Y_{\bar{x}}^i\right]\\
&\qquad+\frac{2}{N}\sum_{i=1}^{N}(X^i-\bar{x}^i)R^i.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\frac{d}{dt}\frac{1}{N}\sum_{i=1}^{N}|X^i-\bar{x}^i|^2\\
&=-\frac{1}{8N}\sum_{i=0}^{N-1}(W_X^i-W_{\bar{x}}^i)\left[(W_X^i)^2-(W_{\bar{x}}^i)^2\right]+\frac{2}{N}\sum_{i=1}^{N}(X^i-\bar{x}^i)R^i\\
&\le-\frac{c}{8N}\sum_{i=0}^{N-1}(W_X^i-W_{\bar{x}}^i)^2+\frac{2}{N}\sum_{i=1}^{N}(X^i-\bar{x}^i)R^i,
\end{aligned}
$$

since $W_{\bar{x}}^i, W_X^i \ge c > 0$. We then apply the following discrete Poincaré inequality (we postpone the proof to the end of the Theorem):

**Lemma 3.5.** *Let* $(u^0,\dots,u^N)\subset\mathbb{R}^N$ *with* $u^0=0$. *Set*

$$\|u\|_2 := \Big(\frac{1}{N}\sum_{i=0}^{N}(u^i)^2\Big)^{\frac{1}{2}};$$

$$\|u'\|_2 := \Big(\frac{1}{N}\sum_{i=0}^{N-1}N^2(u^{i+1}-u^i)^2\Big)^{\frac{1}{2}}.$$

*Then* $\|u\|_2^2 \leq \frac{1}{2}\|u'\|_2^2$.

$$\frac{d}{dt}\frac{1}{N}\sum_{i=1}^{N}|X^i-\bar{x}^i|^2 \leq -\bar{c}\frac{1}{N}\sum_{i=1}^{N}|X^i-\bar{x}^i|^2+\frac{2}{N}\sum_{i=1}^{N}(X^i-\bar{x}^i)R^i.$$

Using that

$$(X^i-\bar{x}^i)R^i \leq \epsilon(X^i-\bar{x}^i)^2+\frac{1}{\epsilon}(R^i)^2,$$

choosing $\epsilon=\bar{c}/4$ we get

$$\frac{d}{dt}\frac{1}{N}\sum_{i=1}^{N}|X^i-\bar{x}^i|^2 \leq -\bar{c}\frac{1}{2N}\sum_{i=1}^{N}|X^i-\bar{x}^i|^2+\frac{2}{N}\sum_{i=1}^{N}(R^i)^2.$$

Recalling that

$$|R^i(t)| \leq \frac{\hat{C}}{N^2},$$

(see Lemma 3.2), we conclude that

$$\frac{d}{dt}\frac{1}{N}\sum_{i=1}^{N}|X^i-\bar{x}^i|^2 \leq -\bar{c}\frac{1}{2N}\sum_{i=1}^{N}|X^i-\bar{x}^i|^2+\frac{2\,\hat{C}^2}{N^4}.$$

By Gronwall Lemma, this implies

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N}|X^i(t)-\bar{x}^i(t)|^2 \leq \frac{1}{N}\sum_{i=1}^{N}|X^i(0)-\bar{x}^i(0)|^2\,e^{-\bar{c}t/2}\\ +\int_0^t e^{-\bar{c}(t-s)/2}\frac{2\,\hat{C}^2}{N^4}\,ds.\end{aligned}$$

In particular, using that the third derivatives of $X(t,\cdot)$ are bounded, we get

$$\frac{1}{N}\sum_{i=1}^{N}|X^i(t)-\bar{x}^i(t)|^2 \leq \frac{1}{N}\sum_{i=1}^{N}|X^i(0)-\bar{x}^i(0)|^2\,e^{-\bar{c}t/2}+\frac{2\hat{C}^2}{\bar{c}N^4},$$

as desired. □

*Proof of Lemma 3.5.* We observe that, since $u^0=0$,

$$u^i=\frac{1}{N}\sum_{k=0}^{i-1}N(u^{k+1}-u^k) \quad \text{for} \quad i=0,\dots,N,$$

hence

$$\begin{aligned}\|u\|_2^2 &= \frac{1}{N}\sum_{i=0}^{N}(u^i)^2 = \frac{1}{N}\sum_{i=0}^{N}\Big(\frac{1}{N}\sum_{k=0}^{i-1}N(u^{k+1}-u^k)\Big)^2\\ &\le \frac{1}{N}\sum_{i=0}^{N}(i-1)\frac{1}{N^2}\sum_{k=0}^{i-1}N^2(u^{k+1}-u^k)^2\\ &= \frac{(N-1)}{2N}\frac{1}{N}\sum_{k=0}^{N-1}N^2(u^{k+1}-u^k)^2 \le \frac{1}{2}\|u'\|_2^2.\end{aligned}$$

□

### 3.3 The Eulerian picture

Let us define $\mu_t^N := \frac{1}{N}\sum_i \delta_{x^i(t)}$. We want to estimate the distance in $MK_1$ between $\mu_t^N$ and the Lebesgue measure on $[0,1]$.

**Theorem 3.6.** *Let $\bar{x}^i$ be a solution of the ODE* (3.6)*, and let $X^i$ be as in* (3.5)*. Assume that $X_0 \in C^{4,\alpha}([0,1])$ and that there exist positive constants $c_0, C_0$ such that*

$$\frac{c_0}{N} \le \bar{x}^i(0) - \bar{x}^{i-1}(0) \le \frac{C_0}{N}, \quad c_0 \le \partial_\theta X_0 \le C_0.$$

*Then there exist two constants $\bar{\bar{c}}, \bar{\bar{C}} > 0$, depending on $c_0, C_0, \|X_0\|_{C^{4,\alpha}([0,1])}$ only, such that*

$$MK_1(\mu_t^N, d\theta) \le e^{-\bar{\bar{c}}t/N^3} + \frac{\bar{\bar{C}}}{N^2} + \frac{1}{4N} \qquad \forall\, t \ge 0.$$

*In particular*

$$MK_1(\mu_t^N, d\theta) \le \frac{1}{4N} + \frac{\bar{\bar{C}}+1}{N^2} \qquad \forall\, t \ge \frac{2N^3\log N}{\bar{\bar{c}}}.$$

*Proof.* Take $X^0(\theta) = \theta$, so that $X(t,\theta) = \theta$ for all $t$, and apply Theorem 3.4: we know that

$$\frac{1}{N}\sum_{i=1}^{N}|X^i(t) - \bar{x}^i(t)|^2 \le \frac{1}{N}\sum_{i=1}^{N}|X^i(0) - \bar{x}^i(0)|^2\, e^{-\bar{c}t/2} + \frac{\bar{C}\,\hat{C}^2}{N^4},$$

hence, since $0 \le \bar{x}^i(0) \le 1$, $0 \le X^i(0) \le 1$,

$$\frac{1}{N}\sum_{i=1}^{N}\left|\bar{x}^i(t) - \frac{i-1/2}{N}\right|^2 \le e^{-\bar{c}t} + \frac{\bar{C}\,\hat{C}^2}{N^4}.$$

Recalling that

$$\bar{x}^i(t) := x^i(N^3 t/8),$$

we get

$$\frac{1}{N}\sum_{i=1}^{N}\left|x^i(t) - \frac{i-1/2}{N}\right|^2 \le e^{-\bar{c}t/N^3} + \frac{\bar{C}\,\hat{C}^2}{N^4}.$$

To control $MK_1(\mu_t^N, d\theta)$, we consider a 1-Lipschitz function $\varphi$ and we estimate

$$\begin{aligned}
\int_0^1 \varphi\, d\mu_t^N - \int_0^1 \varphi\, d\theta &= \frac{1}{N}\sum_{i=1}^{N}\varphi(x^i(t)) - \sum_{i=1}^{N}\int_{(i-1)/N}^{i/N}\varphi\, d\theta \\
&= \frac{1}{N}\sum_{i=1}^{N}\left[\varphi(x^i(t)) - \varphi\left(\frac{i-1/2}{N}\right)\right] \\
&+ \sum_{i=1}^{N}\int_{(i-1)/N}^{i/N}\left[\varphi\left(\frac{i-1/2}{N}\right) - \varphi(\theta)\right] d\theta \\
&\le \frac{1}{N}\sum_{i=1}^{N}\left|x^i(t) - \frac{i-1/2}{N}\right| \\
&+ \sum_{i=1}^{N}\int_{(i-1)/N}^{i/N}\left|\frac{i-1/2}{N} - \theta\right| d\theta \\
&\le \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left|x^i(t) - \frac{i-1/2}{N}\right|^2} + \frac{1}{4N} \\
&\le e^{-\bar{c}t/(2N^3)} + \frac{\bar{C}^{1/2}\,\hat{C}}{N^2} + \frac{1}{4N},
\end{aligned}$$

hence, taking the supremum over all 1-Lipschitz functions we get

$$MK_1(\mu_t^N, d\theta) \le e^{-\bar{c}t/(2N^3)} + \frac{\bar{C}^{1/2}\,\hat{C}}{N^2} + \frac{1}{4N},$$

which proves the result with $\bar{\bar{c}} := \bar{c}/2$ and $\bar{\bar{C}} := \bar{C}^{1/2}\,\hat{C}$. □

# 4 The case $\rho \not\equiv 1$

We consider the case $r = 2$ whit $\rho$ a periodic function of class $C^{3,\alpha}$, where

$$\|\rho\|_{C^{3,\alpha}} := \|\rho\|_{C^3} + \sup_{x\neq y}\frac{|\rho'''(x) - \rho'''(y)|}{|x-y|^\alpha}.$$

We recall that

$$\mathcal{F}[X] = \frac{1}{12}\int_0^1 \rho(X(\theta)(\partial_\theta X(\theta))^3 d\theta,$$

and that the gradient flow PDE for $\mathcal{F}$ for the $L^2$-metric is given in (1.3).

## 4.1 Convergence of the gradient flows

We recall the formula for the gradient of $F_{N,2}$ given in (2.1).

