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▶ To cite this version:

Jean-Jacques Marigo, Agnes Maurel. Second Order Homogenization of Subwavelength Stratified Media Including Finite Size Effect. SIAM Journal on Applied Mathematics, 2017, 77 (2), pp.721 - 743. 10.1137/16M1070542 . hal-01657080

HAL Id: hal-01657080 https://polytechnique.hal.science/hal-01657080

Submitted on 6 Dec 2017

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SECOND ORDER HOMOGENIZATION OF SUBWAVELENGTH STRATIFIED MEDIA INCLUDING FINITE SIZE EFFECT

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JEAN-JACQUES MARIGO* AND AGNÈS MAUREL[†]

Résumé. We present a homogenization method to find the effective behavior of a periodically stratified slab which accounts for the finite size of the slab. The effective behavior is that of a homogeneous anisotropic slab associated with discontinuity conditions, or jump conditions, for the displacement and for the normal stress at the boundaries of the slab. The coefficients entering in the effective homogenized wave equation are related to the geometry and to the composition of the layers only, as in the classical homogenization. Those entering in the jump conditions are related to boundary layer effects and thus they depend also on the properties of the media surrounding the slab. The validity of our homogenization method is inspected in the case of layers associated with

12 Neumann boundary conditions.

13 **Key words.** homogenization, two-scale method, matched asymptotic expansion, stratified me-14 dia, finite size effects, effective jump conditions,

15 **AMS subject classifications.** 34E13, 35B27,74Q10, 74Q15,80M35, 74J20,

1. Introduction. The scattering properties of media stratified at a subwave-16length scale are known to be correctly described by an equivalent homogeneous aniso-17 tropic medium whose effective bulk parameters involve averages of the bulk parame-18 ters of the layers (the parameters entering in the wave equations); for shear horizontal 19 (SH) elastic waves, it is the mass density and the shear modulus, and the averages 20involve also the surface or volume fractions of the layers. This is known since Rytov's 21 work in 1956 [14] and the result has been extended within a rigorous mathematical 22 framework to periodic media using the homogenization theory, see e.q. [3]. 23

24 In its classical form, the homogenization is performed considering that the structured medium occupies the whole space. Obviously in practice, samples of finite thickness 25e are considered and it has been shown that, for small e, the scattering properties of 26the samples were not correctly described by their homogenized counterparts [7, 8, 9]. 27In this paper, we show that the homogenization theory can be adapted to stratified 28 29structures of finite thickness, which yields an equivalent slab whose scattering properties accurately describe those of the actual structure. We establish that, in addition 30 to the bulk parameters entering in the effective wave equation, the homogenization 31 makes interface parameters to appear, which enter in jump conditions at the boun-32 daries of the equivalent slab (Figure 1). While the effective bulk parameters depend 33 34 only on the characteristics of the structure at the microscale, the interface parameters 35 result from boundary layer effects and as such, they depend also on the characteristics of the surrounding media. 36

The homogenization for finite size stratified structures is performed up to the second order in $\varepsilon \equiv kh \ll 1$ (k being the wavelength and h the periodicity of the structure). It is presented in the section 2 and the approach essentially follows the one presented in [1] in the context of solid mechanics. We start with the elastic wave equation for the scalar displacement field $U(\mathbf{X})$ of shear waves written in the harmonic regime

43 (1)
$$\operatorname{div}\left(\mu\boldsymbol{\nabla}U\right) + \rho\omega^2 U = 0,$$

^{*}Laboratoire de Mécanique des Solides, Route de Saclay, 91128 Palaiseau, France, (ma-rigo@lms.polytechnique.fr)

 $^{^\}dagger Institut$ Langevin, 1 rue Jussieu, 75005 Paris, France (agnes.maurel@espci.fr)



