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1 **A NON-INTRUSIVE STRATIFIED RESAMPLER FOR REGRESSION**  
2 **MONTE CARLO: APPLICATION TO SOLVING NON-LINEAR**  
3 **EQUATIONS\***

4 EMMANUEL GOBET<sup>†</sup>, GANG LIU<sup>†</sup>, AND JORGE P. ZUBELLI<sup>‡</sup>

5 **Abstract.** Our goal is to solve certain dynamic programming equations associated to a given  
6 Markov chain  $X$ , using a regression-based Monte Carlo algorithm. More specifically, we assume that  
7 the model for  $X$  is not known in full detail and only a root sample  $X^1, \dots, X^M$  of such process  
8 is available. By a stratification of the space and a suitable choice of a probability measure  $\nu$ , we  
9 design a new resampling scheme that allows to compute local regressions (on basis functions) in each  
10 stratum. The combination of the stratification and the resampling allows to compute the solution to  
11 the dynamic programming equation (possibly in large dimensions) using only a relatively small set  
12 of root paths. To assess the accuracy of the algorithm, we establish non-asymptotic error estimates  
13 in  $L^2(\nu)$ . Our numerical experiments illustrate the good performance, even with  $M = 20 - 40$  root  
14 paths.

15 **Key words.** discrete Dynamic Programming Equations, empirical regression scheme, resam-  
16 pling methods, small-size sample

17 **AMS subject classifications.** 62G08, 62G09, 93Exx

18 **1. Introduction.** Stochastic dynamic programming equations are classic equa-  
19 tions arising in the resolution of nonlinear evolution equations, like in stochastic control  
20 (see [18, 4]) or non-linear PDEs (see [6, 9]). In a discrete-time setting they take  
21 the form:

22 
$$Y_N = g_N(X_N), \quad Y_i = \mathbb{E} [g_i(Y_{i+1}, \dots, Y_N, X_i, \dots, X_N) \mid X_i], \quad i = N - 1, \dots, 0,$$

24 for some functions  $g_N$  and  $g_i$  which depend on the non-linear problem under consid-  
25 eration. Here  $X = (X_0, \dots, X_N)$  is a Markov chain valued in  $\mathbb{R}^d$ , entering also in  
26 the definition of the problem. The aim is to compute the value function  $y_i$  such that  
27  $Y_i = y_i(X_i)$ .

28 Among the popular methods to solve this kind of problem, we are concerned with  
29 Regression Monte Carlo (RMC) methods that take as input  $M$  simulated paths of  $X$ ,  
30 say  $(X^1, \dots, X^M) =: X^{1:M}$ , and provide as output simulation-based approximations  
31  $y_i^{M,\mathcal{L}}$  using Ordinary Least Squares (OLS) within a vector space of functions  $\mathcal{L}$ :

32 
$$y_i^{M,\mathcal{L}} = \arg \inf_{\varphi \in \mathcal{L}} \frac{1}{M} \sum_{m=1}^M \left| g_i(y_{i+1}^{M,\mathcal{L}}(X_{i+1}^m), \dots, y_N^{M,\mathcal{L}}(X_N^m), X_i^m, \dots, X_N^m) - \varphi(X_i^m) \right|^2.$$

33

34 This Regression Monte Carlo methodology has been investigated in [9] to solve Back-  
35 ward Stochastic Differential Equations associated to semi-linear partial differential  
36 equations (PDEs) [16], with some tight error estimates. Generally speaking, it is well  
37 known that the number of simulations  $M$  has to be much larger than the dimension  
38 of the vector space  $\mathcal{L}$  and thus the number of coefficients we are seeking.

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39 In contradistinction, throughout this work, we focus on the case where  $M$  is  
 40 relatively small (a few hundreds) and the simulations are not sampled by the user but  
 41 are directly taken from historical data ( $X^{1:M}$  is called **root sample**), in the spirit of  
 42 [17]. This is the most realistic situation when we collect data and when the model  
 43 which fits the data is unknown.

44 Thus, as a main difference with the aforementioned references:

- 45 • We do not assume that we have full information about the model for  $X$  and  
 46 we do not assume that we can generate as many simulations as needed to  
 47 have convergent Regression Monte Carlo methods.
- 48 • The size  $M$  of the learning samples  $X^1, \dots, X^M$  is relatively small, which  
 49 discards the use of a direct RMC with large dimensional  $\mathcal{L}$ .

50 To overcome these major obstacles, we elaborate on two ingredients:

- 51 1. First, we partition  $\mathbb{R}^d$  in strata  $(\mathcal{H}_k)_k$ , so that the regression functions can  
 52 be computed locally on each stratum  $\mathcal{H}_k$ ; for *small* stratum this allows to  
 53 use only a small dimensional approximation space  $\mathcal{L}_k$ , and therefore it puts  
 54 a lower constraint on  $M$ . In general, this stratification breaks the properties  
 55 for having well-behaved error propagation and we provide a precise way to  
 56 sample in order to be able to aggregate the error estimates in different strata.  
 57 We use a probabilistic distribution  $\nu$  that has good norm-stability properties  
 58 with  $X$  (see Assumptions 3.2 and 4.2).
- 59 2. Second, by assuming a mild model condition on  $X$ , we are able to resample  
 60 from the root sample of size  $M$ , a *training sample* of  $M$  simulations suitable  
 61 for the stratum  $\mathcal{H}_k$ . This resampler is non intrusive in the sense that it only  
 62 requires to know the form of the model but not its coefficients: for example,  
 63 we can handle models with independent increments (discrete inhomogeneous  
 64 Levy process) or Ornstein-Uhlenbeck processes. See Examples 2.1-2.2-2.3-  
 65 2.4. We call this scheme NISR (Non Intrusive Stratified Resampler), it is  
 66 described in Definition 2.1 and Proposition 2.1.

67 The resulting regression scheme is, to the best of our knowledge, completely new.

68 To sum up, the contributions of this work are the following:

- 69 • We design a non-intrusive stratified resample (NISR) scheme that allows to  
 70 sample from  $M$  paths of the root sample restarting from any stratum  $\mathcal{H}_k$ .  
 71 See Section 2.
- 72 • We combine this with regression Monte Carlo schemes, in order to solve one-  
 73 step ahead dynamic programming equations (Section 3), discrete backward  
 74 stochastic differential equations (BSDEs) and semi-linear PDEs (Section 4).
- 75 • In Theorems 3.4 and 4.1, we provide quadratic error estimates of the form

$$76 \quad \text{quadratic error on } y_i \leq \text{approximation error} + \text{statistical error} \\
 77 \quad \quad \quad + \text{interdependency error} .$$

79 The approximation error is related to the best approximation of  $y_i$  on each  
 80 stratum  $\mathcal{H}_k$ , and averaged over all the strata. The statistical error is bounded  
 81 by  $C/M$  with a constant  $C$  which does not depend on the number of strata:  
 82 only relatively small  $M$  is necessary to get low statistical errors. This is in  
 83 agreement with the motivation that the root sample has a relatively small  
 84 size. The interdependency error is an unusual issue, it is related to the strong  
 85 dependency between regression problems (because they all use the same root  
 86 sample). The analysis as well as the framework are original. The error es-  
 87 timates take different forms according to the problem at hand (Section 3 or

Section 4).

- Finally we illustrate the performance of the methods on two types of examples: first, approximation of non-linear PDEs arising in reaction-diffusion biological models (Subsection 5.1) and optimal sequential decision (Subsection 5.2), where we illustrate that root samples of size  $M = 20 - 40$  only can lead to remarkably accurate numerical solutions.

The paper is organized as follows. In Section 2 we present the model structure that leads to the non-intrusive stratified resampler for regression Monte Carlo (NISR), together with the stratification. Main notations will be also introduced. The algorithm is presented in a generic form of dynamic programming equations in Algorithm 1. In Section 3 we analyze the convergence of the algorithm in the case of one-step ahead dynamic programming equations (for instance optimal stopping problems). Section 4 is devoted to the convergence analysis for discrete BSDEs (probabilistic representation of semi-linear PDEs arising in stochastic control problems). Section 5 is devoted to numerical examples. Technical results are postponed to the Appendix.

## 2. Setting and the general algorithm.

**2.1. General dynamic programming equation.** Suppose we have  $N$  discrete dates, and we aim at solving numerically the following dynamic programming equation (DPE for short), written in general form:

$$Y_N = g_N(X_N), \quad Y_i = \mathbb{E}[g_i(Y_{i+1:N}, X_{i:N}) \mid X_i], \quad 0 \leq i < N.$$

Here,  $(X_i)_{0 \leq i \leq N}$  is a Markov chain with state space  $\mathbb{R}^d$ ,  $(Y_i)_{0 \leq i \leq N}$  is a random process taking values in  $\mathbb{R}$  and we use for convenience the generic short notation  $z_{i:N} := (z_i, \dots, z_N)$ . Note that the argument of the conditional expectation is path-dependent, thus allowing greater generality. Had we considered  $Y$  to be multidimensional, the subsequent algorithm and analysis would remain essentially the same.

Later (Sections 3 and 4), specific forms for  $g_i$  will be considered, depending on the model of DPE to solve at hand: it will have an impact on the error estimates that we can derive. However, the description of the algorithm can be the same for all the DPEs, as seen below, and this justifies our choice of unifying the presentation.

Thanks to the Markovian property of  $X$ , under mild assumptions we can easily prove by induction that there exists a measurable function  $y_i$  such that  $Y_i = y_i(X_i)$ , our aim is to compute an approximation of the value functions  $y_i(\cdot)$  for all  $i$ . We assume that a bound on  $y_i$  is available.

ASSUMPTION 2.1 (A priori bound). *The solution  $y_i$  is bounded by a constant  $|y_i|_\infty$ .*

**2.2. Model structure and root sample.** We will represent  $y_i(\cdot)$  through its coefficients on a vector space, and the coefficients will be computed thanks to learning samples of  $X$ .

ASSUMPTION 2.2 (Data). *We have the observation of  $M$  independent paths of  $X$ , which are denoted by  $((X_i^m : 0 \leq i \leq N), 1 \leq m \leq M)$ . We refer to this data as the root sample.*

For our needs, we adopt a representation of the *flow of the Markov chain for different initial conditions*, i.e., the Markov chain  $X^{i,x}$  starting at different times  $i \in \{0, \dots, N\}$  and points  $x \in \mathbb{R}^d$ . Namely, we write

$$(2.1) \quad X_j^{i,x} = \theta_{i,j}(x, U), \quad i \leq j \leq N,$$

where

- 134 •  $U$  is some random vector, called random source,  
 135 •  $\theta_{i,j}$  are (deterministic) measurable functions.

136 We emphasize that, for the sake of convenience,  $U$  is the same for representing all  
 137  $X_j^{i,x}$ ,  $0 \leq i \leq j \leq N$ ,  $x \in \mathbb{R}^d$ .

ASSUMPTION 2.3 (Noise extraction). *We assume that  $\theta_{i,j}$  are known and we can retrieve the random sources  $(U^1, \dots, U^M)$  associated to the root sample  $X^{1:M} = (X^m : 1 \leq m \leq M)$ , i.e.,*

$$X_j^m = X_j^{0,x_0^m,m} = \theta_{0,j}(x_0^m, U^m).$$

138 Observe that this assumption is much less stringent than identifying the distribution  
 139 of the model. We exemplify this now.

EXAMPLE 2.1 (Arithmetic Brownian motion with time dependent parameters). *Let  $(t_i : 0 \leq i \leq N)$  be  $N$  times and define the arithmetic Brownian motion by*

$$X_i = x_0 + \int_0^{t_i} \mu_s ds + \int_0^{t_i} \sigma_s dW_s$$

where  $\mu_t \in \mathbb{R}^d$ ,  $\sigma_t \in \mathbb{R}^{d \times q}$ ,  $W_t \in \mathbb{R}^q$  and  $\mu, \sigma$  are deterministic functions of time. In this case, the random source is given by

$$U := (X_{i+1} - X_i)_{0 \leq i \leq N-1}$$

and the functions by

$$\theta_{ij}(x, U) := x + \sum_{i \leq k < j} U_k.$$

140 This works since  $U_i = \int_{t_i}^{t_{i+1}} \mu_s ds + \int_{t_i}^{t_{i+1}} \sigma_s dW_s$ . The crucial point is that, in order  
 141 to extract  $U$  from  $X$ , we do not assume that  $\mu$  and  $\sigma$  are known.