**Lemma 4.1.** *Let $X(t,\theta)$ be a solution of (1.3)-(1.4) starting from an initial datum $X_0 \in C^{4,\alpha}([0,1])$ for some $\alpha > 0$ with $\partial_\theta X_0 \geq c_0 > 0$, and assume that $0 < \lambda \leq \rho \leq 1/\lambda$. Let $X^i$ be the discrete values of the exact solution at the points $\left(\frac{i-1/2}{N}, t\right)$ as in (3.5). Then*

$$\partial_t X^i - N^3 \frac{\partial F_{N,2}}{\partial x^i}(X^1, \dots, X^N) = R^i$$

*with*

$$|R^i(t)| \leq \frac{\hat{C}}{N^2} \qquad \forall\, t \geq 0,\, \forall\, i = 1, \dots, N, \tag{4.1}$$

*where $\hat{C}$ depends only on $c_0$, $\lambda$, $\|\rho\|_{C^{3,\alpha}([0,1])}$, and $\|X_0\|_{C^{4,\alpha}([0,1])}$.*

*Proof.* As we showed in Proposition 2.3, under our assumptions $\partial_\theta X(t) \geq c > 0$ for all $t$ and the solution $X(t)$ remains of class $C^4$ for all times, with

$$\|X(t)\|_{C^4} \leq C \qquad \forall\, t \geq 0.$$

A Taylor expansion yields

$$X^{i+1} = X^i + \frac{1}{N}\partial_\theta X^i + \frac{1}{2N^2}\partial_{\theta\theta} X^i + \frac{1}{6N^3}\partial_{\theta\theta\theta} X^i + O\left(\frac{\|X(t)\|_{C^4}}{N^4}\right);$$

$$X^{i-1} = X^i - \frac{1}{N}\partial_\theta X^i + \frac{1}{2N^2}\partial_{\theta\theta} X^i - \frac{1}{6N^3}\partial_{\theta\theta\theta} X^i + O\left(\frac{\|X(t)\|_{C^4}}{N^4}\right);$$

$$\rho(y) = \rho(X^i) + \rho'(X^i)\,(y - X^i) + \frac{\rho''(X^i)}{2}\,(y - X^i)^2 + O(|y - X^i|^3),$$

where as before we adopt the convention $X^0 := -X^1$ and $X^{N+1} := 2 - X^N$. In addition, we set

$$\rho(y) := \rho(-y) \quad \text{for } y \in [X^0, 0], \qquad \rho(y) := \rho(2 - y) \quad \text{for } y \in [1, X^{N+1}].$$

Then

$$\begin{aligned}
-\frac{\partial F_{N,2}}{\partial x^i}(X^1, \dots, X^N) &= 2\int_{\frac{X^i+X^{i-1}}{2}}^{\frac{X^i+X^{i+1}}{2}} (y - X^i)\Big[\rho(X^i) + \rho'(X^i)\,(y - X^i) \\
&\qquad + \frac{\rho''(X^i)}{2}\,(y - X^i)^2 + O(|y - X^i|^3)\Big]\,dy \\
&= 2\int_{\frac{X^i+X^{i-1}}{2}}^{\frac{X^i+X^{i+1}}{2}} (y - X^i)\rho(X^i)dy + 2\int_{\frac{X^i+X^{i-1}}{2}}^{\frac{X^i+X^{i+1}}{2}} (y - X^i)\Big[\rho'(X^i)\,(y - X^i) \\
&\qquad + \frac{\rho''(X^i)}{2}\,(y - X^i)^2 + O(|y - X^i|^3)\Big]\,dy
\end{aligned}$$

so that

$$
\begin{aligned}
-\frac{\partial F_{N,2}}{\partial x^i}(X^1,\dots,X^N) = \frac{\rho(X^i)}{4}&\left[(X^{i+1}-X^i)^2-(X^i-X^{i-1})^2\right]\\
+2\rho'(X^i)\frac{1}{24}&\left[(X^{i+1}-X^i)^3-(X^i-X^{i-1})^3\right]\\
+\rho''(X^i)\frac{1}{64}&\left[(X^{i+1}-X^i)^4-(X^i-X^{i-1})^4\right]+O(1/N^5).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
-\frac{\partial F_{N,2}}{\partial x^i}(X^1,\dots,X^N) = \frac{\rho(X^i)}{4}&\left[(X^{i+1}-X^i)^2-(X^i-X^{i-1})^2\right]\\
&+\rho'(X^i)\frac{1}{12}\left[(X^{i+1}-X^i)^3-(X^i-X^{i-1})^3\right]\\
&+\rho''(X^i)\frac{1}{64}\left[(X^{i+1}-X^i)^4-(X^i-X^{i-1})^4\right]\\
&+O(1/N^5).
\end{aligned}
$$

We now use the Taylor expansion for $X$ to see that

$$
(X^{i+1}-X^i)^2-(X^i-X^{i-1})^2 = \frac{2\partial_\theta X^i\,\partial_{\theta\theta}X^i}{N^3}+O(1/N^5),
$$

$$
(X^{i+1}-X^i)^3-(X^i-X^{i-1})^3 = \frac{2(\partial_\theta X^i)^3}{N^3}+O(1/N^5),
$$

$$
(X^{i+1}-X^i)^4-(X^i-X^{i-1})^4 = O(1/N^5),
$$

thus

$$
\begin{aligned}
&-\frac{\partial F_{N,2}}{\partial x^i}(X^1,\dots,X^N)\\
&=\frac{1}{2N^3}\rho(X^i)\partial_\theta X^i\,\partial_{\theta\theta}X^i+\frac{1}{6N^3}\rho'(X^i)(\partial_\theta X^i)^3+O(1/N^5)=O(1/N^5).
\end{aligned}
$$

□

### 4.1.1 The $L^\infty$ stability estimate

Let $X$ be a smooth solution of the continuous gradient flow

$$
\partial_t X = \frac{1}{2}\rho(X)\partial_\theta X\,\partial_{\theta\theta}X+\frac{1}{6}\rho'(X)(\partial_\theta X)^3 \tag{4.2}
$$

and define

$$
X^i(t) := X\left(t,\frac{i-1/2}{N}\right). \tag{4.3}
$$

Recall that, according to Lemma 4.1, $X^i$ solves the following ODE:

$$\dot{X}^i = 2N^3 \int_{\frac{X^i+X^{i-1}}{2}}^{\frac{X^i+X^{i+1}}{2}} (z - X^i)\rho(z)dz + R^i \tag{4.4}$$

where $R^i$ satisfies (4.1) and we are using the conventions $X^0 := -X^1$, $X^{N+1} := 2 - X^N$, and

$$\rho(y) := \rho(-y) \quad \text{for } y \in [X^0, 0], \qquad \rho(y) := \rho(2-y) \quad \text{for } y \in [1, X^{N+1}].$$

We also consider the rescaled discrete solution $\big(\bar{x}^i(t)\big)_{1\le i\le N}$

$$\dot{\bar{x}}^i = 2N^3 \int_{\frac{\bar{x}^i+\bar{x}^{i-1}}{2}}^{\frac{\bar{x}^i+\bar{x}^{i+1}}{2}} (z - \bar{x}^i)\rho(z)dz. \tag{4.5}$$

In the following lemma we prove that, over a time scale $\tau > 0$, $X^i$ gets at most $\eta\tau$ apart from the exact solution of the ODE, where $\eta$ depends both on $\frac{\hat{C}}{N^2}$ and on the initial distance between the two solutions.

**Lemma 4.2.** *Let $\bar{x}^i$ be a solution of the ODE* (4.5)*, and let $X^i$ be as in* (4.3)*. Set*

$$A_t := \max_{i=1,\dots,N} |\bar{x}^i(t) - X^i(t)|.$$

*There exists a time $T > 0$, depending only on $\sup_{t\ge 0} \|X(t)\|_{C^2}$ and $\|\rho'\|_\infty + \|\rho''\|_\infty$, such that, for any $t^* \ge 0$,*

$$A_{t^*+\tau} \le A_{t^*} + \eta\,\tau \qquad \forall\,\tau \in [0,T]$$

*with $\eta := \frac{3\hat{C}}{N^2} + \frac{A_{t^*}}{T}$, where $\hat{C}$ is as in* (4.1)*.*

*Proof.* Let us define

$$A_t^{\pm} := \max_{i=1,\dots,N} \big(\pm[\bar{x}^i(t) - X^i(t)]\big)_+.$$

Notice that $A_t = \max\{A_t^+, A_t^-\}$, and to prove the result it is enough to prove the following stronger statement:

$$A^+_{t^*+\tau} \le A^+_{t^*} + \eta\,\tau \qquad \forall\,\tau \in [0,T], \tag{4.6}$$

$$A^-_{t^*+\tau} \le A^-_{t^*} - \eta\,\tau \qquad \forall\,\tau \in [0,T].$$

Since the arguments for $A^+$ and $A^-$ are completely analogous , we prove only (4.6). Also, without loss of generality we can assume $t^* = 0$. By definition of $A_0^+$, at time 0 the solutions are ordered

$$X^i(0) \le \bar{x}^i(0) + A_0^+.$$

Let us define $Y^i(t) := X^i(t) - A_0^+ - \eta t$ and assume that there exist $t_0 \in \mathbb{R}^+$ defined as $t_0 := \inf_{t\in\mathbb{R}^+} \{Y^i(t) = \bar{x}^i(t)\}$. Then,

$$\dot{\bar{x}}^i(t_0) \leq \dot{Y}^i(t_0) \leq \dot{X}^i(t_0) - \eta. \tag{4.7}$$

Observing that $\bar{x}^{i+1}(t_0) \geq Y^{i+1}(t_0)$ and $\bar{x}^{i-1}(t_0) \geq Y^{i-1}(t_0)$,

$$\begin{aligned}\dot{\bar{x}}^i(t_0) =& 2N^3 \int_{\frac{\bar{x}^i+\bar{x}^{i-1}}{2}}^{\frac{\bar{x}^i+\bar{x}^{i+1}}{2}} (z - \bar{x}^i)\rho(z)dz \\ \geq& 2N^3 \int_{\frac{Y^i+Y^{i-1}}{2}}^{\frac{Y^i+Y^{i+1}}{2}} (z - Y^i)\rho(z)dz.\end{aligned}$$

Performing a change of variable $\omega = z + A_0^+ + \eta t_0$, we have

$$\begin{aligned}\dot{\bar{x}}^i(t_0) &\geq 2N^3 \int_{\frac{Y^i+Y^{i-1}}{2}}^{\frac{Y^i+Y^{i+1}}{2}} (z - Y^i)\rho(z)dz \\ &= 2N^3 \int_{\frac{X^i+X^{i-1}}{2}}^{\frac{X^i+X^{i+1}}{2}} (\omega - X^i)\rho(\omega - A_0^+ - \eta t_0)d\omega.\end{aligned}$$