FIGURE 1. On the left, the wave problem is set for a slab filled with a stratified medium, with the usual continuity conditions on the boundaries of the layers. On the right, it is set for an equivalent slab made of an effective medium (homogeneous and anisotropic) and jump conditions apply at the boundaries of the slab $X_1 = \pm e/2$ (the boundaries at $|X_2| = H/2$ are disregarded or equivalently, we consider $H \to +\infty$).

with $\mu(\mathbf{X})$ the shear modulus and $\rho(\mathbf{X})$ the mass density being spatially dependent of $\mathbf{X} = (X_1, X_2) \in \mathbb{R} \times (-H/2, H/2)$ and ω the frequency (Fig. 1). Eq. (1) can be written using the non dimensional parameters

$$\alpha(\mathbf{X}) \equiv \frac{\mu(\mathbf{X})}{\mu_{\rm m}}, \quad \text{and} \quad \beta(\mathbf{X}) \equiv \frac{\rho(\mathbf{X})}{\rho_{\rm m}},$$

with $(\mu_{\rm m}, \rho_{\rm m})$ the shear modulus and mass density of the medium surrounding the stratified structure, hereafter called the substrate; with $k = \omega \sqrt{\rho_{\rm m}/\mu_{\rm m}}$ the wavenum-

46 ber in the substrate, we get

47 (2)
$$\operatorname{div}\left(\alpha \nabla U\right) + \beta k^2 U = 0,$$

which also applies to acoustic waves, to transverse magnetic or transverse electric polarized electromagnetic waves or to scalar (shear) elastic waves. It follows that the Helmholtz equation $\Delta U + k^2 U = 0$ applies in the substrate by construction. We shall establish that the homogenized wave equation, up to second order, is of the form

52 (3)
$$\operatorname{div}\boldsymbol{\Sigma} + \beta^{\operatorname{eff}} k^2 U = 0, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \alpha_1^{\operatorname{eff}} & 0\\ 0 & \alpha_2^{\operatorname{eff}} \end{pmatrix} \boldsymbol{\nabla} U,$$

where $(\alpha_1^{\text{eff}}, \alpha_2^{\text{eff}}, \beta^{\text{eff}})$ will be defined in subsection 2.2, see (27). Next, for the stratified medium occupying the region $\mathbf{X} = (X_1, X_2) \in (-e/2, e/2) \times (-H/2, H/2)$, the homogenized slab, in which (3) applies, occupies the same region and effective continuity or discontinuity conditions apply at $X_1 = \pm e/2$. While the usual continuities of the displacement U and of the normal stress $\Sigma \cdot \mathbf{N}$ are obtained at the first order, discontinuities of these quantities at second order are established. Specifically, the jump conditions read

60 (4)
$$\begin{cases} \llbracket U \rrbracket = \frac{h\mathcal{B}}{2} \left(\mathbf{\Sigma}^{-} + \mathbf{\Sigma}^{+} \right) \cdot \mathbf{N}, \\ \llbracket \mathbf{\Sigma} \rrbracket \cdot \mathbf{N} = -\frac{h\mathcal{C}}{2} \left(\frac{\partial^{2}U^{-}}{\partial X_{2}^{2}} + \frac{\partial^{2}U^{+}}{\partial X_{2}^{2}} \right). \end{cases}$$

In the above relations, for any field V being discontinuous across a boundary with (V^-, V^+) its values on both sides, we have defined the jump $[\![V]\!] \equiv V^+ - V^-$ (and

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63 the convention \pm refers to the direction of the normal N). The parameters $(\mathcal{B}, \mathcal{C})$ 64 entering in the jump conditions are deduced from elementary problems, being the

⁶⁵ equivalent of the cell problems in the classical homogenization, Eqs. (34) and (35). In

66 the absence of high contrasts resulting in possible resonances in one or several layers

67 (and possible resonances are disregarded here), these problems are static problems

that can be solved once and for all. Finally, the problem of the boundary layers at the

boundaries $X_2 = \pm H/2$ would require a specific treatment and they are disregarded

in the present paper (alternatively, we may consider $H \to +\infty$).