EXAMPLE 2.2 (Levy process). *More generally, we can set  $X_i = \mathbf{X}_{t_i}$  with a time-inhomogeneous Levy process  $\mathbf{X}$ . Then take*

$$U := (X_{i+1} - X_i)_{0 \leq i \leq N-1}, \quad \theta_{ij}(x, U) := x + \sum_{i \leq k < j} U_k.$$

EXAMPLE 2.3 (Geometric Brownian motion with time dependent parameters). *With the same kind of parameters as for Example 2.1, define the geometric Brownian motion (component by component)*

$$X_i = X_0 \exp \left( \int_0^{t_i} \mu_s ds + \int_0^{t_i} \sigma_s dW_s \right).$$

Then, we have that

$$U := \left( \log \left( \frac{X_{i+1}}{X_i} \right) \right)_{0 \leq i \leq N-1}, \quad \theta_{ij}(x, U) := x \prod_{i \leq k < j} \exp(U_k).$$

EXAMPLE 2.4 (Ornstein-Uhlenbeck process with time dependent parameters). *Given  $N$  times  $(t_i : 0 \leq i \leq N)$ , set  $X_i = \mathbf{X}_{t_i}$  where  $\mathbf{X}$  has the following dynamics:*

$$\mathbf{X}_t = \mathbf{x}_0 - \int_0^t A(\mathbf{X}_s - \bar{\mathbf{X}}_s) ds + \int_0^t \Sigma_s dW_s$$

where  $A$  is  $d \times d$ -matrix,  $\mathbf{X}_t$  and  $\bar{\mathbf{X}}_t$  are in  $\mathbb{R}^d$ ,  $\Sigma_t$  is a  $d \times q$ -matrix,  $W_t \in \mathbb{R}^q$ .  $\bar{\mathbf{X}}_t$  and  $\Sigma_t$  are both deterministic functions of time. The explicit solution is

$$\mathbf{X}_t = e^{-A(t-s)}\mathbf{X}_s + e^{-At} \int_s^t e^{Ar} (A\bar{\mathbf{X}}_r dr + \Sigma_r dW_r).$$

Assume that we know  $A$ : in this case, an observation of  $X_{0:N}$  enables to retrieve the random source

$$U := \left( X_j - e^{-A(t_j-t_i)} X_i \right)_{0 \leq i \leq j \leq N}$$

and then

$$\theta_{ij}(x, U) := e^{-A(t_j-t_i)} x + U_{i,j}.$$

142 The noise extraction works since  $U_{i,j} = e^{-At_j} \int_{t_i}^{t_j} e^{Ar} (A\bar{\mathbf{X}}_r dr + \Sigma_r dW_r)$ .

143 As illustrated above, through Assumption 2.2, all we need to know is the general  
144 structure of the Markov chain model but we do not need to estimate all the model  
145 parameters, and sometimes none of them (Examples 2.1, 2.2, 2.3). Our approach is  
146 non intrusive in this sense.

147 **2.3. Stratification and resampling algorithm.** On the one hand, we can rely  
148 on a root sample of size  $M$  only (possibly with a relatively small  $M$ , constrained by  
149 the available data), which is very little to perform accurate Regression Monte-Carlo  
150 methods (usually  $M$  has to be much larger than the dimension of approximation  
151 spaces, as reminded in introduction).

152 On the other hand, we are able to access the random sources so that resampling  
153 the  $M$  paths is possible. The degree of freedom comes from the flexibility of initial  
154 conditions  $(i, x)$ , thanks to the flow representation (2.1). We now explain how we take  
155 advantage of this property.

156 The idea is to resample the model paths for different starting points in different  
157 parts of the space  $\mathbb{R}^d$  and on each part, we will perform a regression Monte Carlo  
158 using  $M$  paths and a low-dimensional approximation space. These ingredients give  
159 the ground reasons for getting accurate results.

160 Let us proceed to the details of the algorithm. We design a stratification approach:  
161 suppose there exist  $K$  strata  $(\mathcal{H}_k)_{1 \leq k \leq K}$  such that

$$162 \quad \mathcal{H}_k \cap \mathcal{H}_l = \emptyset \quad \text{for } k \neq l, \quad \bigcup_{k=1}^K \mathcal{H}_k = \mathbb{R}^d.$$

163 An example for  $\mathcal{H}_k$  is a hypercube of the form  $\mathcal{H}_k = \prod_{l=1}^d [x_{k,l}^-, x_{k,l}^+)$ . Then, we are  
164 given a probability measure  $\nu$  on  $\mathbb{R}^d$  and denote its restriction on  $\mathcal{H}_k$  by

$$165 \quad \nu_k(dx) := \frac{1}{\nu(\mathcal{H}_k)} 1_{\mathcal{H}_k}(x) \nu(dx).$$

166 The measure  $\nu$  will serve as a reference to control the errors. See Paragraph 3.1.2 and  
167 Section 5 for choices of  $\nu$ .

168 **DEFINITION 2.1** (Non-intrusive stratified resampler, NISR for short). *We define the*  
169  *$M$ -sample used for regression at time  $i$  and in the  $k$ -th stratum  $\mathcal{H}_k$ :*

- 170 • let  $(X_i^{i,k,m})_{1 \leq m \leq M}$  be an i.i.d. sample according to the law  $\nu_k$ ;

- for  $j = i + 1, \dots, N$ , set

$$X_j^{i,k,m} = \theta_{i,j}(X_i^{i,k,m}, U^m),$$

171 where  $U^{1:M}$  are the random sources from Assumption 2.3.

172 In view of Assumptions 2.2 and 2.3, the random sources  $U^1, \dots, U^M$  are independent, therefore we easily prove the following.

174 PROPOSITION 2.1. *The  $M$  paths  $(X_{i:N}^{i,k,m}, 1 \leq m \leq M)$  are independent and identically distributed as  $X_{i:N}$  with  $X_i \stackrel{d}{\sim} \nu_k$ .*

176 **2.4. Approximation spaces and regression Monte Carlo schemes.** On each stratum, we approximate the value functions  $y_i$  using basis functions. We can take different kinds of basis functions:

- 179 - **LP<sub>0</sub>** (partitioning estimate):  $\mathcal{L}_k = \text{span}(1_{\mathcal{H}_k})$ ,
- 180 - **LP<sub>1</sub>** (piecewise linear):  $\mathcal{L}_k = \text{span}(1_{\mathcal{H}_k}, x_1 1_{\mathcal{H}_k}, \dots, x_d 1_{\mathcal{H}_k})$ ,
- 181 - **LP<sub>n</sub>** (piecewise polynomial):  $\mathcal{L}_k = \text{span}$ ( all the polynomials of degree less than or equal to  $n$  on  $\mathcal{H}_k$ ).

To simplify the presentation, we assume hereafter that the dimension of  $\mathcal{L}_k$  does not depend on  $k$ , we write

$$\dim(\mathcal{L}_k) =: \dim(\mathcal{L}).$$

183 To compute the approximation of  $y_i$  on each stratum  $\mathcal{H}_k$ , we will use the  $M$  samples of Definition 2.1. Our NISR-regression Monte Carlo algorithm takes the form:

---

**Algorithm 1** General NISR-regression Monte Carlo algorithm

---

```

1: set  $y_N^{(M)}(\cdot) = g_N(\cdot)$ 
2: for  $i = N - 1$  until 0 do
3:   for  $k = 1$  until  $K$  do
4:     sample  $(X_{i:N}^{i,k,m})_{1 \leq m \leq M}$  using the NISR (Definition 2.1)
5:     set  $S^{(M)}(x_{i:N}) = g_i(y_{i+1}^{(M)}(x_{i+1}), \dots, y_N^{(M)}(x_N), x_{i:N})$ 
6:     compute  $\psi_i^{(M),k} = \text{OLS}(S^{(M)}, \mathcal{L}_k, X_{i:N}^{i,k,1:M})$ 
7:     set  $y_i^{(M),k} = \mathcal{T}_{|y_i|_\infty}(\psi_i^{(M),k})$  where  $\mathcal{T}_L$  is the truncation operator,
8:     defined by  $\mathcal{T}_L(x) = -L \vee x \wedge L$ 
9:   end for
10:  set  $y_i^{(M)} = \sum_{k=1}^K y_i^{(M),k} 1_{\mathcal{H}_k}$ 
11: end for

```

---

In the above, the Ordinary Least Squares approximation of the response function  $\tilde{S} : (\mathbb{R}^d)^{N-i+1} \mapsto \mathbb{R}$  in the function space  $\mathcal{L}_k$  using the  $M$  sample  $X_{i:N}^{i,k,1:M}$  is defined and denoted by

$$\text{OLS}(\tilde{S}, \mathcal{L}_k, X_{i:N}^{i,k,1:M}) = \arg \inf_{\varphi \in \mathcal{L}_k} \frac{1}{M} \sum_{m=1}^M |\tilde{S}(X_{i:N}^{i,k,m}) - \varphi(X_i^{i,k,m})|^2.$$

185 The main difference with the usual regression Monte-Carlo schemes (see [8] for  
186 instance) is that here we use the common random numbers  $U^{1:M}$  for all the regression  
187 problems. This is the effect of resampling. The convergence analysis becomes more  
188 delicate because we lose nice independence properties. Figure 1 describes a key part  
189 in the algorithm, namely the process of using the root paths to generate new paths.

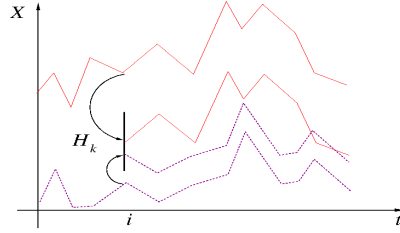


FIG. 1. Description of the use of the root paths to produce new paths in an arbitrary hypercube.

190 **3. Convergence analysis in the case of the one-step ahead dynamic**  
 191 **programming equation.** We consider here the case

192 
$$Y_N = g_N(X_N), \quad Y_i = \mathbb{E} [g_i(Y_{i+1}, X_i, \dots, X_N) \mid X_i], \quad 0 \leq i < N,$$

194 where we need the value of  $Y_{i+1}$  (one step ahead) to compute the value  $Y_i$  (at the  
 195 current date) through a conditional expectation. To compare with Algorithm 1, we  
 196 take  $g_i(Y_{i+1:N}, X_{i:N}) = g_i(Y_{i+1}, X_{i:N})$ .

Equations of this form are quite natural when solving optimal stopping problems  
 in the Markovian case. Indeed, if  $V_i$  is the related value function at time  $i$ , i.e., the  
 essential supremum over stopping times  $\tau \in \{i, \dots, N\}$  of a reward process  $f_\tau(X_\tau)$ ,  
 then  $V_i = \max(Y_i, f_i(X_i))$  where  $Y_i$  is the continuation value defined by

$$Y_i = \mathbb{E} [\max(Y_{i+1}, f_{i+1}(X_{i+1})) \mid X_i] ,$$

see [18] for instance. This corresponds to our setting with

$$g_i(y_{i+1}, x_{i:N}) = \max(y_{i+1}, f_{i+1}(x_{i+1})) .$$

197 Similar dynamic programming equations appear in stochastic control problems. See  
 198 [4].

199 **3.1. Standing assumptions.** The following assumptions enable us to provide  
 200 error estimates (Theorem 3.4 and Corollary 3.1) for the convergence of Algorithm 1.

201 **3.1.1. Assumptions on  $g_i$ .**

202 ASSUMPTION 3.1 (Functions  $g_i$ ). *Each function  $g_i$  is Lipschitz w.r.t. the variable*  
 203  *$y_{i+1}$ , with Lipschitz constant  $L_{g_i}$  and  $C_{g_i} := \sup_{x_{i:N}} |g_i(0, x_{i:N})| < +\infty$ .*

204 It is then easy to justify that  $y_i$  (such that  $y_i(X_i) = Y_i$ ) is bounded (Assumption 2.1).

205 **3.1.2. Assumptions on the distribution  $\nu$ .** We assume a condition on the  
 206 probability measure  $\nu$  and the Markov chain  $X$ , which ensures a suitable stability in  
 207 the propagation of errors.

208 ASSUMPTION 3.2 (norm-stability). *There exists a constant  $\underline{C}_{(3.1)} \geq 1$  such that*  
 209 *for any  $\varphi : \mathbb{R}^d \mapsto \mathbb{R} \in L^2(\nu)$  and any  $0 \leq i \leq N - 1$ , we have*

210 (3.1) 
$$\int_{\mathbb{R}^d} \mathbb{E} [\varphi^2(X_{i+1}^{i,x})] \nu(dx) \leq \underline{C}_{(3.1)} \int_{\mathbb{R}^d} \varphi^2(x) \nu(dx).$$

211 We now provide some examples of distribution  $\nu$  where the above assumption  
 212 holds, in connection with Examples 2.1, 2.2 and 2.4.