By the fundamental theorem of calculus

$$\begin{aligned}\rho(\omega - A_0^+ - \eta t_0) &= \rho(\omega) - (A_0^+ + \eta t_0)\left(\int_0^1 \rho'(\omega + s(A_0^+ + \eta t_0))ds\right) \\ &:= \rho(\omega) - a(\omega),\end{aligned}$$

so

$$\begin{aligned}\dot{\bar{x}}^i(t_0) \geq& 2N^3 \int_{\frac{X^i+X^{i-1}}{2}}^{\frac{X^i+X^{i+1}}{2}} (\omega - X^i)\rho(\omega)d\omega \\ &- 2N^3(A_0^+ + \eta t_0) \int_{\frac{X^i+X^{i-1}}{2}}^{\frac{X^i+X^{i+1}}{2}} (\omega - X^i)a(\omega)d\omega.\end{aligned}$$

If we recall that $X^i$ solves the ODE (4.4) we have

$$\begin{aligned}\dot{\bar{x}}^i(t_0) &\geq \dot{X}^i - R^i - 2N^3(A_0^+ + \eta t_0) \int_{\frac{X^i+X^{i-1}}{2}}^{\frac{X^i+X^{i+1}}{2}} (\omega - X^i)a(\omega)d\omega \\ &= \dot{X}^i - R^i - 2N^3(A_0^+ + \eta t_0) \int_{\frac{X^i+X^{i-1}}{2}}^{\frac{X^i+X^{i+1}}{2}} (\omega - X^i)\left(a(\omega) - a(X^i)\right)d\omega \\ &+ 2N^3(A_0^+ + \eta t_0) \int_{\frac{X^i+X^{i-1}}{2}}^{\frac{X^i+X^{i+1}}{2}} (\omega - X^i)a(X^i)d\omega \\ &:= \dot{X}^i - R^i - T_1 + T_2.\end{aligned}$$

For $T_1$ we observe that, since $|X^{i+1} - X^i| \le C/N$ for all $i$,

$$|T_1| \le CN^3(A_0^+ + \eta t_0)\|a'\|_\infty \int_{\frac{X^i+X^{i-1}}{2}}^{\frac{X^i+X^{i+1}}{2}} |\omega - X^i|^2 d\omega \le C(A_0^+ + \eta t_0)\|\rho''\|_\infty.$$

For $T_2$ we use the Taylor expansion for $X$:

$$X^{i+1} = X^i + \frac{\partial_\theta X^i}{N} + O\left(\frac{1}{N^2}\right);$$

$$X^{i-1} = X^i - \frac{\partial_\theta X^i}{N} + O\left(\frac{1}{N^2}\right).$$

Thus,

$$\begin{aligned} T_2 &\le CN^3(A_0^+ + \eta t_0)\|\rho'\|_\infty \int_{\frac{X^i+X^{i-1}}{2}}^{\frac{X^i+X^{i+1}}{2}} (\omega - X^i) d\omega \\ &= CN^3(A_0^+ + \eta t_0)\|\rho'\|_\infty \Bigg[ -\frac{1}{2}\left(-\frac{\partial_\theta X^i}{N} + O\left(\frac{1}{N^2}\right)\right)^2 \\ &\qquad + \frac{1}{2}\left(\frac{\partial_\theta X^i}{N} + O\left(\frac{1}{N^2}\right)\right)^2 \Bigg] \\ &\le C(A_0^+ + \eta t_0)\|\rho'\|_\infty. \end{aligned}$$

Then

$$\dot{\bar{x}}^i(t_0) \ge \dot{X}^i - |R^i| - C(A_0^+ + \eta t_0)\left(\|\rho''\|_\infty + \|\rho'\|_\infty\right),$$

that combined with (4.7) and (4.1) gives

$$\begin{aligned} \eta &\le C(A_0^+ + \eta t_0)\left(\|\rho''\|_\infty + \|\rho'\|_\infty\right) + |R^i| \\ &\le C(A_0^+ + \eta t_0)\left(\|\rho''\|_\infty + \|\rho'\|_\infty\right) + \frac{\hat{C}}{N^2}. \end{aligned}$$

We now show that there exists a time $T > 0$, depending on $\sup_{t\ge 0}\|X(t)\|_{C^2}$ and $\|\rho'\|_\infty + \|\rho''\|_\infty$ only, such that $t_0 > T$. This will prove that (4.6) holds on $[0, T]$.

Assume by contradiction that $t_0 \le T$. Then the above estimate gives

$$\eta \le C\left(\frac{A_0^+}{T} + \eta\right) T\left(\|\rho''\|_\infty + \|\rho'\|_\infty\right) + \frac{\hat{C}}{N^2}.$$

Choosing $T$ sufficiently small so that

$$CT\left(\|\rho''\|_\infty + \|\rho'\|_\infty\right) \le \frac{1}{2}$$

we get

$$\eta \leq \frac{1}{2}\left(\frac{A_0^+}{T} + \eta\right) + \frac{\hat{C}}{N^2},$$

or equivalently

$$\eta \leq \frac{A_0^+}{T} + 2\frac{\hat{C}}{N^2}.$$

This contradicts the definition of $\eta$ and proves the result. □

### 4.1.2 The $L^2$ stability estimate

**Lemma 4.3.** *Let $\bar{x}^i$ be a solution of the ODE* (4.5)*, and let $X^i$ be as in* (4.3)*. Let $0 \leq T_1 \leq T_2 \leq \infty$, and assume that there exist two positive constants $c_0, C_0$ such that*

$$\frac{c_0}{N} \leq \bar{x}^i(t) - \bar{x}^{i-1}(t) \leq \frac{C_0}{N}, \quad \frac{c_0}{N} \leq X^i(t) - X^{i-1}(t) \leq \frac{C_0}{N}, \qquad \forall\, t \in [T_1, T_2].$$

*Then, there exists $\varepsilon_0 = \varepsilon_0(c_0, C_0) > 0$ such that, if $\|\rho'\|_{L^\infty} + \|\rho''\|_{L^\infty} \leq \varepsilon_0$ then one can find two constants $\bar{c}, \bar{C} > 0$, depending only on $c_0$, such that*

$$\frac{1}{N}\sum_{i=1}^{N}\left(\bar{x}^i(t) - X^i(t)\right)^2 \leq e^{-\bar{c}(t-T_1)}\frac{1}{N}\sum_{i=1}^{N}\left(\bar{x}^i(T_1) - X^i(T_1)\right)^2 + \bar{C}\left(\frac{\hat{C}}{N^2}\right)^2$$

*for all $t \in [T_1, T_2]$.*

*Proof.* We compute

$$
\begin{aligned}
&\frac{d}{dt}\frac{1}{N}\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)^2 = \\
&4N^2\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)\left[\int_{\frac{\bar{x}^i+\bar{x}^{i-1}}{2}}^{\frac{\bar{x}^i+\bar{x}^{i+1}}{2}}(z-\bar{x}^i)\rho(z)dz - \int_{\frac{X^i+X^{i-1}}{2}}^{\frac{X^i+X^{i+1}}{2}}(z-X^i)\rho(z)dz\right] \\
&+\frac{2}{N}\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)R^i \\
&= 4N^2\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)\left[\int_{\frac{\bar{x}^i+\bar{x}^{i-1}}{2}}^{\bar{x}^i}(z-\bar{x}^i)\rho(z)dz\right. \\
&+\int_{\bar{x}^i}^{\frac{\bar{x}^i+\bar{x}^{i+1}}{2}}(z-\bar{x}^i)\rho(z)dz - \int_{\frac{X^i+X^{i-1}}{2}}^{X^i}(z-X^i)\rho(z)dz \\
&\left.-\int_{X^i}^{\frac{X^i+X^{i+1}}{2}}(z-X^i)\rho(z)dz\right] + \frac{2}{N}\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)R^i \\
&:= 4N^2\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)\left[A_{\bar{x}^i} + B_{\bar{x}^i} - A_{X^i} - B_{X^i}\right] + \frac{2}{N}\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)R^i.
\end{aligned}
$$

For $A_{\bar{x}^i}$ and $B_{\bar{x}^i}$ we have

$$
\begin{aligned}
A_{\bar{x}^i} &= \int_{\frac{\bar{x}^i+\bar{x}^{i-1}}{2}}^{\bar{x}^i}(z-\bar{x}^i)\rho(\bar{x}^{i-1})dz + \int_{\frac{\bar{x}^i+\bar{x}^{i-1}}{2}}^{\bar{x}^i}(z-\bar{x}^i)\big(\rho(z)-\rho(\bar{x}^{i-1})\big)dz \\
&= -\frac{\rho(\bar{x}^{i-1})}{8}(\bar{x}^i-\bar{x}^{i-1})^2 + \int_{\frac{\bar{x}^i+\bar{x}^{i-1}}{2}}^{\bar{x}^i}(z-\bar{x}^i)\big(\rho(z)-\rho(\bar{x}^{i-1})\big)dz \\
&:= D_{\bar{x}}^{i-1} + E_{\bar{x}}^{i,2}.
\end{aligned}
$$

$$
\begin{aligned}
B_{\bar{x}^i} &= \int_{\bar{x}^i}^{\frac{\bar{x}^i+\bar{x}^{i+1}}{2}}(z-\bar{x}^i)\rho(\bar{x}^i)dz + \int_{\bar{x}^i}^{\frac{\bar{x}^i+\bar{x}^{i+1}}{2}}(z-\bar{x}^i)\big(\rho(z)-\rho(\bar{x}^i)\big)dz \\
&= \frac{\rho(\bar{x}^i)}{8}(\bar{x}^{i+1}-\bar{x}^i)^2 + \int_{\bar{x}^i}^{\frac{\bar{x}^i+\bar{x}^{i+1}}{2}}(z-\bar{x}^i)\big(\rho(z)-\rho(\bar{x}^i)\big)dz \\
&:= D_{\bar{x}}^{i} + E_{\bar{x}}^{i,1}.
\end{aligned}
$$

Analogously we can set $A_{X^i} := D_X^{i-1} + E_X^{i,2}$ and $B_{X^i} := D_X^i + E_X^{i,1}$. In this

way we have

$$
\begin{aligned}
&\frac{d}{dt}\frac{1}{N}\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)^2 \\
&= 4N^2\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)\left[A_{\bar{x}^i} + B_{\bar{x}^i} - A_{X^i} - B_{X^i}\right] + \frac{2}{N}\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)R^i \\
&= 4N^2\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)\left[D^i_{\bar{x}} - D^{i-1}_{\bar{x}} - D^i_X + D^i_X\right] \\
&+ 4N^2\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)\left[E^{i,1}_{\bar{x}} - E^{i,1}_X + E^{i,2}_{\bar{x}} - E^{i,2}_X\right] + \frac{2}{N}\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)R^i \\
&= T_1 + T_2 + \frac{2}{N}\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)R^i.
\end{aligned}
$$

Let us estimate $T_1$ and $T_2$ separately. First,

$$
\begin{aligned}
T_1 &= 4N^2\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)\left[D^i_{\bar{x}} - D^{i-1}_{\bar{x}} - D^i_X + D^i_X\right] \\
&= 4N^2\left(\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)\left(D^i_{\bar{x}} - D^i_X\right) - \sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)\left(D^{i-1}_{\bar{x}} - D^{i-1}_X\right)\right)
\end{aligned}
$$

Using the discrete version of the integration by parts we obtain

$$
\begin{aligned}
T_1 &= 4N^2\left(\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)\left(D^i_{\bar{x}} - D^i_X\right) - \sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)\left(D^{i-1}_{\bar{x}} - D^{i-1}_X\right)\right) \\
&= 4N^2\left(\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)\left(D^i_{\bar{x}} - D^i_X\right) - \sum_{i=1}^{N}\left(\bar{x}^{i+1} - X^{i+1}\right)\left(D^i_{\bar{x}} - D^i_X\right)\right) \\
&= 4N^2\left(\sum_{i=1}^{N}\left(\left(\bar{x}^i - \bar{x}^{i+1}\right) - \left(X^i - X^{i+1}\right)\right)\left(D^i_{\bar{x}} - D^i_X\right)\right)
\end{aligned}
$$

Recalling the definitions of $D^i_{\bar{x}}$ and $D^i_X$ we have

$$
\begin{aligned}
T_1 = &-\frac{N^2}{4}\Bigg(\sum_{i=1}^{N}\Big(\big(\bar{x}^{i+1}-\bar{x}^i\big)-\big(X^{i+1}-X^i\big)\Big) \\
&\qquad\qquad \times\Big(\rho(\bar{x}^i)(\bar{x}^{i+1}-\bar{x}^i)^2-\rho(X^i)(X^{i+1}-X^i)^2\Big)\Bigg) \\
= &-\frac{N^2}{4}\Bigg(\sum_{i=1}^{N}\Big[\big(\bar{x}^{i+1}-\bar{x}^i\big)-\big(X^{i+1}-X^i\big)\Big] \\
&\qquad\qquad \times\Big[\rho(\bar{x}^i)\Big((\bar{x}^{i+1}-\bar{x}^i)^2-(X^{i+1}-X^i)^2\Big)\Big]\Bigg) \\
&+\frac{N^2}{4}\Bigg(\sum_{i=1}^{N}\Big[\big(\bar{x}^{i+1}-\bar{x}^i\big)-\big(X^{i+1}-X^i\big)\Big] \\
&\qquad\qquad \times\big(\rho(\bar{x}^i)-\rho(X^i)\big)\big(X^{i+1}-X^i\big)^2\Bigg) \\
=: &\, T_{1,1}+T_{1,2}.
\end{aligned}
$$

Notice that, since $\|\rho'\|_\infty \le \varepsilon_0$ and $\|\rho\|_{L^1}=1$, we have $\rho \ge 1/2$ provided $\varepsilon_0$ is small enough. Hence, recalling that $\bar{x}^{i+1}-\bar{x}^i \ge \frac{c_0}{N}$ and $X^{i+1}-X^i \ge \frac{c_0}{N}$, we can estimate the first term

$$
\begin{aligned}
T_{1,1} &\le -\frac{N^2}{8}\sum_{i=1}^{N}\Big[\big(\bar{x}^{i+1}-\bar{x}^i\big)-\big(X^{i+1}-X^i\big)\Big]^2 \\
&\qquad\qquad \times\Big((\bar{x}^{i+1}-\bar{x}^i)+(X^{i+1}-X^i)\Big) \\
&\le -\frac{c_0}{4}N\sum_{i=1}^{N}\Big[\big(\bar{x}^{i+1}-\bar{x}^i\big)-\big(X^{i+1}-X^i\big)\Big]^2 \\
&= -\frac{c_0}{4N}\sum_{i=1}^{N}\Big[N\big(\bar{x}^{i+1}-\bar{x}^i\big)-N\big(X^{i+1}-X^i\big)\Big]^2.
\end{aligned}
$$

Hence, recalling that $X^{i+1}-X^i \le \frac{C_0}{N}$,

$$
\begin{aligned}
|T_{1,2}| &\le \|\rho'\|_{L^\infty}\frac{N^2}{4}\sum_{i=1}^{N}\Big|\big(\bar{x}^{i+1}-\bar{x}^i\big)-\big(X^{i+1}-X^i\big)\Big| \\
&\qquad\qquad \times\big(\bar{x}^i-X^i\big)\big(X^{i+1}-X^i\big)^2 \\
&\le \frac{C_0^2}{N}\|\rho'\|_{L^\infty}\sum_{i=1}^{N}\Big|N\big(\bar{x}^{i+1}-\bar{x}^i\big)-N\big(X^{i+1}-X^i\big)\Big|\big|\bar{x}^i-X^i\big|.
\end{aligned}
$$

Using the inequality $ab \leq a^2 + b^2$ we get

$$|T_{1,2}| \leq \frac{C_0^2}{N}\|\rho'\|_{L^\infty} \sum_{i=1}^{N} \left[ N\left(\bar{x}^{i+1} - \bar{x}^i\right) - N\left(X^{i+1} - X^i\right) \right]^2$$
$$+ \frac{C_0^2}{N}\|\rho'\|_{L^\infty} \sum_{i=1}^{N} \left(\bar{x}^i - X^i\right)^2.$$

Let us now consider $T_2$.

$$T_2 = 4N^2 \sum_{i=1}^{N} \left(\bar{x}^i - X^i\right) \left[ E_{\bar{x}}^{i,1} - E_X^{i,1} + E_{\bar{x}}^{i,2} - E_X^{i,2} \right]$$
$$= 4N^2 \sum_{i=1}^{N} \left(\bar{x}^i - X^i\right) \left[ E_{\bar{x}}^{i,1} - E_X^{i,1} \right] + 4N^2 \sum_{i=1}^{N} \left(\bar{x}^i - X^i\right) \left[ E_{\bar{x}}^{i,2} - E_X^{i,2} \right]$$
$$:= T_{2,1} + T_{2,2}.$$

Let us first focus on the differences $E_{\bar{x}}^{i,1} - E_X^{i,1}$ and $E_{\bar{x}}^{i,2} - E_X^{i,2}$. Keeping in mind the definitions of $E_{\bar{x}}^{i,1}$ and $E_X^{i,1}$ we have

$$E_{\bar{x}}^{i,1} - E_X^{i,1} =$$
$$\int_{\bar{x}^i}^{\frac{\bar{x}^i + \bar{x}^{i+1}}{2}} (z - \bar{x}^i)\left(\rho(z) - \rho(\bar{x}^i)\right) dz - \int_{X^i}^{\frac{X^i + X^{i+1}}{2}} (z - X^i)\left(\rho(z) - \rho(X^i)\right) dz.$$

Performing the change of variable $\omega = z - \bar{x}^i$, $\omega = z - X^i$ respectively, we get

$$E_{\bar{x}}^{i,1} - E_X^{i,1} = \int_0^{\frac{\bar{x}^{i+1} - \bar{x}^i}{2}} \omega\left(\rho(\omega + \bar{x}^i) - \rho(\bar{x}^i)\right) d\omega$$
$$- \int_0^{\frac{X^{i+1} - X^i}{2}} \omega\left(\rho(\omega + X^i) - \rho(X^i)\right) d\omega.$$

Adding and subtracting

$$\int_0^{\frac{\bar{x}^{i+1} - \bar{x}^i}{2}} \omega\left(\rho(\omega + X^i) - \rho(X^i)\right) d\omega$$

we have

$$E_{\bar{x}}^{i,1} - E_X^{i,1} = \int_0^{\frac{\bar{x}^{i+1} - \bar{x}^i}{2}} \omega\Big[\rho(\omega + \bar{x}^i) - \rho(\bar{x}^i) - \rho(\omega + X^i) + \rho(X^i)\Big] d\omega$$
$$- \int_{\frac{\bar{x}^{i+1} - \bar{x}^i}{2}}^{\frac{X^{i+1} - X^i}{2}} \omega\left(\rho(\omega + X^i) - \rho(X^i)\right) d\omega.$$

By the fundamental theorem of calculus and recalling that $(\bar{x}^{i+1}-\bar{x}^i)\leq \frac{C_0}{N}$, $(X^{i+1}-X^i)\leq \frac{C_0}{N}$ we obtain the following estimate

$$
\begin{aligned}
|E^{i,1}_{\bar{x}}-E^{i,1}_{X}| &= \Bigg| \int_0^{\frac{\bar{x}^{i+1}-\bar{x}^i}{2}} \omega^2 \left[\int_0^1 \rho'(\bar{x}^i+s\omega)ds - \int_0^1 \rho'(X^i+s\omega)ds\right] d\omega \\
&\quad - \int_{\frac{\bar{x}^{i+1}-\bar{x}^i}{2}}^{\frac{X^{i+1}-X^i}{2}} \omega\left(\rho(\omega+X^i)-\rho(X^i)\right) d\omega \Bigg| \\
&\leq \frac{C_0}{N^3}\|\rho''\|_{L^\infty}|\bar{x}^i-X^i| + \|\rho'\|_{L^\infty}\left|\int_{\frac{\bar{x}^{i+1}-\bar{x}^i}{2}}^{\frac{X^{i+1}-X^i}{2}} \omega^2 d\omega\right| \\
&= \frac{C_0}{N^3}\|\rho''\|_{L^\infty}|\bar{x}^i-X^i| \\
&\quad + \frac{\|\rho'\|_{L^\infty}}{8}\left|\left(X^{i+1}-X^i\right)^3-\left(\bar{x}^{i+1}-\bar{x}^i\right)^3\right|.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|T_{2,1}| &= 4N^2\sum_{i=1}^N \left|\bar{x}^i-X^i\right|\left|E^{i,1}_{\bar{x}}-E^{i,1}_{X}\right| \\
&\leq \frac{C}{N}\|\rho''\|_{L^\infty}\sum_{i=1}^N\left(\bar{x}^i-X^i\right)^2 \\
&\quad + \frac{N^2}{2}\|\rho'\|_{L^\infty}\sum_{i=1}^N|\bar{x}^i-X^i|\left|\left(X^{i+1}-X^i\right)^3-\left(\bar{x}^{i+1}-\bar{x}^i\right)^3\right|.
\end{aligned}
$$