213 PROPOSITION 3.1. Let  $\alpha = (\alpha^1, \dots, \alpha^d) \in ]0, +\infty[^d$  and assume that  $X_{i+1}^{i,x} = x +$   
 214  $U_i$  (as in Examples 2.1 and 2.2) with  $\mathbb{E} \left[ \prod_{j=1}^d e^{\alpha^j |U_i^j|} \right] < +\infty$ . Then, the tensor-  
 215 product Laplace distribution  $\nu(dx) := \prod_{j=1}^d \frac{\alpha^j}{2} e^{-\alpha^j |x^j|} dx$  satisfies Assumption 3.2.

216 *Proof.* The L.H.S. of (3.1) writes

$$217 \quad \mathbb{E} \left[ \int_{\mathbb{R}^d} \varphi^2(x + U_i) \nu(dx) \right] = \mathbb{E} \left[ \int_{\mathbb{R}^d} \varphi^2(x) \prod_{j=1}^d \frac{\alpha^j}{2} e^{-\alpha^j |x^j - U_i^j|} dx \right]$$

$$218 \quad \leq \mathbb{E} \left[ \int_{\mathbb{R}^d} \varphi^2(x) \prod_{j=1}^d \frac{\alpha^j}{2} e^{-\alpha^j |x^j| + \alpha^j |U_i^j|} dx \right]$$

219

220 which leads to the announced inequality (3.1) with  $\underline{C}_{(3.1)} := \mathbb{E} \left[ \prod_{j=1}^d e^{\alpha^j |U_i^j|} \right]$ .  $\square$

221 PROPOSITION 3.2. Let  $k > 0$  and assume that  $X_{i+1}^{i,x} = Dx + U_i$  for a diago-  
 222 nal invertible matrix  $D := \text{diag}(D^1, \dots, D^d)$  (a form similar to Example 2.4) with  
 223  $\mathbb{E} \left[ (1 + |U_i|)^{d(k+1)} \right] < +\infty$ . Then, the tensor-product Pareto-type distribution  $\nu(dx) :=$   
 224  $\prod_{j=1}^d \frac{k}{2} (1 + |x^j|)^{-k-1} dx$  satisfies Assumption 3.2.

225 *Proof.* The L.H.S. of (3.1) equals

$$226 \quad \mathbb{E} \left[ \int_{\mathbb{R}^d} \varphi^2(Dx + U_i) \nu(dx) \right]$$

$$227 \quad = \mathbb{E} \left[ \int_{\mathbb{R}^d} \varphi^2(x) \det(D^{-1}) \prod_{j=1}^d \frac{k}{2} (1 + |(x^j - U_i^j)/D^j|)^{-k-1} dx \right]$$

$$228 \quad (3.2) \quad \leq \int_{\mathbb{R}^d} \varphi^2(x) \det(D^{-1}) \prod_{j=1}^d \frac{k}{2} \left( \mathbb{E} \left[ (1 + |(x^j - U_i^j)/D^j|)^{-d(k+1)} \right] \right)^{1/d} dx.$$

229

On the set  $\{|U_i^j| \leq |x^j|/2\}$  we have  $(1 + |(x^j - U_i^j)/D^j|) \geq (1 + (|x^j| - |U_i^j|)/D^j) \geq$   
 $(1 + |x^j|/(2D^j))$ . On the complementary set  $\{|U_i^j| > |x^j|/2\}$ , the random variable  
 inside the  $j$ -th expectation in (3.2) is bounded by 1 and furthermore

$$\mathbb{P} \left( |U_i^j| > |x^j|/2 \right) \leq \frac{\mathbb{E} \left[ (1 + 2|U_i^j|)^{d(k+1)} \right]}{(1 + |x^j|)^{d(k+1)}}.$$

230 By gathering the two cases, we observe that we have shown that the  $j$ -th expectation  
 231 in (3.2) is bounded by  $\text{Cst}(1 + |x^j|)^{-d(k+1)}$ , for any  $x^j$ , whence the advertised result.  $\square$

232 *Remarks.*

- 233 • Since we will apply the inequality (3.1) only to functions in a finite dimen-  
 234 sional space, the norm equivalence property of finite dimensional space may  
 235 also give the existence of a constant  $\underline{C}_{(3.1)}$ . But the constant built in this  
 236 way could depend on the finite dimensional space (and may blow up when its  
 237 dimension increases) while here the constant is valid for any  $\varphi$ .
- 238 • The previous examples on  $\nu$  are related to distributions with independent  
 239 components: this is especially convenient when one has to sample  $\nu$  restricted  
 240 to hypercubes  $\mathcal{H}_k$ , since we are reduced to independent one-dimensional sim-  
 241 ulations.

242 • In Proposition 3.2, had the matrix  $D$  been symmetric instead of diagonal, we  
 243 would have applied an appropriate rotation to the density  $\nu$ .

244 **3.1.3. Covering number of an approximation space.** To analyze how the  
 245  $M$ -samples  $(X_{i:N}^{i,k,m}, 1 \leq m \leq M)$  from NISR approximates the exact distribution of  
 246  $X_{i:N}$  with  $X_i \stackrel{d}{\sim} \nu_k$  over test functions in the space  $\mathcal{L}_k$ , we will invoke concentration of  
 247 measure inequalities (uniform in  $\mathcal{L}_k$ ). This is possible thanks to complexity estimates  
 248 related to  $\mathcal{L}_k$ , expressed in terms of covering numbers. Note that the concept of  
 249 covering numbers is mainly used to introduce Assumption 3.3 and it intervenes in the  
 250 main theorems only through the proof of Proposition 3.5.

251 We briefly recall the definition of a covering number of a dictionary of functions  $\mathcal{G}$ ,  
 252 see [10, Chapter 9] for more details. For a dictionary  $\mathcal{G}$  of functions from  $\mathbb{R}^d$  to  $\mathbb{R}$  and  
 253 for  $M$  points  $x^{1:M} := \{x^{(1)}, \dots, x^{(M)}\}$  in  $\mathbb{R}^d$ , an  $\varepsilon$ -cover ( $\varepsilon > 0$ ) of  $\mathcal{G}$  w.r.t. the  $L^1$ -  
 254 empirical norm  $\|g\|_1 := \frac{1}{M} \sum_{m=1}^M |g(x^{(m)})|$  is a finite collection of functions  $g_1, \dots, g_n$   
 255 such that for any  $g \in \mathcal{G}$ , we can find a  $j \in \{1, \dots, n\}$  such that  $\|g - g_j\|_1 \leq \varepsilon$ .  
 256 The smallest possible integer  $n$  is called the  $\varepsilon$ -covering number and is denoted by  
 257  $\mathcal{N}_1(\varepsilon, \mathcal{G}, x^{1:M})$ .

ASSUMPTION 3.3 (Covering the approximation space). *There exist three constants*

$$\alpha_{(3.3)} \geq \frac{1}{4}, \quad \beta_{(3.3)} > 0, \quad \gamma_{(3.3)} \geq 1$$

258 *such that for any  $B > 0, \varepsilon \in (0, \frac{4}{15}B]$  and stratum index  $1 \leq k \leq K$ , the minimal size*  
 259 *of an  $\varepsilon$ -covering number of  $\mathcal{T}_B \mathcal{L}_k := \{\mathcal{T}_B \varphi : \varphi \in \mathcal{L}_k\}$  is bounded as follows:*

260 (3.3) 
$$\mathcal{N}_1(\varepsilon, \mathcal{T}_B \mathcal{L}_k, x^{1:M}) \leq \alpha_{(3.3)} \left( \frac{\beta_{(3.3)} B}{\varepsilon} \right)^{\gamma_{(3.3)}}$$

261 *independently of the points sample  $x^{1:M}$ .*

262 We assume that the above constants do not depend on  $k$ , mainly for the sake of  
 263 simplicity. In the error analysis (see also Proposition A.1), the constants  $\alpha_{(3.3)}$  and  
 264  $\beta_{(3.3)}$  appear in log and thus, they have a small impact on error bounds. On the  
 265 contrary,  $\gamma_{(3.3)}$  appears as a multiplicative factor and we seek to have the smallest  
 266 estimate.

267 PROPOSITION 3.3. *In the case of approximation spaces  $\mathcal{L}_k$  like  $\mathbf{LP}_0$ ,  $\mathbf{LP}_1$  or  $\mathbf{LP}_n$ ,*  
 268 *Assumption 3.3 is satisfied with the following parameters: for any given  $\eta > 0$ , we*  
 269 *have*

	$\alpha_{(3.3)}$	$\beta_{(3.3)}$	$\gamma_{(3.3)}$
$\mathbf{LP}_0$	1	7/5	1
$\mathbf{LP}_1$	3	$[4c_\eta 6^\eta]^{1/(1+\eta)} e$	$(d+2)(1+\eta)$
$\mathbf{LP}_n$	3	$[4c_\eta 6^\eta]^{1/(1+\eta)} e$	$((d+1)^n + 1)(1+\eta)$

270 where  $c_\eta = \sup_{x \geq \frac{45\varepsilon}{2}} x^{-\eta} \log(x)$ .

272 The proof is postponed to the Appendix.

**3.2. Main result: Error estimate.** We are now in the position to state a  
 convergence result, expressed in terms of the quadratic error of the best approximation  
 of  $y_i$  on the stratum  $\mathcal{H}_k$ :

$$T_{i,k} := \inf_{\varphi \in \mathcal{L}_k} |y_i - \varphi|_{\nu_k}^2 \quad \text{where} \quad |\varphi|_{\nu_k}^2 := \int_{\mathbb{R}^d} |\varphi|^2(x) \nu_k(dx).$$

Our goal is to find an upper bound for the error  $\mathbb{E} \left[ |y_i^{(M)} - y_i|_\nu^2 \right]$  where

$$|\varphi|_\nu^2 := \int_{\mathbb{R}^d} |\varphi|^2(x) \nu(dx).$$

Note that the above expectation is taken over all the random variables, including the random sources  $U^{1:M}$ , i.e., we estimate the quadratic error averaged on the root sample.

**THEOREM 3.4.** *Assume Assumptions 2.2-2.3-3.2-3.3 and define  $y_i^{(M)}$  as in Algorithm 1. Then, for any  $\varepsilon > 0$ , we have*

$$\begin{aligned} \mathbb{E} \left[ |y_i^{(M)} - y_i|_\nu^2 \right] &\leq 4(1 + \varepsilon) L_{g_i}^2 \underline{C}_{(3.1)} \mathbb{E} \left[ |y_{i+1}^{(M)} - y_{i+1}|_\nu^2 \right] + 2 \sum_{k=1}^K \nu(H_k) T_{i,k} + 4c_{(3.8)}(M) \frac{|y_i|_\infty^2}{M} \\ &\quad + 2\left(1 + \frac{1}{\varepsilon}\right) \frac{\dim(\mathcal{L})}{M} (C_{g_i} + L_{g_i} |y_{i+1}|_\infty)^2 + 8(1 + \varepsilon) L_{g_i}^2 c_{(3.7)}(M) \frac{|y_{i+1}|_\infty^2}{M}. \end{aligned}$$

We emphasize that whenever useful, the constant  $4(1 + \varepsilon) L_{g_i}^2 \underline{C}_{(3.1)}$  could be reduced to  $(1 + \delta)(1 + \varepsilon) L_{g_i}^2 \underline{C}_{(3.1)}$  (for any given  $\delta > 0$ ) by slightly adapting the proof: namely, the term  $4 = 2^2$  comes from two applications of deviation inequalities stated in Proposition A.1. These inequalities are valid with  $(1 + \delta)^{\frac{1}{2}}$  instead of 2, up to modifying the constants  $c_{(A.2)}(M)$  and  $c_{(A.3)}(M)$ .

As a very significant difference with usual Regression Monte-Carlo methods (see [9, Theorem 4.11]), in our algorithm there is no competition between the bias term (approximation error) and the variance term (statistical error), while in usual algorithms as the dimension of the approximation space  $K \dim(\mathcal{L})$  goes to infinity, the statistical term (of size  $\frac{K \dim(\mathcal{L})}{M}$ ) blows up. This significant improvement comes from the stratification which gives rise to decoupled and low-dimensional regression problems.

Since  $y_N^{(M)} = y_N$ , we easily derive global error bounds.