Recalling that $0\leq(\bar{x}^{i+1}-\bar{x}^i)\leq\frac{C_0}{N}$ and $0\leq(X^{i+1}-X^i)\leq\frac{C_0}{N}$ we see that

$$
\left|\left(X^{i+1}-X^i\right)^3-\left(\bar{x}^{i+1}-\bar{x}^i\right)^3\right| \leq \frac{C}{N^2}\left|\left(X^{i+1}-X^i\right)-\left(\bar{x}^{i+1}-\bar{x}^i\right)\right|,
$$

therefore

$$
\begin{aligned}
|T_{2,1}| &\leq \frac{C}{N}\|\rho''\|_{L^\infty}\sum_{i=1}^N\left(\bar{x}^i-X^i\right)^2 \\
&\quad + \frac{C}{N}\|\rho'\|_{L^\infty}\sum_{i=1}^N|\bar{x}^i-X^i|\left|N\left(\bar{x}^{i+1}-\bar{x}^i\right)-N\left(X^{i+1}-X^i\right)\right| \\
&\leq \frac{C}{N}\|\rho''\|_{L^\infty}\sum_{i=1}^N\left(\bar{x}^i-X^i\right)^2 \\
&\quad + \frac{C}{N}\|\rho'\|_{L^\infty}\left[\sum_{i=1}^N|\bar{x}^i-X^i|^2+\sum_{i=1}^N\left|N\left(\bar{x}^{i+1}-\bar{x}^i\right)-N\left(X^{i+1}-X^i\right)\right|^2\right].
\end{aligned}
$$

Let us now estimate $E^{i,2}_{\bar{x}} - E^{i,2}_{X}$. By definition we have

$$E^{i,2}_{\bar{x}} - E^{i,2}_{X} = \int_{\frac{\bar{x}^i+\bar{x}^{i-1}}{2}}^{\bar{x}^i} (z-\bar{x}^i)\left(\rho(z)-\rho(\bar{x}^{i-1})\right) dz$$
$$- \int_{\frac{X^i+X^{i-1}}{2}}^{X^i} (z-X^i)\left(\rho(z)-\rho(X^{i-1})\right) dz.$$

With the substitutions $\omega = z - \bar{x}^{i-1}$ and $\omega = z - X^{i-1}$ respectively, we get

$$E^{i,2}_{\bar{x}} - E^{i,2}_{X} = \int_{\frac{\bar{x}^i-\bar{x}^{i-1}}{2}}^{\bar{x}^i-\bar{x}^{i-1}} \left(\omega+\bar{x}^{i-1}-\bar{x}^i\right)\left(\rho(\omega+\bar{x}^{i-1})-\rho(\bar{x}^{i-1})\right) d\omega$$
$$- \int_{\frac{X^i-X^{i-1}}{2}}^{X^i-X^{i-1}} \left(\omega+X^{i-1}-X^i\right)\left(\rho(\omega+X^{i-1})-\rho(X^{i-1})\right) d\omega.$$

Adding and subtracting

$$- \int_{\frac{X^i-X^{i-1}}{2}}^{X^i-X^{i-1}} \left(\omega+X^{i-1}-X^i\right)\left(\rho(\omega+X^{i-1})-\rho(X^{i-1})\right) d\omega$$

we get

$$|E^{i,2}_{\bar{x}} - E^{i,2}_{X}| \leq \Bigg| \int_{\frac{\bar{x}^i-\bar{x}^{i-1}}{2}}^{\bar{x}^i-\bar{x}^{i-1}} \left(\omega+\bar{x}^{i-1}-\bar{x}^i\right)\Big(\rho(\omega+\bar{x}^{i-1})-\rho(\bar{x}^{i-1})$$
$$-\rho(\omega+X^{i-1})+\rho(X^{i-1})\Big) d\omega \Bigg|$$
$$+ \Bigg| - \int_{\frac{X^i-X^{i-1}}{2}}^{X^i-X^{i-1}} \left(\omega+X^{i-1}-X^i\right)\left(\rho(\omega+X^{i-1})-\rho(X^{i-1})\right) d\omega$$
$$+ \int_{\frac{\bar{x}^i-\bar{x}^{i-1}}{2}}^{\bar{x}^i-\bar{x}^{i-1}} \left(\omega+X^{i-1}-X^i\right)\left(\rho(\omega+X^{i-1})-\rho(X^{i-1})\right) d\omega \Bigg|.$$

Arguing as we did for the first term in $E^{i,1}_{\bar{x}} - E^{i,1}_{X}$, the first term in $E^{i,2}_{\bar{x}} - E^{i,2}_{X}$ is controlled by

$$\|\rho''\|_\infty \int_{\frac{\bar{x}^i-\bar{x}^{i-1}}{2}}^{\bar{x}^i-\bar{x}^{i-1}} \left|\omega+\bar{x}^{i-1}-\bar{x}^i\right| \left|X^{i-1}-\bar{x}^{i-1}\right| \omega^2\, d\omega,$$

and recalling that $|\bar{x}^{i-1}-\bar{x}^i| \leq C_0/N$, the above term is bounded by

$$\frac{C}{N^3}\|\rho''\|_\infty \left|X^{i-1}-\bar{x}^{i-1}\right|.$$

Concerning the second term in $E^{i,2}_{\bar{x}} - E^{i,2}_{X}$, using that

$$\left|\int_{a/2}^{a} - \int_{b/2}^{b}\right| \leq \left|\int_{0}^{a} - \int_{0}^{b}\right| + \left|\int_{0}^{a/2} - \int_{0}^{b/2}\right| = \left|\int_{a}^{b}\right| + \left|\int_{a/2}^{b/2}\right|$$

we get

$$\begin{aligned}
&\Bigg|\int_{\frac{\bar{x}^i-\bar{x}^{i-1}}{2}}^{\bar{x}^i-\bar{x}^{i-1}} \left(\omega + X^{i-1} - X^i\right)\left(\rho(\omega + X^{i-1}) - \rho(X^{i-1})\right) d\omega \\
&- \int_{\frac{X^i-X^{i-1}}{2}}^{X^i-X^{i-1}} \left(\omega + X^{i-1} - X^i\right)\left(\rho(\omega + X^{i-1}) - \rho(X^{i-1})\right) d\omega\Bigg| \\
&\leq \left|\int_{\frac{\bar{x}^i-\bar{x}^{i-1}}{2}}^{\frac{X^i-X^{i-1}}{2}} \left(\omega + X^{i-1} - X^i\right)\left(\rho(\omega + X^{i-1}) - \rho(X^{i-1})\right) d\omega\right| \\
&+ \left|\int_{\bar{x}^i-\bar{x}^{i-1}}^{X^i-X^{i-1}} \left(\omega + X^{i-1} - X^i\right)\left(\rho(\omega + X^{i-1}) - \rho(X^{i-1})\right) d\omega\right|. \\
&\leq \|\rho'\|_{L^\infty}\Bigg[\Big|\int_{\frac{\bar{x}^i-\bar{x}^{i-1}}{2}}^{\frac{X^i-X^{i-1}}{2}} \left|\omega + X^{i-1} - X^i\right| |\omega|\, d\omega\Big| \\
&\qquad + \Big|\int_{\bar{x}^i-\bar{x}^{i-1}}^{X^i-X^{i-1}} \left|\omega + X^{i-1} - X^i\right| |\omega|\, d\omega\Big|\Bigg].
\end{aligned}$$

We now notice that in the last term the second integral is bounded by the first integral hence we can bound it by

$$\begin{aligned}
&2\|\rho'\|_{L^\infty}\int_{\bar{x}^i-\bar{x}^{i-1}}^{X^i-X^{i-1}} \omega^2\, d\omega + 2\|\rho'\|_{L^\infty}(X^i - X^{x-i})\int_{\bar{x}^i-\bar{x}^{i-1}}^{X^i-X^{i-1}} \omega\, d\omega \\
&\leq C\|\rho'\|_{L^\infty}\Big|(X^i - X^{i-1})^3 - (\bar{x}^i - \bar{x}^{i-1})^3\Big| \\
&\qquad + C\|\rho'\|_{L^\infty}(X^{i-1} - X^i)\Big|(X^i - X^{i-1})^2 - (\bar{x}^i - \bar{x}^{i-1})^2\Big|.
\end{aligned}$$

Hence, arguing as for $T_{2,1}$, we obtain

$$\begin{aligned}
|T_{2,2}| \leq \frac{C}{N}\|\rho''\|_{L^\infty}\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)^2 + \frac{C}{N}\|\rho'\|_{L^\infty}\Bigg[\sum_{i=1}^{N}|\bar{x}^i - X^i|^2 \\
+ \sum_{i=1}^{N}\left|N\left(\bar{x}^{i+1} - \bar{x}^i\right) - N\left(X^{i+1} - X^i\right)\right|^2\Bigg].
\end{aligned}$$

Combining all these bounds together, we get

$$
\begin{aligned}
&\frac{d}{dt}\frac{1}{N}\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)^2 \\
&= T_1 + T_2 + \frac{2}{N}\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)R^i \\
&= T_{1,1} + T_{1,2} + T_{2,1} + T_{2,2} + \frac{2}{N}\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)R^i \\
&\leq -\frac{c_0\lambda}{2N}\sum_{i=1}^{N}\Big[N\left(\bar{x}^{i+1} - \bar{x}^i\right) - N\left(X^{i+1} - X^i\right)\Big]^2 \\
&\qquad + \frac{C}{N}\Big(\|\rho'\|_{L^\infty} + \|\rho''\|_{L^\infty}\Big)\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)^2 \\
&\qquad + \frac{C}{N}\Big(\|\rho'\|_{L^\infty} + \|\rho''\|_{L^\infty}\Big)\sum_{i=1}^{N}\Big[N\left(\bar{x}^{i+1} - \bar{x}^i\right) - N\left(X^{i+1} - X^i\right)\Big]^2 \\
&\qquad + \frac{2}{N}\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)R^i.
\end{aligned}
$$