**COROLLARY 3.1.** *Under the assumptions and notations of Theorem 3.4, there exists a constant  $C_{(3.4)}(N)$  (depending only on  $N$ ,  $\sup_{0 \leq i < N} L_{g_i}$ ,  $\underline{C}_{(3.1)}$ ), such that for any  $j \in \{0, \dots, N - 1\}$ ,*

$$\begin{aligned} \mathbb{E} \left[ |y_j^{(M)} - y_j|_\nu^2 \right] &\leq C_{(3.4)}(N) \sum_{i=j}^{N-1} \left[ \sum_{k=1}^K \nu(H_k) T_{i,k} \right. \\ &\quad \left. + \frac{1}{M} \left( c_{(3.8)}(M) |y_i|_\infty^2 + \dim(\mathcal{L}) (C_{g_i} + L_{g_i} |y_{i+1}|_\infty)^2 + L_{g_i}^2 c_{(3.7)}(M) |y_{i+1}|_\infty^2 \right) \right]. \end{aligned}$$

It is easy to see that if  $4(1 + \varepsilon) L_{g_i}^2 \underline{C}_{(3.1)} \leq 1$ , then interestingly  $C_{(3.4)}(N)$  can be taken uniformly in  $N$ . This case corresponds to a small Lipschitz constant of  $g_i$ . In the case  $4(1 + \varepsilon) L_{g_i}^2 \underline{C}_{(3.1)} \gg 1$ , the above error estimates deteriorate quickly as  $N$  increases. We shall discuss that in Section 4 which deals with BSDEs and where we propose a different scheme that allows both large Lipschitz constant and large  $N$ .

**3.3. Proof of Theorem 3.4.** Let us start by setting up some useful notations:

$$\begin{aligned} S(x_{i:N}) &:= g_i(y_{i+1}(x_{i+1}), x_{i:N}), & \psi_i^k &:= \text{OLS}(S, \mathcal{L}_k, X_{i:N}^{i,k,1:M}), \\ |f|_{i,k,M}^2 &:= \frac{1}{M} \sum_{m=1}^M f^2(X_{i:N}^{i,k,m}) \end{aligned}$$

308

 309 (or  $|f|_{i,k,M}^2 := \frac{1}{M} \sum_{m=1}^M f^2(X_i^{i,k,m})$  when  $f$  depends only on one argument).

 We first aim at deriving a bound on  $\mathbb{E} \left[ |y_i^{(M)} - y_i|_{i,k,M}^2 \right]$ . First of all, note that

$$|y_i^{(M)} - y_i|_{i,k,M}^2 = \left| \mathcal{T}_{|y_i|_\infty}(\psi_i^{(M),k}) - \mathcal{T}_{|y_i|_\infty}(y_i) \right|_{i,k,M}^2 \leq |\psi_i^{(M),k} - y_i|_{i,k,M}^2$$

310 since the truncation operator is 1-Lipschitz. Now we define

311 (3.5) 
$$\mathbb{E} \left[ S(X_{i:N}^{i,k,m}) | X_i^{i,k,1:M} \right] = \mathbb{E} \left[ S(X_{i:N}^{i,k,m}) | X_i^{i,k,m} \right] = y_i(X_i^{i,k,m})$$

 312 where the first equality is due to the independence of the paths  $(X_{i:N}^{i,k,m}, 1 \leq m \leq M)$   
 313 (Proposition 2.1) and where the last equality stems from the definition of  $y_i$ .

According to [9, Proposition 4.12] which allows to interchange conditional expectation and OLS, we have

$$\mathbb{E} \left[ \psi_i^k(\cdot) | X_i^{i,k,1:M} \right] = \text{OLS}(y_i, \mathcal{L}_k, X_{i:N}^{i,k,1:M}).$$

 314 Since the expected values  $\left( \mathbb{E} \left[ \psi_i^k(X_i^{i,k,m}) | X_i^{i,k,1:M} \right] \right)_{1 \leq m \leq M}$  can be seen as the projec-  
 315 tions of  $(y_i(X_i^{i,k,m}))_{1 \leq m \leq M}$  on the subspace of  $\mathbb{R}^M$  spanned by  $\{(\varphi(X_i^{i,k,m}))_{1 \leq m \leq M}, \varphi \in$   
 316  $\mathcal{L}_k\}$  and  $(\psi_i^{(M),k}(X_i^{i,k,m}))_{1 \leq m \leq M}$  is an element in this subspace, Pythagoras theorem  
 317 yields

318 
$$|\psi_i^{(M),k} - y_i|_{i,k,M}^2 = \left| \psi_i^{(M),k} - \mathbb{E} \left[ \psi_i^k(\cdot) | X_i^{i,k,1:M} \right] \right|_{i,k,M}^2 + \left| \mathbb{E} \left[ \psi_i^k(\cdot) | X_i^{i,k,1:M} \right] - y_i \right|_{i,k,M}^2$$
 319 
$$= \left| \psi_i^{(M),k} - \mathbb{E} \left[ \psi_i^k(\cdot) | X_i^{i,k,1:M} \right] \right|_{i,k,M}^2 + \inf_{\varphi \in \mathcal{L}_k} |\varphi - y_i|_{i,k,M}^2.$$

 321 For any given  $\phi \in \mathcal{L}_k$ , we have

322 
$$\mathbb{E} \left[ \inf_{\varphi \in \mathcal{L}_k} |\varphi - y_i|_{i,k,M}^2 \right] \leq \mathbb{E} \left[ |\phi - y_i|_{i,k,M}^2 \right] = \mathbb{E} \left[ \frac{1}{M} \sum_{m=1}^M |\phi(X_i^{i,k,m}) - y_i(X_i^{i,k,m})|^2 \right]$$
 323 
$$= \int_{\mathbb{R}^d} |\phi(x) - y_i(x)|^2 \nu_k(dx).$$
 324

 Taking the infimum over all functions  $\phi$  on the R.H.S. gives

$$\mathbb{E} \left[ \inf_{\varphi \in \mathcal{L}_k} |\varphi - y_i|_{i,k,M}^2 \right] \leq T_{i,k}.$$

 325 So, for any  $\varepsilon > 0$ , we have

326 
$$\mathbb{E} \left[ |\psi_i^{(M),k} - y_i|_{i,k,M}^2 \right] \leq T_{i,k} + (1 + \varepsilon) \mathbb{E} \left[ |\psi_i^{(M),k} - \psi_i^k|_{i,k,M}^2 \right]$$
 327 
$$+ (1 + \frac{1}{\varepsilon}) \mathbb{E} \left[ \left| \psi_i^k - \mathbb{E} \left[ \psi_i^k(\cdot) | X_i^{i,k,1:M} \right] \right|_{i,k,M}^2 \right].$$
 328

 329 By [9, Proposition 4.12], the last term is bounded by  $\frac{\dim(\mathcal{L})}{M} (C_{g_i} + L_{g_i} |y_{i+1}|_\infty)^2$  where  
 330  $(C_{g_i} + L_{g_i} |y_{i+1}|_\infty)^2$  clearly bounds the conditional variance of  $S(X_{i:N}^{i,k})$ . This is the

331 statistical error contribution. Here, we have used the independence of  $(X_{i:N}^{i,k,m}, 1 \leq$   
 332  $m \leq M)$  (Proposition 2.1).

333 The control of the term  $\mathbb{E} \left[ |\psi_i^{(M),k} - \psi_i^k|_{i,k,M}^2 \right]$  is possible due to the linearity and  
 334 stability of OLS [9, Proposition 4.12]:

$$335 \quad |\psi_i^{(M),k} - \psi_i^k|_{i,k,M}^2 \leq |S^{(M)} - S|_{i,k,M}^2 \leq L_{g_i}^2 \frac{1}{M} \sum_{m=1}^M (y_{i+1}^{(M)} - y_{i+1})^2 (X_{i+1}^{i,k,m}),$$

336

337 where we have taken advantage of the Lipschitz property of  $g_i$  w.r.t. the component  
 338  $y_{i+1}$ . So far we have shown

$$339 \quad \mathbb{E} \left[ |y_i^{(M)} - y_i|_{i,k,M}^2 \right] \leq T_{i,k} + (1 + \varepsilon) L_{g_i}^2 \mathbb{E} \left[ \frac{1}{M} \sum_{m=1}^M (y_{i+1}^{(M)} - y_{i+1})^2 (X_{i+1}^{i,k,m}) \right]$$

$$340 \quad (3.6) \quad + (1 + \frac{1}{\varepsilon}) \frac{\dim(\mathcal{L})}{M} (C_{g_i} + L_{g_i} |y_{i+1}|_\infty)^2.$$

341

342 This shows a relation between the errors at time  $i$  and time  $i + 1$ , but measured in  
 343 different norms. In order to retrieve the same  $L^2(\nu)$ -norm and continue the analysis,  
 344 we will use the norm-stability property (Assumption 3.2) and the following result  
 345 about concentration of measures. The proof is a particular case of Proposition A.1 in  
 346 the Appendix, with  $\psi(x) = (-2|y_{i+1}|_\infty \vee x \wedge 2|y_{i+1}|_\infty)^2$ ,  $B = |y_{i+1}|_\infty$ ,  $\mathcal{K} = \mathcal{L}_k$ ,  $\eta =$   
 347  $y_{i+1}$ .

348 PROPOSITION 3.5. Define  $(c_{(3.7)}(M), c_{(3.8)}(M))$  by considering  $(c_{(A.2)}(M), c_{(A.3)}(M))$   
 349 from Proposition A.1 with the values  $(\alpha_{(3.3)}, \beta_{(3.3)}, \gamma_{(3.3)})$  instead of  $(\alpha, \beta, \gamma)$ . Then  
 350 we have

$$351 \quad \mathbb{E} \left[ \frac{1}{M} \sum_{m=1}^M (y_{i+1}^{(M)} - y_{i+1})^2 (X_{i+1}^{i,k,m}) \right] \leq 2\mathbb{E} \left[ |y_{i+1}^{(M)}(X_{i+1}^{i,\nu_k}) - y_{i+1}(X_{i+1}^{i,\nu_k})|^2 \right]$$

$$352 \quad (3.7) \quad + 4c_{(3.7)}(M) \frac{|y_{i+1}|_\infty^2}{M},$$

$$353 \quad (3.8) \quad \mathbb{E} \left[ |y_i^{(M)} - y_i|_{i,\nu_k}^2 \right] \leq 2\mathbb{E} \left[ |y_i^{(M)} - y_i|_{i,k,M}^2 \right] + 4c_{(3.8)}(M) \frac{|y_i|_\infty^2}{M}.$$

354

355 Multiply both sides of Equation (3.7) by  $\nu(H_k)$ , sum over  $k$ , and use the norm-stability  
 356 property (Assumption 3.2): it readily follows that

$$357 \quad \sum_{k=1}^K \nu(H_k) \mathbb{E} \left[ \frac{1}{M} \sum_{m=1}^M (y_{i+1}^{(M)} - y_{i+1})^2 (X_{i+1}^{i,k,m}) \right]$$

$$358 \quad \leq 2\mathbb{E} \left[ |y_{i+1}^{(M)}(X_{i+1}^{i,\nu}) - y_{i+1}(X_{i+1}^{i,\nu})|^2 \right] + 4c_{(3.7)}(M) \frac{|y_{i+1}|_\infty^2}{M}$$

$$359 \quad \leq 2C_{(3.1)} \mathbb{E} \left[ |y_{i+1}^{(M)} - y_{i+1}|_\nu^2 \right] + 4c_{(3.7)}(M) \frac{|y_{i+1}|_\infty^2}{M}.$$

360

361 Similarly, we can get from Equation (3.8) that

$$362 \quad \mathbb{E} \left[ |y_i^{(M)} - y_i|_\nu^2 \right] \leq 2 \sum_{k=1}^K \nu(H_k) \mathbb{E} \left[ |y_i^{(M)} - y_i|_{i,k,M}^2 \right] + 4c_{(3.8)}(M) \frac{|y_i|_\infty^2}{M}.$$

363

364 Finally, by combining the above estimates with (3.6), we get

$$\begin{aligned}
365 \quad \mathbb{E} \left[ |y_i^{(M)} - y_i|_\nu^2 \right] &\leq 4c_{(3.8)}(M) \frac{|y_i|_\infty^2}{M} + 2 \sum_{k=1}^K \nu(H_k) T_{i,k} + 2 \left(1 + \frac{1}{\varepsilon}\right) \frac{\dim(\mathcal{L})}{M} (C_{g_i} + L_{g_i} |y_{i+1}|_\infty)^2 \\
366 \quad &+ 2(1 + \varepsilon) L_{g_i}^2 \left( 2C_{(3.1)} \mathbb{E} \left[ |y_{i+1}^{(M)} - y_{i+1}|_\nu^2 \right] + 4c_{(3.7)}(M) \frac{|y_{i+1}|_\infty^2}{M} \right). \\
367
\end{aligned}$$

368 This links  $\mathbb{E} \left[ |y_i^{(M)} - y_i|_\nu^2 \right]$  with  $\mathbb{E} \left[ |y_{i+1}^{(M)} - y_{i+1}|_\nu^2 \right]$  as announced.