Hence, recalling that $\|\rho'\|_{L^\infty} + \|\rho''\|_{L^\infty} \leq \varepsilon_0$, we can choose $\varepsilon_0$ small (the smallness depending only on $c_0, C_0, \lambda$) so that $C\Big(\|\rho'\|_{L^\infty} + \|\rho''\|_{L^\infty}\Big) \leq c_0\lambda/2$ to obtain

$$
\begin{aligned}
\frac{d}{dt}\frac{1}{N}\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)^2 \leq -\frac{c_0}{8N}\sum_{i=1}^{N}\Big[N\left(\bar{x}^{i+1} - \bar{x}^i\right) - N\left(X^{i+1} - X^i\right)\Big]^2 \\
+ \frac{C}{N}\varepsilon_0\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)^2 + \frac{2}{N}\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)R^i.
\end{aligned}
$$

We now use the discrete Poincaré inequality (see Lemma 3.5) to get

$$
\frac{1}{2}\sum_{i=1}^{N}\Big[N\left(\bar{x}^{i+1} - \bar{x}^i\right) - N\left(X^{i+1} - X^i\right)\Big]^2 \geq \sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)^2,
$$

so that assuming $\varepsilon_0$ small enough we conclude

$$\begin{aligned}\frac{d}{dt}\frac{1}{N}\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)^2 &\leq -\frac{c_0}{N}\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)^2 + \frac{C}{N}\varepsilon_0\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)^2 \\ &\quad + \frac{2}{N}\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)R^i \\ &\leq -\frac{2c_0}{3N}\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)^2 + \frac{2}{N}\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)R^i\end{aligned}$$

Finally, using the bound

$$2\left(\bar{x}^i - X^i\right)R^i \leq \epsilon(\bar{x}^i - X^i)^2 + \frac{1}{\epsilon}|R^i|^2$$

with $\epsilon := 2c_0/3$, and recalling that $|R^i| \leq \hat{C}/N^2$ we conclude

$$\frac{d}{dt}\frac{1}{N}\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)^2 \leq -\frac{c_0}{6N}\sum_{i=1}^{N}\left(\bar{x}^i - X^i\right)^2 + \frac{3}{2c_0}\left(\frac{\hat{C}}{N^2}\right)^2.$$

Integrating this differential inequality over $[T_1, t]$ with $t \leq T_2$, by Gronwall Lemma we obtain

$$\frac{1}{N}\sum_{i=1}^{N}\left(\bar{x}^i(t) - X^i(t)\right)^2 \leq e^{-\bar{c}(t-T_1)}\frac{1}{N}\sum_{i=1}^{N}\left(\bar{x}^i(T_1) - X^i(T_1)\right)^2 + \bar{C}\left(\frac{\hat{C}}{N^2}\right)^2$$

for some constants $\bar{c}, \bar{C} > 0$ depending only on $c_0$, as desired. □

## 4.2 The convergence results

Combining the results in the previous sections, we can now prove that if a continuous and a discrete solution are close up to $1/N^2$ at time zero, then they remain close for all time. As one can see from the proof, it is crucial that the discrete scheme has a error of order $\frac{1}{N^2}$ (see Lemma 4.1).

**Theorem 4.4.** *Let $\bar{x}^i$ be a solution of the ODE* (4.5)*, and let $X^i$ be as in* (4.3)*. Assume that $X_0 \in C^{4,\alpha}([0,1])$, $X_0(0) = 0$, $X_0(1) = 1$, and that $a_0 \leq \partial_\theta X_0 \leq A_0$ for some positive constants $a_0, A_0$. Also, suppose that*

$$|X^i(0) - \bar{x}^i(0)| \leq \frac{C'}{N^2} \qquad \forall\, i = 1, \ldots, N. \tag{4.8}$$

*for some positive constant $C'$.*

*Then, there exists $\varepsilon_1 \equiv \varepsilon_1\big(a_0, A_0, \|\rho\|_{C^{3,\alpha}([0,1])}, \|X_0\|_{C^{4,\alpha}([0,1])}\big) > 0$ such that, if $\|\rho'\|_\infty + \|\rho''\|_\infty \leq \varepsilon_1$ we have*

$$\frac{1}{N}\sum_{i=1}^{N}\left(\bar{x}^i(t) - X^i(t)\right)^2 \leq \frac{\bar{\bar{C}}}{N^4} \qquad \forall\, t \in [0, \infty).$$

*Proof.* The idea of the proof is the following: we want to prove the discrete gradient flow and the continuous one are $L^2$ close for all times. This is exactly what is claimed in Lemma 4.3 which, on the other hand, is based on the assumption $\frac{c_0}{N} \leq \bar{x}^i(t) - \bar{x}^{i-1}(t) \leq \frac{C_0}{N}$, $c_0, C_0 \in \mathbb{R}^+$. Unfortunately, a priori, these assumptions may not hold for every time. However, by carefully combining Lemmas 4.2 and 4.3 by an induction argument, we can show that these assumptions actually holds for all times.

*Basis for the induction.* First we observe that, by Proposition 2.3, there exist two positive constants $a$ and $A$ such that

$$a \leq \partial_\theta X(t) \leq A \qquad \forall\, t \geq 0. \tag{4.9}$$

Recalling the definition of $X^i$ in (4.3), we can infer the following inequalities at the discrete level:

$$\frac{a}{N} \leq X^i(t) - X^{i-1}(t) \leq \frac{A}{N} \qquad \forall\, t \geq 0,\ \forall\, i. \tag{4.10}$$

Let us now focus on the assumption

$$\frac{c_0}{N} \leq \bar{x}^i(t) - \bar{x}^{i-1}(t) \leq \frac{C_0}{N}, \qquad c_0, C_0 \in \mathbb{R}^+.$$

Using Lemma 4.2 we have

$$|\bar{x}^i(t) - X^i(t)| \leq |\bar{x}^i(0) - X^i(0)| + \eta t \qquad \forall\, t \in [0, T].$$

Keeping in mind the definition of $\eta$ and (4.8) we have

$$|\bar{x}^i(t) - X^i(t)| \leq \frac{C'}{N^2} + \frac{3\hat{C}}{N^2} t + \frac{C'}{N^2}\frac{t}{T} \qquad \forall\, t \in [0, T],$$

so by the triangle inequality and (4.10) we obtain

$$\frac{a}{N} - 2\left(\frac{2C'}{N^2} + \frac{3\hat{C}}{N^2} T\right) \leq \bar{x}^i(t) - \bar{x}^{i-1}(t) \leq \frac{A}{N} + 2\left(\frac{2C'}{N^2} + \frac{3\hat{C}}{N^2} T\right)$$

for $t \in [0, T]$. In particular, by choosing $N$ large enough (depending only on $a, A, \hat{C}, C', T$), we can ensure that

$$\frac{a}{2N} \leq \bar{x}^i(t) - \bar{x}^{i-1}(t) \leq \frac{2A}{N} \qquad \forall\, t \in [0, T]. \tag{4.11}$$

*Inductive step.* Our goal is to show that if the above property holds for all $t \in [0, \alpha T]$ then it holds for all $t \in [0, (\alpha + 1)T]$. Let us apply Lemma 4.3 on $[0, \alpha T]$ and (4.8) to get

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N} \left(\bar{x}^i(t) - X^i(t)\right)^2 &\leq \frac{e^{-\bar{c}t}}{N}\sum_{i=1}^{N} \left(\bar{x}^i(0) - X^i(0)\right)^2 + \bar{C}\left(\frac{\hat{C}}{N^2}\right)^2 \\ &\leq \frac{\bar{\bar{C}}}{N^4} \qquad \forall\, t \in [0, \alpha T]\end{aligned}$$

for some constant $\bar{\bar{C}}$ depending only on $\bar{C}, \hat{C}, C'$. Hence, since

$$|\bar{x}^i(t) - X^i(t)| \leq \sqrt{\sum_{i=1}^{N} \big(\bar{x}^i(t) - X^i(t)\big)^2} \qquad \forall\, t \in [0, \alpha T],\ \forall\, i,$$

we obtain in particular,

$$|\bar{x}^i(\alpha T) - X^i(\alpha T)| \leq \sqrt{\frac{\bar{\bar{C}}}{N^3}} \qquad \forall\, i = 1, \ldots, N.$$

Applying again Lemma 4.2 with $\alpha T$ as initial time, we now get

$$\begin{aligned} |\bar{x}^i(\alpha T + t) - X^i(\alpha T + t)| &\leq |\bar{x}^i(\alpha T) - X^i(\alpha T)| + \eta \alpha T \\ &\leq \sqrt{\frac{\bar{\bar{C}}}{N^3}} + \frac{3\hat{C}}{N^2}\alpha T + \sqrt{\frac{\bar{\bar{C}}}{N^3}}\frac{t}{\alpha T} \qquad \forall\, t \in [0, \alpha T]. \end{aligned}$$

Hence, by (4.10) and the triangle inequality,

$$\begin{aligned} \frac{a}{N} - 2\left(2\sqrt{\frac{\bar{\bar{C}}}{N^3}} - \frac{3\hat{C}}{N^2}\alpha T\right) &\leq \bar{x}^i(t) - \bar{x}^{i-1}(t) \\ &\leq \frac{A}{N} + 2\left(2\sqrt{\frac{\bar{\bar{C}}}{N^3}} - \frac{3\hat{C}}{N^2}\alpha T\right) \end{aligned} \tag{4.12}$$

for each $t \in [\alpha T, (\alpha + 1)T]$. Then, if $N$ is big enough so that

$$2\sqrt{\frac{\bar{\bar{C}}}{N^3}} + \frac{3\hat{C}}{N^2}\alpha T \leq \frac{a}{4N} \tag{4.13}$$

we have

$$\frac{a}{2N} \leq \bar{x}^i(t) - \bar{x}^{i-1}(t) \leq \frac{2A}{N} \qquad \forall\, t \in [\alpha T, (\alpha + 1)T].$$

Recalling the inequality (4.11) we get

$$\frac{a}{2N} \leq \bar{x}^i(t) - \bar{x}^{i-1}(t) \leq \frac{2A}{N} \qquad \forall\, t \in [0, (\alpha + 1)T].$$

This concludes the inductive step and, in particular, Lemma 4.3 applied on $[0, \infty)$ proves the desired estimate for $N \geq N_0$ for some large number $N_0$.