**4. Convergence analysis for the solution of BSDEs with the MDP representation.** Let us consider the semi-linear final value problem for a parabolic PDE of the form

$$\begin{cases} \partial_t u(t, x) + \frac{1}{2} \Delta u(t, x) + f(t, u(t, x), x) = 0, & t < 1, x \in \mathbb{R}^d, \\ u(1, x) = g(x). \end{cases}$$

This is a simple form of the Hamilton-Jacobi-Bellman equation of stochastic control problems [13]. Under fairly mild assumptions (see [16] for instance), the solution to the above PDE is related to a Backward Stochastic Differential Equation  $(\mathcal{Y}, \mathcal{Z})$  driven by a  $d$ -dimensional Brownian motion  $W$ . Namely,

$$\mathcal{Y}_t = g(W_1) + \int_t^1 f(s, \mathcal{Y}_s, W_s) ds - \int_t^1 \mathcal{Z}_s dW_s$$

and  $\mathcal{Y}_t = u(t, W_t)$ ,  $\mathcal{Z}_t = \nabla u(t, W_t)$ . Needless to say, the Laplacian  $\Delta$  and the process  $W$  could be replaced by a more general second order operator and its related diffusion process, and that  $f$  could depend on the gradient  $Z$  as well. We stick to the above setting which is consistent with this work. Taking conditional expectation reduces to

$$\mathcal{Y}_t = \mathbb{E} \left[ g(W_1) + \int_t^1 f(s, \mathcal{Y}_s, W_s) ds \mid W_t \right].$$

369 There are several time discretization schemes of  $\mathcal{Y}$  (explicit or implicit Euler  
370 schemes, high order schemes [6]) but here we follow the Multi-Step Forward Dynamic  
371 Programming (MDP for short) Equation of [9], which allows a better error propagation  
372 compared to the One-Step Dynamic Programming Equation:

$$\begin{aligned}
373 \quad Y_i &= \mathbb{E} \left[ g_N(X_N) + \frac{1}{N} \sum_{j=i+1}^N f_j(Y_j, X_j, \dots, X_N) \mid X_i \right] = y_i(X_i), \quad 0 \leq i < N. \\
374
\end{aligned}$$

Here, we consider a more general path-dependency on  $f_j$ , actually this does not affect the error analysis. In comparison with Algorithm 1, we take

$$g_i(y_{i+1:N}, x_{i:N}) = g_N(x_N) + \frac{1}{N} \sum_{j=i+1}^N f_j(y_j, x_{j:N}).$$

375 In [2] similar discrete BSDEs appear but with an external noise. That corresponds to  
376 time-discretization of Backward Doubly SDEs, which in turn are related to stochastic  
377 semi-linear PDEs.

378 **4.1. Standing assumptions.** We shall now describe the main assumptions that  
 379 are needed in the methodology proposed in this paper.

380 **4.1.1. Assumptions on  $f_i$  and  $g_N$ .**

381 ASSUMPTION 4.1 (Functions  $f_i$  and  $g_N$ ). *Each  $f_i$  is Lipschitz w.r.t.  $y_i$ , with Lip-*  
 382 *schitz constant  $L_{f_i}$  and  $C_{f_i} = \sup_{x_{i:N}} |f_i(0, x_{i:N})| < +\infty$ . Moreover  $g_N$  is bounded.*

383 The reader can easily check that  $y_i$  is bounded.

384 **4.1.2. Assumptions on the distribution  $\nu$ .**

385 ASSUMPTION 4.2 (norm-stability). *There exists a constant  $\underline{C}_{(4.1)} \geq 1$  such that*  
 386 *for any  $\varphi \in L^2(\nu)$  and any  $0 \leq i < j \leq N$ , we have*

$$387 \quad (4.1) \quad \int_{\mathbb{R}^d} \mathbb{E} \left[ \varphi^2(X_j^{i,x}) \right] \nu(dx) \leq \underline{C}_{(4.1)} \int_{\mathbb{R}^d} |\varphi(x)|^2 \nu(dx).$$

388 It is straightforward to extend Propositions 3.1 and 3.2 to fulfill the above assumption.

**4.2. Main result: error estimate.** We express the error in terms of the best  
 local approximation error and the averaged one:

$$T_{i,k} := \inf_{\varphi \in \mathcal{L}_k} |y_i - \varphi|_{\nu_k}^2, \quad \nu(T_{i,\cdot}) := \sum_{k=1}^K \nu(H_k) T_{i,k}.$$

389 In this discrete time BSDE context, Theorem 3.4 becomes the following.

THEOREM 4.1. *Assume Assumptions 2.2-2.3-3.3-4.2 and define  $y_i^{(M)}$  as in Algo-*  
 rithm 1. Set

$$\bar{\mathcal{E}}(Y, M, i) := \mathbb{E} \left[ |y_i^{(M)} - y_i|_{\nu}^2 \right] = \sum_{k=1}^K \nu(\mathcal{H}_k) \mathbb{E} \left[ |y_i^{(M)} - y_i|_{\nu_k}^2 \right].$$

390 Define

$$391 \quad \delta_i = 4c_{(3.8)}(M) \frac{|y_i|_{\infty}^2}{M} + 2\nu(T_{i,\cdot}) + 16 \frac{1}{N} \sum_{j=i+1}^{N-1} L_{f_j}^2 c_{(3.7)}(M) \frac{|y_j|_{\infty}^2}{M} +$$

$$392 \quad + 4 \frac{\dim(\mathcal{L})}{M} \left( |y_N|_{\infty} + \frac{1}{N} \sum_{j=i+1}^N (C_{f_j} + L_{f_j} |y_j|_{\infty}) \right)^2.$$

393 Then, letting  $L_f := \sup_j L_{f_j}$ , we have

$$395 \quad \bar{\mathcal{E}}(Y, M, i) \leq \delta_i + 8\underline{C}_{(4.1)} L_f^2 \exp \left( 8\underline{C}_{(4.1)} L_f^2 \right) \frac{1}{N} \sum_{j=i+1}^{N-1} \delta_j.$$

396 The above general error estimates become simpler when the parameters are uniform  
 397 in  $i$ .

399 COROLLARY 4.1. *Under the assumptions of Theorem 4.1 and assuming that  $C_{f_i}, L_{f_i}$*   
 400 *and  $|y_i|_{\infty}$  are bounded uniformly in  $i$  and  $N$ , there exists a constant  $C_{(4.2)}$  (independ-*  
 401 *ent of  $N$  and of approximation spaces  $\mathcal{L}_k$ ) such that*

$$(4.2)$$

$$402 \quad \bar{\mathcal{E}}(Y, M, i) \leq C_{(4.2)} \left( \frac{c_{(3.8)}(M) + c_{(3.7)}(M) + \dim(\mathcal{L})}{M} + \nu(T_{i,\cdot}) + \frac{1}{N} \sum_{j=i+1}^{N-1} \nu(T_{j,\cdot}) \right).$$

403

404 We observe that this upper bound is expressed in a quite convenient form to let  
 405  $N \rightarrow +\infty$  and  $K \rightarrow +\infty$ . As a major difference with the usual Regression Monte  
 406 Carlo schemes, the impact of the statistical error (through the parameter  $M$ ) is not  
 407 affected by the number  $K$  of strata.

408 **4.3. Proof of Theorem 4.1.** We follow the arguments of the proof of Theo-  
 409 rem 3.4 with the following notation:

$$410 \quad S(x_{i:N}) := g_N(x_N) + \frac{1}{N} \sum_{j=i+1}^N f_j(y_j(x_j), x_{j:N}),$$

$$411 \quad S^{(M)}(x_{i:N}) := g_N(x_N) + \frac{1}{N} \sum_{j=i+1}^N f_j(y_j^{(M)}(x_j), x_{j:N}),$$

$$412 \quad \psi_i^k := \text{OLS}(S, \mathcal{L}_k, X_{i:N}^{i,k,1:M}), \quad \psi_i^{(M),k} := \text{OLS}(S^{(M)}, \mathcal{L}_k, X_{i:N}^{i,k,1:M}).$$

414 The beginning of the proof is similar and we obtain (here, there is no need to optimize  
 415  $\varepsilon$  and we take  $\varepsilon = 1$ )

$$416 \quad \mathbb{E} \left[ |y_i^{(M)} - y_i|_{i,k,M}^2 \right] \leq \mathbb{E} \left[ |\psi_i^{(M),k} - y_i|_{i,k,M}^2 \right]$$

$$417 \quad \leq T_{i,k} + 2\mathbb{E} \left[ |\psi_i^{(M),k} - \psi_i^k|_{i,k,M}^2 \right] + 2\mathbb{E} \left[ \left| \psi_i^k - \mathbb{E} \left[ \psi_i^k(\cdot) | X_i^{i,k,1:M} \right] \right|_{i,k,M}^2 \right].$$

419 The last term is a statistical error term, which can be controlled as follows:

$$420 \quad \mathbb{E} \left[ \left| \psi_i^k - \mathbb{E} \left[ \psi_i^k(\cdot) | X_i^{i,k,1:M} \right] \right|_{i,k,M}^2 \right] \leq \frac{\dim(\mathcal{L})}{M} \left( |y_N|_\infty + \frac{1}{N} \sum_{j=i+1}^N (C_{f_j} + L_{f_j} |y_j|_\infty) \right)^2$$

422 where  $(\dots)^2$  is a rough bound of the conditional variance of  $S(X_{i:N}^{i,k})$ .

423 We handle the control of the term  $\mathbb{E} \left[ |\psi_i^{(M),k} - \psi_i^k|_{i,k,M}^2 \right]$  as in Theorem 3.4 but  
 424 the results are different because the dynamic programming equation differs:

$$425 \quad \mathbb{E} \left[ |\psi_i^{(M),k} - \psi_i^k|_{i,k,M}^2 \right] \leq \mathbb{E} \left[ |S^{(M)} - S|_{i,k,M}^2 \right]$$

$$426 \quad \leq \mathbb{E} \left[ \frac{1}{M} \sum_{m=1}^M \left( \frac{1}{N} \sum_{j=i+1}^{N-1} L_{f_j} |y_j^{(M)} - y_j| (X_j^{i,k,m}) \right)^2 \right]$$

$$427 \quad \leq \mathbb{E} \left[ \frac{1}{M} \sum_{m=1}^M \frac{1}{N} \sum_{j=i+1}^{N-1} L_{f_j}^2 |y_j^{(M)} - y_j|^2 (X_j^{i,k,m}) \right].$$

429 We multiply the above by  $\nu(H_k)$ , sum over  $k$ , apply the *extended* Proposition 3.5  
 430 valid also for the problem at hand, and the Assumption 4.2. Then, it follows that

$$431 \quad \sum_{k=1}^K \nu(H_k) \mathbb{E} \left[ |\psi_i^{(M),k} - \psi_i^k|_{i,k,M}^2 \right] \leq 2 \frac{1}{N} \sum_{j=i+1}^{N-1} L_{f_j}^2 \left( \underline{C}_{(4.1)} \mathbb{E} \left[ |y_j^{(M)} - y_j|_\nu^2 \right] + 2c_{(3.7)}(M) \frac{|y_j|_\infty^2}{M} \right).$$

433 On the other hand, from Equation (3.8) we have

$$434 \quad \mathbb{E} \left[ |y_i^{(M)} - y_i|_\nu^2 \right] \leq 2 \sum_{k=1}^K \nu(H_k) \mathbb{E} \left[ |y_i^{(M)} - y_i|_{i,k,M}^2 \right] + 4c_{(3.8)}(M) \frac{|y_i|_\infty^2}{M}.$$



435

436 Now collect the different estimates: it writes

$$\begin{aligned}
437 \quad \bar{\mathcal{E}}(Y, M, i) &:= \mathbb{E} \left[ |y_i^{(M)} - y_i|_\nu^2 \right] \leq 4c_{(3.8)}(M) \frac{|y_i|_\infty^2}{M} + 2 \sum_{k=1}^K \nu(H_k) T_{i,k} \\
438 \quad &+ 8 \frac{1}{N} \sum_{j=i+1}^{N-1} L_{f_j}^2 \left( \underline{C}_{(4.1)} \mathbb{E} \left[ |y_j^{(M)} - y_j|_\nu^2 \right] + 2c_{(3.7)}(M) \frac{|y_j|_\infty^2}{M} \right) + \\
439 \quad &+ 4 \frac{\dim(\mathcal{L})}{M} \left( |y_N|_\infty + \frac{1}{N} \sum_{j=i+1}^N (C_{f_j} + L_{f_j} |y_j|_\infty) \right)^2 \\
440 \quad &:= \delta_i + 8\underline{C}_{(4.1)} \frac{1}{N} \sum_{j=i+1}^{N-1} L_{f_j}^2 \bar{\mathcal{E}}(Y, M, j).
\end{aligned}$$

441

442 It takes the form of a discrete Gronwall lemma, which easily allows to derive the  
443 following upper bound (see [2, Appendix A.3]):

$$\begin{aligned}
444 \quad \bar{\mathcal{E}}(Y, M, i) &\leq \delta_i + 8\underline{C}_{(4.1)} \frac{1}{N} \sum_{j=i+1}^{N-1} \Gamma_{i,j} L_{f_j}^2 \delta_j, \\
445 \quad \text{where } \Gamma_{i,j} &:= \begin{cases} \prod_{i < k < j} (1 + 8\underline{C}_{(4.1)} \frac{1}{N} L_{f_k}^2), & \text{for } i+1 < j, \\ 1, & \text{otherwise.} \end{cases}
\end{aligned}$$

446

447 Using now  $L_f = \sup_j L_{f_j}$ , we get  $\Gamma_{i,j} \leq \exp \left( \sum_{i < k < j} 8\underline{C}_{(4.1)} \frac{1}{N} L_{f_k}^2 \right) \leq \exp(8\underline{C}_{(4.1)} L_f^2)$ .