Notice that the case $N \leq N_0$ is trivial since (using that $0 \leq \bar{x}^i, X^i \leq 1$)

$$\frac{1}{N}\sum_{i=1}^{N} \big(\bar{x}^i(t) - X^i(t)\big)^2 \leq 1 \leq \frac{N_0^4}{N^4} \qquad \forall\, t \in [0, \infty).$$

□

## 4.3 The Eulerian description

In order to get a convergence result in Eulerian variable, we will also need a full stability result in $L^2$ in the continuous case. The following result holds:

**Proposition 4.5.** *Assume that $\rho : [0,1] \to (0,\infty)$ is a periodic probability density of class $C^2$ and let $X_1, X_2$ be two solutions of the equation* (4.2) *satisfying* (1.4) *and*

$$0 < c_0 \leq \partial_\theta X_i(0,\theta) \leq C_0, \qquad i = 1,2. \tag{4.14}$$

*There exists $\varepsilon_0 \equiv \varepsilon_0(c_0, C_0)$ as in Lemma 4.3 such that, under the condition $\|\rho'\|_{L^\infty} + \|\rho''\|_{L^\infty} \leq \varepsilon_0$, one has*

$$\int_0^1 |X_1(t,\theta) - X_2(t,\theta)|^2 \, d\theta \leq \left( \int_0^1 |X_1(0,\theta) - X_2(0,\theta)|^2 \, d\theta \right) e^{-\bar{c}t}$$

*for all $t \geq 0$, for some $\bar{c} \equiv \bar{c}(c_0)$.*

*Proof.* The proof of this result follows the lines of the proof of Proposition 3.1, with the difference that we have to get rid of the extra terms using the smallness of $\|\rho'\|_{L^\infty} + \|\rho''\|_{L^\infty}$. Also, this result could also be obtained as a consequence of Lemma 4.3 letting $N \to \infty$. However, since the proof is relatively short, we give it for the convenience of the reader.

We begin by noticing that since $\int_0^1 \rho(x)dx = 1$, if $\|\rho'\|_\infty$ is sufficiently small it follows that $1/2 \leq \rho \leq 2$, so the monotonicity condition (4.14) implies that

$$0 < c_1 \leq \partial_\theta X_i(t) \leq C_1, \qquad i = 1,2, \text{ for all } t \geq 0 \tag{4.15}$$

for some constants $c_1, C_1$ depending only on $c_0, C_0$ (see Proposition 2.3). Also, we notice that (4.2) can be equivalently rewritten as

$$\partial_t X = \frac{1}{4}\partial_\theta\big(\rho(X)(\partial_\theta X)^2\big) - \frac{1}{12}\rho'(X)(\partial_\theta X)^3.$$

Then, since $X_2 - X_1$ vanishes at the boundary, we compute

$$
\begin{aligned}
&\frac{d}{dt}\int_0^1 |X_1 - X_2|^2\, d\theta \\
&= \frac{1}{2}\int_0^1 (X_1 - X_2)\left(\partial_\theta\big(\rho(X_1)(\partial_\theta X_1)^2\big) - \partial_\theta\big(\rho(X_2)(\partial_\theta X_2)^2\big)\right) d\theta \\
&\qquad - \frac{1}{6}\int_0^1 (X_1 - X_2)\left(\rho'(X_1)(\partial_\theta X_1)^3 - \rho'(X_2)(\partial_\theta X_2)^3\right) d\theta \\
&= -\frac{1}{2}\int_0^1 \partial_\theta(X_1 - X_2)\left(\big(\rho(X_1)(\partial_\theta X_1)^2\big) - \big(\rho(X_2)(\partial_\theta X_2)^2\big)\right) d\theta \\
&\qquad - \frac{1}{6}\int_0^1 (X_1 - X_2)\left(\rho'(X_1)(\partial_\theta X_1)^3 - \rho'(X_2)(\partial_\theta X_2)^3\right) d\theta \\
&= -\frac{1}{2}\int_0^1 \rho(X_2)\partial_\theta(X_1 - X_2)\left((\partial_\theta X_1)^2 - (\partial_\theta X_2)^2\right) d\theta \\
&\qquad - \frac{1}{2}\int_0^1 [\rho(X_1) - \rho(X_2)]\,\partial_\theta(X_1 - X_2)\,(\partial_\theta X_1)^2\, d\theta \\
&\qquad - \frac{1}{6}\int_0^1 \rho'(X_2)(X_1 - X_2)\left((\partial_\theta X_1)^3 - (\partial_\theta X_2)^3\right) d\theta \\
&\qquad - \frac{1}{6}\int_0^1 [\rho'(X_1) - \rho'(X_2)]\,\rho(X_1 - X_2)\,(\partial_\theta X_1)^3\, d\theta \\
&=: T_{1,1} + T_{1,2} + T_{2,1} + T_{2,2}.
\end{aligned}
$$

Recalling that $1/2 \le \rho$, using (4.15) we get

$$
\begin{aligned}
T_{1,1} &\le -\frac{1}{2}\int_0^1 \rho(X_2)\left(\partial_\theta(X_1 - X_2)\right)^2\left((\partial_\theta X_1) + (\partial_\theta X_2)\right) d\theta \\
&\le -\frac{c_1}{2}\int_0^1 \left(\partial_\theta(X_1 - X_2)\right)^2 d\theta.
\end{aligned}
$$

Using again (4.15) we bound

$$
\begin{aligned}
|T_{1,2}| &\le \frac{C_1^2}{2}\|\rho'\|_\infty \int_0^1 |X_1 - X_2|\,|\partial_\theta X_1 - \partial_\theta X_2|\, d\theta \\
&\le \frac{C_1^2}{2}\|\rho'\|_\infty \int_0^1 (X_1 - X_2)^2\, d\theta + \frac{C_1^2}{2}\|\rho'\|_\infty \int_0^1 (\partial_\theta X_1 - \partial_\theta X_2)^2\, d\theta,
\end{aligned}
$$

$$
\begin{aligned}
|T_{2,1}| &\le \frac{C_1^2}{2}\|\rho'\|_\infty \int_0^1 |X_1 - X_2|\,|\partial_\theta X_1 - \partial_\theta X_2|\, d\theta \\
&\le \frac{C_1^2}{2}\|\rho'\|_\infty \int_0^1 (X_1 - X_2)^2\, d\theta + \frac{C_1^2}{2}\|\rho'\|_\infty \int_0^1 (\partial_\theta X_1 - \partial_\theta X_2)^2\, d\theta,
\end{aligned}
$$

$$
|T_{2,2}| \le \frac{C_1^3}{6}\|\rho''\|_\infty \int_0^1 (X_1 - X_2)^2\, d\theta.
$$

Hence, combining all together, if both $\|\rho'\|_\infty$ and $\|\rho''\|_\infty$ are sufficiently small, using Poincaré inequality (see Lemma 3.5 and let $N \to \infty$), we obtain

$$\begin{aligned}\frac{d}{dt}\int_0^1 |X_1 - X_2|^2\, d\theta &\le -\frac{c_1}{4}\int_0^1 \big(\partial_\theta(X_1 - X_2)\big)^2\, d\theta \\ &\qquad + C\big(\|\rho'\|_\infty + \|\rho''\|_\infty\big)\int_0^1 (X_1 - X_2)^2\, d\theta \\ &\le -\frac{c_1}{2}\int_0^1 (X_1 - X_2)^2\, d\theta \\ &\qquad + C\big(\|\rho'\|_\infty + \|\rho''\|_\infty\big)\int_0^1 (X_1 - X_2)^2\, d\theta \\ &\le -\frac{c_1}{4}\int_0^1 (X_1 - X_2)^2\, d\theta,\end{aligned}$$

and the result follows by Gronwall's inequality. □

**Theorem 4.6.** *Let $\rho : [0,1] \to (0,\infty)$ be a periodic probability density of class $C^{3,\alpha}$. Let $x^i$ be a solution of the discrete gradient flow starting from an initial datum satisfying*

$$\left| x^i(0) - X_0\left(0, \frac{i - 1/2}{N}\right)\right| \le \frac{C'}{N^2} \qquad \forall\, i = 1, \dots, N,$$

*where $X_0 \in C^{4,\alpha}([0,1])$, $X_0(0) = 1$, $X_0(1) = 1$, and $0 < c_0 \le \partial_\theta X_0 \le C_0$. Then there exist three constants $\varepsilon_1 \equiv \varepsilon_1\big(c_0, C_0, \|\rho\|_{C^{3,\alpha}([0,1])}, \|X_0\|_{C^{4,\alpha}([0,1])}\big)$ as in Theorem 4.4; $\bar{\bar{c}} \equiv \bar{\bar{c}}(c_0) > 0, \bar{\bar{C}} \equiv \bar{\bar{C}}(c_0) > 0$, such that,*

$$MK_1(\mu_t^N, \gamma\rho^{1/3}\, d\theta) \le \bar{\bar{C}}\, e^{-\bar{\bar{c}}t/N^3} + \frac{\bar{\bar{C}}}{N} \qquad \forall\, t \ge 0,$$

*where*

$$\gamma := \frac{1}{\int_0^1 \rho^{1/3}(x)dx}$$

*provided that $\|\rho'\|_\infty + \|\rho''\|_\infty \le \varepsilon_1$. In particular*

$$MK_1(\mu_t^N, \gamma\rho^{1/3}\, d\theta) \le \frac{2\bar{\bar{C}}}{N} \qquad \textit{for all } t \ge \frac{N^3 \log N}{\bar{\bar{c}}}.$$

*Proof.* Let $\bar{X}$ satisfy

$$\partial_\theta \bar{X} = \frac{1}{\gamma\rho^{1/3} \circ \bar{X}}, \quad \bar{X}(0) = 0.$$