448

This completes the proof.

449

450 **5. Numerical tests.** We shall now illustrate the methodology in two numerical  
451 examples coming from practical problems. The first one concerns a reaction-diffusion  
452 PDE connected to spatially distributed populations, whereas the second one deals  
453 with a stochastic control problem.

453

454 **5.1. An Application to Reaction-Diffusion Models in Spatially Dis-**  
455 **tributed Populations.** In this section we consider a biologically motivated example  
456 to illustrate the strength of the stratified resampling regression methodology pre-  
457 sented in the previous sections. We selected an application to spatially distributed  
458 populations that evolve under reaction diffusion equations. Besides the theoretical  
459 challenges behind the models, it has recently attracted attention due to its impact  
460 in the spread of infectious diseases [14, 15] and even to the modeling of Wolbachia  
461 infected mosquitoes in the fight of disease spreading *Aedes aegypti* [3, 11].

461

462 The use of reaction diffusion models to describe the population dynamics of a sin-  
463 gle species or genetic trait expanding into new territory dominated by another one goes  
464 back to the work of R. A. Fisher [7] and A. Kolmogorov et al. [12]. The mathematical  
465 model behind it is known as the (celebrated) Fisher-Kolmogorov-Petrovski-Piscounov  
466 (FKPP) equation.

466

In a conveniently chosen scale it takes the form, in dimension 1,

$$467 \quad (5.1) \quad \partial_t u + \partial_x^2 u + au(1-u) = 0, \quad u(T, x) = h(x), \quad x \in \mathbb{R}, t \leq T,$$

468

where  $u = u(t, x)$  refers to the proportion of members of an invading species in a

469

spatially distributed population on a straight line. The equation is chosen with time

470 running backwards and as a final value problem to allow direct connection with the  
 471 standard probabilistic interpretation.

472 It is well known [1] that for any arbitrary positive  $C$ , if we define

473 (5.2) 
$$h(x) := \left(1 + C \exp\left(\pm \frac{\sqrt{6a}}{6}x\right)\right)^{-2}$$

then

$$u(t, x) = \left(1 + C \exp\left(\frac{5a}{6}(t - T) \pm \frac{\sqrt{6a}}{6}x\right)\right)^{-2}$$

474 is a traveling wave solution to Equation (5.1). The behavior of  $h(x)$  as  $x \rightarrow \pm\infty$  is  
 475 either one or zero according to the sign chosen inside the exponential. Thus describing  
 476 full dominance of the invading species or its absence.

477 The probabilistic formulation goes as follows: Introduce the system, as in Sec-  
 478 tion 4,

479 
$$dP_s = \sqrt{2}dW_s ,$$
  
 480 
$$dY_s = -f(Y_s)ds + Z_s dW_s, \text{ where } f(x) = ax(1 - x), \text{ and } Z_s = \sqrt{2}\partial_x u(s, P_s)$$
  
 481 
$$Y_T = u(T, P_T) = h(P_T) .$$

482 Then, the process  $Y_t = \mathbb{E} \left[ Y_T + \int_t^T f(Y_s)ds | P_t \right]$  satisfies  $Y_t = u(t, P_t)$ .

483 To test the algorithms presented herein, we shall start with the following more  
 484 general parabolic PDE

485 (5.3) 
$$\partial_t W + \sum_{1 \leq i, j \leq d} A_{ij} \partial_{y_i} \partial_{y_j} W + aW(1 - W) = 0 , t \leq T, \text{ and } y \in \mathbb{R}^d .$$

486 Here, the matrix  $A$  is chosen as an arbitrary *positive-definite* constant  $d \times d$  matrix.  
 487 Furthermore, we choose, for convenience, the final condition

488 (5.4) 
$$W(T, y) = h(y' \Sigma^{-1} \theta) ,$$

489 where  $\Sigma = \Sigma' = \sqrt{A}$  and  $\theta$  is arbitrary unit vector. We stress that this special choice  
 490 of the final condition has the sole purpose of bypassing the need of solving Equa-  
 491 tion (5.4) by other numerical methods for comparison with the present methodology.  
 492 Indeed, the fact that we are able to exhibit an explicit solution to Equation (5.3) with  
 493 final condition (5.4) allows an easy checking of the accuracy of the method. We also  
 494 stress that the method developed in this work does not require an explicit knowledge  
 495 of the diffusion coefficient matrix  $A$  of Equation (5.3) since we shall make use of the  
 496 observed paths. Yet the knowledge of the function  $W \mapsto aW(1 - W)$  is crucial.

497 It is easy to see that if  $u = u(t, x)$  satisfies Equation (5.1) with final condition  
 498 given by Equation (5.2) then

499 (5.5) 
$$W(t, y) := u(t, y' \Sigma^{-1} \theta)$$

500 satisfies Equation (5.3) with final condition (5.4).

501 An interpretation of the methodology proposed here is the following: If we were  
 502 able to observe the trajectories performed by a small number of free individuals ac-  
 503 cording to the diffusion process associated to Equation (5.3), even if we did not know  
 504 the explicit form of the diffusion (i.e., we did not have a good calibration of the co-  
 505 variance matrix) we could use such trajectories to produce a reliable solution to the  
 506 final value problem (5.3) and (5.4).

We firstly present some numerical results in dimension 1 (Tables 1-2-3). We have tested both the one-step (Section 3) and multi-step schemes (Section 4). The final time  $T$  is fixed to 1 and we use time discretization  $t_i = \frac{i}{N}T, 0 \leq i \leq N$  with  $N = 10$  or 20. We divide the real line  $\mathbb{R}$  into  $K$  subintervals  $(I_i)_{1 \leq i \leq K}$  by fixing  $A = 25$  and dividing  $[-A, A]$  into  $K - 2$  equal length intervals and then adding  $(-\infty, -A)$  and  $(A, +\infty)$ . We implement our method by using piecewise constant estimation on each interval. Then finally we get a piecewise constant estimation of  $u(0, y)$ , noted as  $\hat{u}(0, y)$ . Then we approximate the squared  $L^2(\nu)$  error of our estimation by

$$\sum_{1 \leq k \leq K} |u(0, y_k) - \hat{u}(0, y_k)|^2 \nu(I_k)$$

where  $y_k$  is chosen as the middle point of the rectangle if  $I_k$  is finite and the boundary point if  $I_k$  is infinite. We take  $\nu(dx) = \frac{1}{2}e^{-|x|}dx$  and we use the restriction of  $\nu$  on  $I_k$  to sample initial points. The squared  $L^2(\nu)$  norm of  $u(0, \cdot)$  is around 0.25. Finally remark that the error of our method includes three parts: time discretization error, approximation error due to the use of piecewise constant estimation on hypercubes and statistical error due to the randomness of trajectories. In the following tables,  $M$  is the number of trajectories that we use (i.e., the root sample).

We observe in Tables 1 and 3 that the approximation error (visible for small  $K$ ) contributes much more to the global error for the one-step scheme, compared to the multi-step one. This observation is in agreement with those of [5, 9].

When  $N$  gets larger with fixed  $K$  and  $M$  (Tables 1 and 2), we may observe an increase of the global error for the one-step scheme, this is coherent with the estimates of Corollary 3.1.

	$K = 10$	$K = 20$	$K = 50$	$K = 100$	$K = 200$	$K = 400$
$M = 20$	0.0993	0.0253	0.0038	0.0014	0.0014	0.0019
$M = 40$	0.0997	0.0252	0.0034	9.01e-04	5.16e-04	6.17e-04
$M = 80$	0.0993	0.0249	0.0029	6.15e-04	3.92e-04	3.91e-04
$M = 160$	0.0990	0.0248	0.0029	3.15e-04	1.57e-04	1.71e-04
$M = 320$	0.0990	0.0248	0.0028	2.47e-04	1.02e-04	1.19e-04
$M = 640$	0.0990	0.0246	0.0028	2.26e-04	5.46e-05	4.94e-05

TABLE 1  
Average squared  $L^2$  errors with 50 macro runs,  $N = 10$ , one-step scheme.

	$K = 10$	$K = 20$	$K = 50$	$K = 100$	$K = 200$	$K = 400$
$M = 20$	0.1031	0.0299	0.0073	0.0018	0.0011	0.0012
$M = 40$	0.1031	0.0294	0.0066	0.0014	7.86e-04	7.28e-04
$M = 80$	0.1027	0.0293	0.0065	0.0010	3.18e-04	3.86e-04
$M = 160$	0.1027	0.0294	0.0064	8.91e-04	2.46e-04	1.04e-04
$M = 320$	0.1026	0.0293	0.0064	8.39e-04	1.42e-04	7.03e-05
$M = 640$	0.1027	0.0292	0.0063	8.04e-04	8.16e-05	5.60e-05

TABLE 2  
Average squared  $L^2$  errors with 50 macro runs,  $N = 20$ , one-step scheme.

519

Table 4 below describes numerical results in dimension 2. The final time  $T$  is fixed to 1 and we use the time discretization  $t_i = \frac{i}{N}T, 0 \leq i \leq N$  with  $N = 10$ . We divide the real line  $\mathbb{R}$  into  $K$  subintervals  $(I_i)_{1 \leq i \leq K}$  by fixing  $A = 25$  and dividing

	$K = 10$	$K = 20$	$K = 50$	$K = 100$	$K = 200$	$K = 400$
$M = 20$	0.0484	0.0066	0.0017	0.0015	0.0011	0.0013
$M = 40$	0.0488	0.0058	8.45e-04	5.81e-04	6.35e-04	5.68e-04
$M = 80$	0.0478	0.0053	4.33e-04	2.96e-04	3.45e-04	4.06e-04
$M = 160$	0.0481	0.0051	2.98e-04	2.23e-04	1.71e-04	1.08e-04
$M = 320$	0.0479	0.0051	1.79e-04	6.48e-05	8.38e-05	1.04e-04
$M = 640$	0.0478	0.0050	1.50e-04	6.49e-05	6.66e-05	5.70e-05

TABLE 3

Average squared  $L^2$  errors with 50 macro runs,  $N = 10$ , multi-step scheme.

$[-A, A]$  into  $K - 2$  equal length intervals and then adding  $(-\infty, -A)$  and  $(A, +\infty)$ . We take  $\Sigma = [1, \beta; \beta, 1]$  with  $\beta = 0.25$  and  $\theta = \frac{[1;1]}{\sqrt{2}}$ . We implement our method by using piecewise constant estimation on each finite (or infinite) rectangle  $I_i \times I_j$ . Then finally we get a piecewise constant estimation of  $W(0, y)$ , noted as  $\hat{W}(0, y)$ . Then we approximate the squared  $L^2(\nu \otimes \nu)$  error of our estimation by

$$\sum_{1 \leq k_1 \leq K, 1 \leq k_2 \leq K} |W(0, y_{k_1}, y_{k_2}) - \hat{W}(0, y_{k_1}, y_{k_2})|^2 \nu \otimes \nu(I_{k_1} \times I_{k_2})$$

520 where  $(y_{k_1}, y_{k_2})$  is chosen as the middle point of the rectangle if  $I_{k_1} \times I_{k_2}$  is finite and  
 521 the boundary point if one or both of  $I_{k_1}$  and  $I_{k_2}$  are infinite. We take  $\nu(dx) = \frac{1}{2}e^{-|x|}dx$   
 522 and we use the restriction of  $\nu \otimes \nu$  on  $I_i \times I_j$  to sample initial points. The squared  
 523  $L^2(\nu \otimes \nu)$  norm of  $W(0, \cdot, \cdot)$  is around 0.25.

	$K = 10$	$K = 20$	$K = 50$	$K = 100$	$K = 200$
$M = 20$	0.0592	0.0167	0.0027	0.0018	0.0010
$M = 40$	0.0588	0.0163	0.0022	5.34e-04	5.00e-04
$M = 80$	0.0588	0.0160	0.0019	3.74e-04	2.98e-04
$M = 160$	0.0586	0.0160	0.0018	3.08e-04	9.16e-05
$M = 320$	0.0586	0.0159	0.0017	1.1e-04	9.24e-05

TABLE 4

Average squared  $L^2$  errors with 50 macro runs,  $N = 10$ , one-step scheme.

524 As for the previous case in dimension 1, we observe that when  $K$  is small, it  
 525 is useless to increase  $M$ . This is because in such case the approximation error is  
 526 dominant. But when  $K$  is large enough, the performance of our method improves  
 527 when  $M$  becomes larger, since this time it is the statistical error which becomes  
 528 dominant and larger  $M$  means smaller statistical error.

529 In the perspective of a given root sample ( $M$  fixed), it is recommended to take  $K$   
 530 large: indeed, in agreement with Theorems 3.4 and 4.1, we observe from the numerical  
 531 results that the global error decreases up to the statistical error term (depending on  
 532  $M$  but not  $K$ ). In this way, for  $M = 20$  (resp.  $M = 40$ ) the relative squared  $L^2$  error  
 533 is about 0.4% (resp. 0.22%).

534 **5.2. Travel agency problem: when to offer travels, according to cur-**  
 535 **rency and weather forecast....** In this section we illustrate the stratified resampler  
 536 methodology in the solution of an optimal investment problem. The underlying model  
 537 will have two sources of stochasticity, one related to the weather and the other one  
 538 to the exchange rate. The corresponding stochastic processes shall be denoted by  $X_t^1$   
 539 and  $X_t^2$ .

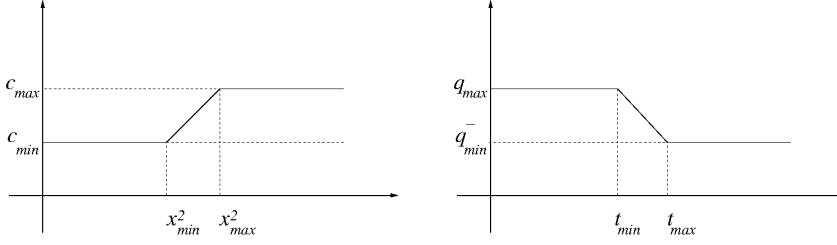


FIG. 2. Pictorial description of the cost function  $c$  (left) and of the campaign effectiveness  $q$  (right).

540 We envision the following situation: A travel agency wants to launch a campaign  
 541 for the promotion of vacations in a warm region abroad during the Fall-Winter season.  
 542 Such travel agency would receive a fixed value  $\underline{c}$  in local currency from the customers  
 543 and on the other hand would have to pay the costs  $c = c(\exp(X_{\tau+1/12}^2))$  in a future  
 544 time  $\tau + 1/12$ , where  $\tau$  is the launching time of the campaign and  $X_{\tau+1/12}^2$  is the  
 545 prevailing logarithm of exchange rate one month after the launching, with the time  
 546 unit set to be one year. The initial time  $t = 0$  is by convention October 1st. In other  
 547 words, the costs are fixed to the traveler and variable for the agency. A pictorial  
 548 description of the cost function is presented in Figure 2.

549 The effectiveness of the campaign will depend on the local temperature  $(t -$   
 550  $0.25)^2 \times 240 + X_t^1$  (in Celsius) and will be denoted by  $q((t - 0.25)^2 \times 240 + X_t^1) \exp(-|t -$   
 551  $1/6|)$ , where  $(t - 0.25)^2 \times 240$  represents the seasonal component and  $X_t^1$  represents  
 552 the random part. Its purpose is to capture the idea that if the local temperature  
 553 is very low, then people would be more interested in spending some days in a warm  
 554 region, whereas if the weather is mild then people would just stay at home. A pictorial  
 555 description of the function  $q$  is presented in Figure 2. The second part of this function  
 556  $\exp(-|t - 1/6|)$  is created to represent the fact that there are likely more registrations  
 557 at beginning of December for the period of new year holidays.

558 Thus, our problem consists of finding the function  $v$  defined by

$$559 \quad v(X_0^1, X_0^2) = \operatorname{ess\,sup}_{\tau \in \mathcal{T}} \mathbb{E} \left[ q((\tau - 0.25)^2 \times 240 + X_\tau^1) e^{-|\tau - 1/6|} \left( \underline{c} - c(e^{X_{\tau+1/12}^2}) \right) \mid X_0^1, X_0^2 \right]$$

$$560 \quad = \operatorname{ess\,sup}_{\tau \in \mathcal{T}} \mathbb{E} \left[ q((\tau - 0.25)^2 \times 240 + X_\tau^1) e^{-|\tau - 1/6|} \left( \underline{c} - \mathbb{E} \left[ c(e^{X_{\tau+1/12}^2}) \mid X_\tau^2 \right] \right) \mid X_0^1, X_0^2 \right],$$

562 where  $\mathcal{T}$  denotes the set of stopping times valued in the weeks of the Fall-Winter  
 563 seasons  $\{\frac{k}{48}, k = 0, 1, \dots, 24\}$ , which corresponds to possible weekly choices for the  
 564 travel agency to launch the campaign. The above function  $v$  models the optimal  
 565 expected benefit for the travel agency and the optimal  $\tau$  gives the best launching  
 566 time. We shall assume, for simplicity, that the processes  $X^1$  and  $X^2$  are uncorrelated  
 567 since we do not expect much influence of the weather on the exchange rate or vice-  
 568 versa.

569 The problem is tackled by formulating it as a dynamic programming one related  
 570 to optimal stopping problems (as exposed in Section 3) using a mean-reversion process  
 571 for the underlying process  $X^1$  and a drifted Brownian motion for  $X^2$ . Their dynamics  
 572 are given as follows:

$$573 \quad dX_t^1 = -aX_t^1 dt + \sigma_1 dW_t, X_0^1 = 0, \quad X_t^2 = -\frac{\sigma_2^2}{2} t + \sigma_2 B_t.$$

574

	$K = 10$	$K = 20$	$K = 50$	$K = 100$
$M = 20$	0.1827	0.0512	0.0349	0.0269
$M = 40$	0.1982	0.0361	0.0249	0.0114
$M = 80$	0.2063	0.0325	0.0051	0.0047
$M = 160$	0.1928	0.0264	0.0058	0.0067

TABLE 5

Average squared  $L^2$  errors with 20 macro runs. Simple regression.

	$K = 10$	$K = 20$	$K = 50$	$K = 100$
$M = 20$	0.1711	0.0458	0.0436	0.0252
$M = 40$	0.1648	0.0361	0.0130	0.0169
$M = 80$	0.1534	0.0273	0.0109	0.0085
$M = 160$	0.1510	0.0296	0.0048	0.0058

TABLE 6

Average squared  $L^2$  errors with 20 macro runs. Nested regression.

575 The cost function  $c$  is chosen piecewise linear so that we can get  $\mathbb{E}(c(e^{X_\tau^2+1/12})|X_\tau^2)$   
 576 explicitly as a function of  $X_\tau^2$  using the Black-Scholes formula in mathematical fi-  
 577 nance. Thus we can run our method in two different ways: either using this explicit  
 578 expression and apply directly the regression scheme of Section 3; or first estimating  
 579  $\mathbb{E}(c(e^{X_\tau^2+1/12})|X_\tau^2)$  by stratified regression then plugging the estimate in our method  
 580 again to get a final estimation. We refer to these two different ways as simple regres-  
 581 sion and nested regression. The latter case corresponds to a coupled two-component  
 582 regression problem (that could be mathematically analyzed very similarly to Sec-  
 583 tion 3).

584 The parameter’s values are given as:  $a = 2, \sigma_1 = 10, \sigma_2 = 0.2, \underline{c} = 3, x_{min}^2 =$   
 585  $e^{-0.5}, x_{max}^2 = e^{0.5}, c_{min} = 1, c_{max} = c_{min} + x_{max}^2 - x_{min}^2, t_{min} = 0, t_{max} = 15, q_{min} =$   
 586  $1, q_{max} = 4$ . We use the restriction of  $\mu(dx) = \frac{k}{2}(1+|x|)^{-k-1}dx$  with  $k = 6$  to sample  
 587 point for  $X^1$  and the restriction of  $\nu(dx) = \frac{1}{2}e^{-|x|}dx$  to sample points for  $X^2$ . Note  
 588 that  $k = 6$  means that, in the error estimation, more weight is distributed to the  
 589 region around  $X_0^1 = 0$ , which is the real interesting information for the travel agency.

590 We will firstly run our method with  $M = 320$  and  $K = 300$  to get a reference value  
 591 for  $v$  then our estimators will be compared to this reference value in a similar way as  
 592 in the previous example. The squared  $L^2(\mu \otimes \nu)$  norm of our reference estimation is  
 593 32.0844. The results are displayed in the Tables 5 and 6.

594 As in Subsection 5.1 and in agreement with Theorem 3.4, we observe an improved  
 595 accuracy as  $K$  and  $M$  increases, independently of each other. The relative error is  
 596 rather small even for small  $M$ .

597 Interestingly, the nested regression algorithm (which is the most realistic scheme  
 598 to use in practice when the model is unknown) is as accurate as the scheme using the  
 599 explicit form of the internal conditional expectation  $\mathbb{E}[c(e^{X_\tau^2+1/12}) | X_\tau^2]$ . Surpris-  
 600 ingly, the simple regression scheme takes much more time than the nested regression  
 601 one because of the numerous evaluations of the Gaussian CDF in the Black-Scholes  
 602 formula.

603 **Appendix A. Appendix.**

**A.1. Proof of Proposition 3.3.** Consider first the case of the partitioning  
 estimate  $(LP_0)$  and let  $\varepsilon \in (0, \frac{4}{15}B]$ . We use an  $\varepsilon$ -cover in the  $L^\infty$ -norm, which

simply reduces to cover  $[-B, B]$  with intervals of size  $2\epsilon$ . A solution is to take the interval center defined by  $h_j = -B + \epsilon + 2\epsilon j$ ,  $0 \leq j \leq n$ , where  $n$  is the smallest integer such that  $h_n \geq B$  (i.e.,  $n = \lceil \frac{B}{\epsilon} - \frac{1}{2} \rceil$ ). Thus, we obtain

$$\mathcal{N}_1(\epsilon, \mathcal{T}_B \mathcal{L}_k, x^{1:M}) \leq n + 1 \leq \frac{B}{\epsilon} + \frac{3}{2} \leq \frac{7}{5} \frac{B}{\epsilon}$$

604 where we use the constraint on  $\epsilon$ .

In the case of general vector space of dimension  $K$ , from [10, Lemma 9.2, Theorem 9.4 and Theorem 9.5], we obtain

$$\mathcal{N}_1(\epsilon, \mathcal{T}_B \mathcal{K}, x^{1:M}) \leq 3 \left( \frac{4eB}{\epsilon} \log \left( \frac{6eB}{\epsilon} \right) \right)^{K+1}$$

whenever  $\epsilon < B/2$ . For  $\epsilon$  as in the statement of Assumption 3.3, we have  $\frac{6eB}{\epsilon} \geq \frac{45e}{2}$ . Let  $\eta > 0$ , since  $\log(x) \leq c_\eta x^\eta$  for any  $x \geq \frac{45e}{2}$  with  $c_\eta = \sup_{x \geq \frac{45e}{2}} \frac{\log(x)}{x^\eta}$ , we get

$$\mathcal{N}_1(\epsilon, \mathcal{T}_B \mathcal{K}, x^{1:M}) \leq 3 \left( [4c_\eta 6^\eta]^{1/(1+\eta)} \frac{eB}{\epsilon} \right)^{(K+1)(1+\eta)}.$$

605 For  $\mathbf{LP}_1$  and  $\mathbf{LP}_n$ , we have respectively  $K = d + 1$  and  $K = (d + 1)^n$ , therefore the  
606 announced result. Whenever useful, the choice  $\eta = 1$  gives  $\beta_{(3.3)} \leq 3.5$ .