Then $\bar{X}$ is a stationary solution of (4.2) satisfying also the boundary condition (1.4), hence by Proposition 4.5 we deduce that

$$\int_0^1 |X(t) - \bar{X}|^2\, d\theta \le Ce^{-\bar{c}t},$$

where $X(t)$ is the solution of (4.2) starting from $X_0$. We then apply Theorem 4.4 to deduce that

$$\frac{1}{N}\sum_{i=1}^{N}\left(x^i(t)-X^i(t/N^3)\right)^2 \le \frac{\bar{\bar{C}}}{N^4} \qquad \forall\, t\in[0,\infty),$$

where $X^i(t) := X\left(t, \frac{i-1/2}{N}\right)$. Combining these two estimates and observing that $\bar{X}_{\#}d\theta = \gamma\,\rho^{1/3}\,d\theta$, the result follows by arguing as in the proof of Theorem 3.6. □

# A From the discrete to the continuous case

In order to obtain a continuous version of the functional

$$F_{N,r}(x^1,\dots,x^N) = \int_0^1 \min_{1\le i\le N}|x^i-y|^r\rho(y)\,dy,$$

with $0\le x^1\le\ldots\le x^N\le 1$, we define

$$x^{i+1/2} := \frac{x^i+x^{i+1}}{2},$$

where by convention $x^0=0$ and $x^{N+1}=1$. Then the expression for the minimum becomes

$$\min_{1\le j\le N}|y-x^j|^r = \begin{cases} |y-x^i|^r & \text{for } y\in(x^{i-1/2},x^{i+1/2}),\\ |y|^r & \text{for } y\in(0,x^{1/2}),\\ |y-1|^r & \text{for } y\in(x^{N+1/2},1),\end{cases}$$

and $F_{N,r}$ is given by

$$F_{N,r}(x^1,\dots,x^N) = \\ \sum_{i=1}^{N}\int_{x^{i-1/2}}^{x^{i+1/2}}|y-x^i|^r\rho(y)dy + \int_0^{x^{1/2}}|y|^r\rho(y)dy + \int_{x^{N+1/2}}^1|y-1|^r\rho(y)dy.$$

Assume that

$$x^i = X\left(\frac{i-1/2}{N}\right), \qquad i=1,\dots,N$$

with $X:[0,1]\to[0,1]$ a smooth non-decreasing map. Then a Taylor expansion yields

$$F_{N,r}(x^1,\dots,x^N) = \frac{C_r}{N^r}\int_0^1\rho(X(\theta))|\partial_\theta X(\theta)|^{r+1}d\theta + O\Big(\frac{1}{N^{r+1}}\Big),$$

where $C_r = \frac{1}{2^r(r+1)}$ and $O\left(\frac{1}{N^{r+1}}\right)$ depends on the smoothness of $\rho$ and $X$ (for instance, $\rho\in C^1$ and $X\in C^2$ is enough). Hence

$$N^rF_{N,r}(x^1,\dots,x^N) \longrightarrow C_r\int_0^1\rho(X(\theta))|\partial_\theta X(\theta)|^{r+1}d\theta := \mathcal{F}[X]$$

as $N\to\infty$.

# B The Hessian of $\mathcal{F}[X]$

Assume $\lambda \le \rho \le \frac{1}{\lambda}$, and let $X, Y \in L^2([0,1])$ with $0 \le c \le \partial_\theta X \le C$ and $|\partial_\theta Y| \le C$. Then

$$D^2\mathcal{F}[X](Y,Y) = 6\int_0^1 \rho(X)\,\partial_\theta X\,(\partial_\theta Y)^2\,d\theta + 6\int_0^1 \rho'(X)\,(\partial_\theta X)^2\,(\partial_\theta Y)\,Y\,d\theta + \int_0^1 \rho''(X)\,(\partial_\theta X)^3\,Y^2\,d\theta.$$

## B.1 Convexity under a smallness assumption on $\rho'$ and $\rho''$

We want to prove that the Hessian of $\mathcal{F}$ is positive definite provided that

$$\|\rho'\|_\infty + \|\rho''\|_\infty \ll 1.$$

We first observe that

$$D^2\mathcal{F}[X](Y,Y) = \frac{d^2}{d^2\varepsilon}\bigg|_{\varepsilon=0}\mathcal{F}(X+\varepsilon Y) \ge 6\lambda\,c\int_0^1 (\partial_\theta Y)^2\,d\theta - 6C^2\|\rho'\|_\infty\int_0^1 |\partial_\theta Y|\,|Y|\,d\theta - C^3\|\rho''\|_\infty\int_0^1 Y^2\,d\theta.$$

Observe that if both $\rho'$ and $\rho''$ are small, we can control both the second and third term by the first one using Cauchy-Schwarz and Poincaré inequalities. In particular one sees that the Hessian is positive at "points" $X$ which are uniformly monotone and Lipschitz [3].

Indeed, using Cauchy-Schwarz,

$$D^2\mathcal{F}[X](Y,Y) \ge -C^3\|\rho''\|_\infty\int_0^1 Y^2\,d\theta + 6\lambda\,c\int_0^1 (\partial_\theta Y)^2\,d\theta - 3C^2\|\rho'\|_\infty\Big[\int_0^1 Y^2\,d\theta + \int_0^1 (\partial_\theta Y)^2\,d\theta\Big].$$

Hence, if $3\|\rho'\|C^2 \le 3\lambda c$ we have

$$\begin{aligned} D^2\mathcal{F}[X](Y,Y) &\ge 3\lambda\,c\int_0^1 (\partial_\theta Y)^2\,d\theta + \Big[-C^3\|\rho''\|_\infty + 3C^2\|\rho'\|_\infty\Big]\int_0^1 Y^2\,d\theta \\ &\ge 6\lambda c\int_0^1 Y^2\,d\theta + \Big[C^3\|\rho''\|_\infty - 3C^2\|\rho'\|_\infty\Big]\int_0^1 Y^2\,d\theta, \end{aligned}$$

where for the second inequality we used Poincaré (see for instance Lemma 3.5 and let $N \to \infty$). Thus, if $3\lambda c > C^3\|\rho''\|_\infty - 3C^2\|\rho'\|_\infty$ it follows that the Hessian of $\mathcal{F}$ is positive definite.

[3] Recall that $0 \le c \le \partial_\theta X \le C$

## B.2 Lack of convexity without a smallness assumption

In this section it will be convenient to specify the dependence of $\mathcal{F}$ on $\rho$, so we denote

$$\mathcal{F}_\rho(X) := \int_0^1 \rho(X)\,|\partial_\theta X|^3\,d\theta.$$

To build a counterexample, we first construct a density $\bar\rho$ and a Lipschitz function $Y$ such that $D^2\mathcal{F}_{\bar\rho}(X)[Y,Y] < 0$ with $X(t,\theta) = \theta$. Although the density $\bar\rho$ will not be smooth nor strictly positive, by an approximation argument we will eventually obtain a counterexample also with a smooth positive density.

Let $\varepsilon > 0$ be a small number and define

$$\bar\rho(\theta) := \begin{cases} 1 & \text{for } \theta \in \left[\frac12 - \varepsilon, \frac12 + \varepsilon\right] \\ 0 & \text{for } \theta \in [0,1] \setminus \left[\frac12 - \varepsilon, \frac12 + \varepsilon\right], \end{cases}$$

and let $Y(t,\theta)$ be a Lipschitz function in $[0,1]$ that coincides with $|\theta - \frac12| + 1$ in $\left[\frac12 - \varepsilon, \frac12 + \varepsilon\right]$. Then, recalling the formula for the Hessian of $\mathcal{F}$,

$$D^2\mathcal{F}_{\bar\rho}(X)[Y,Y] = 6\int_0^1 \bar\rho\,(\partial_\theta Y)^2\,d\theta + 6\int_0^1 \bar\rho'\,\partial_\theta Y\,Y\,d\theta + \int_0^1 \bar\rho''\,Y^2\,d\theta.$$

Integrating by parts we have

$$\begin{aligned} D^2\mathcal{F}_{\bar\rho}(X)[Y,Y] &= 6\int_0^1 \bar\rho\,(\partial_\theta Y)^2\,d\theta - 6\int_0^1 \bar\rho\,(\partial_\theta Y)^2 - 6\int_0^1 \bar\rho\,\partial_\theta^2 Y\,Y\,d\theta \\ &\quad + 2\int_0^1 \bar\rho\Big[(\partial_\theta Y)^2 + \partial_\theta^2 Y\,Y\Big]\,d\theta \\ &= 2\int_0^1 \bar\rho\,(\partial_\theta Y)^2\,d\theta - 4\int_0^1 \bar\rho\,\partial_\theta^2 Y\,Y\,d\theta. \end{aligned}$$

Recalling the definitions of $Y$ and $\bar\rho$ we have

$$D^2\mathcal{F}_{\bar\rho}(X)[Y,Y] = 2\int_{\frac12-\varepsilon}^{\frac12+\varepsilon} (\partial_\theta Y)^2\,d\theta - 4Y\left(\frac12\right) = 4\varepsilon - 4 < 0 \qquad \text{for } \varepsilon < 1.$$

In order to build a counterexample with a smooth positive density, we first extend $\bar\rho$ by periodicity on the whole real line and define $\rho_\delta := \bar\rho * \varphi_\delta$, with

$$\varphi_\delta(\theta) = \frac{\exp^{-\frac{|\theta|^2}{2\delta}}}{\sqrt{2\pi\delta}}.$$

Then, by the same computation as above we have

$$D^2\mathcal{F}_{\rho_\delta}(X)[Y,Y] = 2\int_0^1 \rho_\delta\,(\partial_\theta Y)^2\,d\theta - 4\int_0^1 \rho_\delta\,\partial_\theta^2 Y\,Y\,d\theta,$$

and since $\rho_\delta \to \bar{\rho}$ in $L^1$ and $\rho_\delta(1/2) \to \bar{\rho}(1/2)$, we conclude that

$$D^2\mathcal{F}_{\rho_\delta}(X)[Y,Y] \to D^2\mathcal{F}_{\bar{\rho}}(X)[Y,Y] \text{as } \delta \to 0.$$

In particular, by choosing $\delta > 0$ sufficiently small, we have obtained that the Hessian of $\mathcal{F}_{\rho_\delta}$ in the direction $Y$ is negative when $X(\theta) = \theta$ and $\rho_\delta \in C^\infty([0,1])$ and satisfies $1 \geq \rho_\delta > 0$.

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