607 For the partitioning estimate (case  $\mathbf{LP}_0$ ), we could also use this estimate with  
608  $K = 1$  but with the first arguments, we get better parameters (especially for  $\gamma$ ).

### 609 A.2. Probability of uniform deviation.

610 LEMMA A.1 ([9, Lemma B.2]). *Let  $\mathcal{G}$  be a countable set of functions  $g : \mathbb{R}^d \mapsto$   
611  $[0, B]$  with  $B > 0$ . Let  $\mathcal{X}, \mathcal{X}^{(1)}, \dots, \mathcal{X}^{(M)}$  ( $M \geq 1$ ) be i.i.d.  $\mathbb{R}^d$  valued random  
612 variables. For any  $\alpha > 0$  and  $\epsilon \in (0, 1)$  one has*

$$\begin{aligned} 613 \quad & \mathbb{P} \left( \sup_{g \in \mathcal{G}} \frac{\frac{1}{M} \sum_{m=1}^M g(\mathcal{X}^{(m)}) - \mathbb{E}[g(\mathcal{X})]}{\alpha + \frac{1}{M} \sum_{m=1}^M g(\mathcal{X}^{(m)}) + \mathbb{E}[g(\mathcal{X})]} > \epsilon \right) \\ 614 \quad & \leq 4\mathbb{E} \left[ \mathcal{N}_1 \left( \frac{\alpha\epsilon}{5}, \mathcal{G}, \mathcal{X}^{1:M} \right) \right] \exp \left( - \frac{3\epsilon^2 \alpha M}{40B} \right), \end{aligned}$$

$$\begin{aligned} 615 \quad & \mathbb{P} \left( \sup_{g \in \mathcal{G}} \frac{\mathbb{E}[g(\mathcal{X})] - \frac{1}{M} \sum_{m=1}^M g(\mathcal{X}^{(m)})}{\alpha + \frac{1}{M} \sum_{m=1}^M g(\mathcal{X}^{(m)}) + \mathbb{E}[g(\mathcal{X})]} > \epsilon \right) \\ 616 \quad & \leq 4\mathbb{E} \left[ \mathcal{N}_1 \left( \frac{\alpha\epsilon}{8}, \mathcal{G}, \mathcal{X}^{1:M} \right) \right] \exp \left( - \frac{6\epsilon^2 \alpha M}{169B} \right). \end{aligned}$$

### 618 A.3. Expected uniform deviation.

619 PROPOSITION A.1. *For finite  $B > 0$ , let  $\mathcal{G} := \{\psi(\mathcal{T}_B \phi(\cdot) - \eta(\cdot)) : \phi \in \mathcal{K}\}$ , where  
620  $\psi : \mathbb{R} \rightarrow [0, \infty)$  is Lipschitz continuous with  $\psi(0) = 0$  and Lipschitz constant  $L_\psi$ ,  
621  $\eta : \mathbb{R}^d \rightarrow [-B, B]$ , and  $\mathcal{K}$  is a finite  $K$ -dimensional vector space of functions with*

$$622 \quad (\text{A.1}) \quad \mathcal{N}_1(\epsilon, \mathcal{T}_B \mathcal{K}, \mathcal{X}^{1:M}) \leq \alpha \left( \frac{\beta B}{\epsilon} \right)^\gamma \quad \text{for } \epsilon \in \left( 0, \frac{4}{15} B \right]$$

623 *for some positive constants  $\alpha, \beta, \gamma$  with  $\alpha \geq 1/4$  and  $\gamma \geq 1$ . Then, for  $\mathcal{X}^{(1)}, \dots, \mathcal{X}^{(M)}$   
624 i.i.d. copies of  $\mathcal{X}$ , we have*

$$625 \quad \mathbb{E} \left[ \sup_{g \in \mathcal{G}} \left( \frac{1}{M} \sum_{m=1}^M g(\mathcal{X}^{(m)}) - 2 \int_{\mathbb{R}^d} g(x) \mathbb{P} \circ \mathcal{X}^{-1}(dx) \right)_+ \right] \leq c_{(\text{A.2})}(M) \frac{BL_\Psi}{M}$$

(A.2)

$$626 \quad \text{with } c_{(\text{A.2})}(M) := 120 \left( 1 + \log(4\alpha) + \gamma \log \left( \left( 1 + \frac{\beta}{16} \right) M \right) \right),$$

$$627 \quad \mathbb{E} \left[ \sup_{g \in \mathcal{G}} \left( \int_{\mathbb{R}^d} g(x) \mathbb{P} \circ \mathcal{X}^{-1}(\mathrm{d}x) - \frac{2}{M} \sum_{m=1}^M g(\mathcal{X}^{(m)}) \right)_+ \right] \leq c_{(\text{A.3})}(M) \frac{BL_\Psi}{M}$$

(A.3)

$$628 \quad \text{with } c_{(\text{A.3})}(M) := \frac{507}{2} \left( 1 + \log(4\alpha) + \gamma \log \left( \left( 1 + \frac{8\beta}{169} \right) M \right) \right).$$

630 *Proof.* The idea is to adapt the arguments of [9, Proposition 4.9].

631  $\triangleright$  We first show (A.2). Set  $\mathcal{Z} := \sup_{g \in \mathcal{G}} \left( \frac{1}{M} \sum_{m=1}^M g(\mathcal{X}^{(m)}) - 2 \int_{\mathbb{R}^d} g(x) \mathbb{P} \circ \mathcal{X}^{-1}(\mathrm{d}x) \right)_+$ . Let us find an upper bound for  $\mathbb{P}(\mathcal{Z} > \varepsilon)$  in order to bound  $\mathbb{E}[\mathcal{Z}] = \int_0^\infty \mathbb{P}(\mathcal{Z} > \varepsilon) \mathrm{d}\varepsilon$ . Using the equality

$$634 \quad \mathbb{P}(\mathcal{Z} > \varepsilon) = \mathbb{P} \left( \exists g \in \mathcal{G} : \frac{\frac{1}{M} \sum_{m=1}^M g(\mathcal{X}^{(m)}) - \int_{\mathbb{R}^d} g(x) \mathbb{P} \circ \mathcal{X}^{-1}(\mathrm{d}x)}{2\varepsilon + \int_{\mathbb{R}^d} g(x) \mathbb{P} \circ \mathcal{X}^{-1}(\mathrm{d}x) + \frac{1}{M} \sum_{m=1}^M g(\mathcal{X}^{(m)})} > \frac{1}{3} \right),$$

635 and that the elements of  $\mathcal{G}$  take values in  $[0, 2BL_\psi]$ , it follows from Lemma A.1 that

$$636 \quad \mathbb{P}(\mathcal{Z} > \varepsilon) \leq 4\mathbb{E} \left[ \mathcal{N}_1 \left( \frac{2\varepsilon}{15}, \mathcal{G}, \mathcal{X}^{1:M} \right) \right] \exp \left( - \frac{\varepsilon M}{120BL_\psi} \right).$$

637 Define  $\mathcal{T}_B\mathcal{K}$  as in Proposition A.1. Since  $|\psi(\phi_1(x) - \eta(x)) - \psi(\phi_2(x) - \eta(x))| \leq$   
 638  $L_\psi |\phi_1(x) - \phi_2(x)|$  for all  $x \in \mathbb{R}^d$  and all  $(\phi_1, \phi_2)$ , it follows that

$$639 \quad \mathcal{N}_1 \left( \frac{2\varepsilon}{15}, \mathcal{G}, \mathcal{X}^{1:M} \right) \leq \mathcal{N}_1 \left( \frac{2\varepsilon}{15L_\psi}, \mathcal{T}_B\mathcal{K}, \mathcal{X}^{1:M} \right).$$

640 Due to Equation (A.1), we deduce that

$$641 \quad (\text{A.4}) \quad \mathbb{P}(\mathcal{Z} > \varepsilon) \leq 4\alpha \left( \frac{15\beta BL_\psi}{2\varepsilon} \right)^\gamma \exp \left( - \frac{\varepsilon M}{120BL_\psi} \right)$$

642 whenever  $\frac{2\varepsilon}{15L_\psi} \leq \frac{4}{15}B$ , i.e.,  $\varepsilon \leq 2BL_\psi$ . On the other hand,  $\mathbb{P}(\mathcal{Z} > \varepsilon) = 0$  for all  
 643  $\varepsilon > 2BL_\psi$ . Setting  $a = \frac{15\beta BL_\psi}{2}$ ,  $b = \frac{1}{120BL_\psi}$ , it follows from (A.4) that

$$644 \quad \mathbb{P}(\mathcal{Z} > \varepsilon) \leq 4\alpha \left( \frac{a}{\varepsilon} \right)^\gamma \exp(-bM\varepsilon), \quad \forall \varepsilon > 0.$$

645 Fix  $\varepsilon_0$  to be some finite value (to be determined later) such that

$$646 \quad (\text{A.5}) \quad \varepsilon_0 \geq \frac{a}{M(1+ab)}.$$

647 It readily follows that

$$648 \quad \mathbb{E}[\mathcal{Z}] = \int_0^\infty \mathbb{P}(\mathcal{Z} > \varepsilon) \mathrm{d}\varepsilon \leq \varepsilon_0 + \int_{\varepsilon_0}^\infty 4\alpha \left( \frac{a}{\varepsilon} \right)^\gamma \exp(-bM\varepsilon) \mathrm{d}\varepsilon$$

$$649 \quad \leq \varepsilon_0 + \frac{4\alpha}{bM} (M(1+ab))^\gamma \exp(-bM\varepsilon_0).$$

650



651 We choose  $\varepsilon_0 = \frac{1}{bM} \log(4\alpha((1+ab)M)^\gamma)$ : It satisfies (A.5) since

$$652 \quad \frac{1}{bM} \log(4\alpha((1+ab)M)^\gamma) \geq \frac{a \log(1+ab)}{M ab} \geq \frac{a}{M} \frac{1}{1+ab}$$

654 (use  $\alpha \geq 1/4$ ,  $\gamma \geq 1$ ,  $M \geq 1$  and  $\log(1+x) \geq x/(1+x)$  for all  $x \geq 0$ ). Moreover, this  
655 choice of  $\varepsilon_0$  implies that

$$656 \quad (\text{A.6}) \quad \mathbb{E}[\mathcal{Z}] \leq \frac{1}{bM} \left( 1 + \log(4\alpha) + \gamma \log((1+ab)M) \right) \\ 657 \quad = \frac{120BL_\psi}{M} \left( 1 + \log(4\alpha) + \gamma \log\left(1 + \frac{\beta}{16}M\right) \right). \\ 658$$

659 The inequality (A.2) is proved.

660  $\triangleright$  We now justify (A.3) by similar arguments. Set  $\mathcal{Z} := \sup_{g \in \mathcal{G}} \left( \int_{\mathbb{R}^d} g(x) \mathbb{P} \circ \right.$   
661  $\left. \mathcal{X}^{-1}(dx) - \frac{2}{M} \sum_{m=1}^M g(\mathcal{X}^{(m)}) \right)_+$ . From Lemma A.1, we get

$$662 \quad \mathbb{P}(\mathcal{Z} > \varepsilon) \leq 4\mathbb{E} \left[ \mathcal{N}_1\left(\frac{\varepsilon}{12}, \mathcal{G}, \mathcal{X}^{1:M}\right) \right] \exp\left(-\frac{2\varepsilon M}{507BL_\psi}\right).$$

663 Since  $\mathcal{N}_1\left(\frac{\varepsilon}{12}, \mathcal{G}, \mathcal{X}^{1:M}\right) \leq \mathcal{N}_1\left(\frac{\varepsilon}{12L_\psi}, \mathcal{T}_B\mathcal{K}, \mathcal{X}^{1:M}\right)$  and thanks to (A.1), we derive

$$664 \quad (\text{A.7}) \quad \mathbb{P}(\mathcal{Z} > \varepsilon) \leq 4\alpha \left( \frac{12\beta BL_\psi}{\varepsilon} \right)^\gamma \exp\left(-\frac{2\varepsilon M}{507BL_\psi}\right)$$

665 whenever  $\frac{\varepsilon}{12L_\psi} \leq \frac{4}{15}B$ . For other values of  $\varepsilon$  the above probability is zero, therefore  
666 (A.7) holds for any  $\varepsilon > 0$ . The end of the computations is now very similar to the  
667 previous case: we finally get the inequality (A.6) for the new  $\mathcal{Z}$  with adjusted values  
668  $a = 12\beta BL_\psi$ ,  $b = \frac{2}{507BL_\psi}$ . Thus inequality (A.3) is thus proved.  $\square$

669

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