Validations of our homogenization method are presented in section 3 by compa-71rison with full wave numerics. We restrict ourselves to the case of layers associated 72with Neumann boundary conditions; it corresponds to cracks or voids in elasticity, to 73 sound hard inclusions in acoustics or to perfectly conducting metals in electromagne-74 tism. This case allows for explicit expressions of the interface parameters $(\mathcal{B}, \mathcal{C})$, Eq. 75(53) (see also subsection S2.1). The limitations of the present approach to small slab 76 thicknesses are discussed in Appendix A. Finally, we report in section S1 the case of 77 a stratified slab with one boundary being free of stresses (associated with Neumann 78 79 boundary condition) and details on the numerical resolutions are given in section S2.

2. Up to second-order homogenization. In this section, we shall work on 80 a problem simplified with respect to the one in Fig. 1 in the sense that we consider 81 a single interface (also, we shall work in dimensionless coordinates). We define $x_1 =$ 82 $k(X_1 - e/2), x_2 = kX_2$, which means that we focus on a region near the boundary 83 of the stratified medium at $X_1 = e/2$ in the original problem in Fig. 1(a). But now, 84 we assume that the stratified medium occupies the region $x_1 < 0$, Fig. 2. Doing 85 so, we assumed implicitly that the wave passing through the stratified slab in the 86 configuration of the Fig. 1 feels the boundaries and the bulk of the stratified medium. 87 To anticipate, this means that the slab is thick enough, and thick means that the 88 evanescent fields at both boundaries of the slab do not interact. If it is not the case, one 89 90 should consider the whole thin slab in the asymptotic analysis, as done in [5, 6, 9, 11](this is discussed further in Appendix A).



FIGURE 2. Single interface between the stratified medium occupying the region $x_1 < 0$ and the substrate occupying the region $x_1 > 0$. The usual continuity conditions apply at the boundaries between the layers $(u^{\varepsilon}, \sigma_2^{\varepsilon})$ and at the boundaries between the layers and the substrate $(u^{\varepsilon}, \sigma_1^{\varepsilon})$ at $x_1 = 0$.

91

We shall define the actual problem for $\mathbf{x} = (x_1, x_2) \in \mathbb{R} \times (-kH/2, kH/2)$. With the periodicity of the stratified medium $\varepsilon = kh \ll 1$, the solution of the problem depends on ε and we make this dependence to appear explicitly, by denoting $a^{\varepsilon}(\mathbf{x}) \equiv$ $\sigma(\mathbf{X}), b^{\varepsilon}(\mathbf{x}) \equiv \beta(\mathbf{X})$ and $u^{\varepsilon}(\mathbf{x}) \equiv U(\mathbf{X}), \sigma^{\varepsilon}(\mathbf{x}) \equiv k^{-1}\alpha(\mathbf{X})\nabla U(\mathbf{X})$. In **x**-coordinate, 96 (2) reads

97 (5a)
$$\operatorname{div}\boldsymbol{\sigma}^{\varepsilon}(\mathbf{x}) + b^{\varepsilon}(\mathbf{x})u^{\varepsilon}(\mathbf{x}) = 0,$$

(5b)
$$\boldsymbol{\sigma}^{\varepsilon}(\mathbf{x}) = a^{\varepsilon}(\mathbf{x})\boldsymbol{\nabla}u^{\varepsilon}(\mathbf{x}),$$

100 and

101 (6)
$$a^{\varepsilon}(\mathbf{x}) = \begin{cases} 1, & x_1 > 0, \\ a\left(\frac{x_2}{\varepsilon}\right), & x_1 < 0. \end{cases} \quad b^{\varepsilon}(\mathbf{x}) = \begin{cases} 1, & x_1 > 0, \\ b\left(\frac{x_2}{\varepsilon}\right), & x_1 < 0. \end{cases}$$

The functions a and b are 1-periodic and piecewise constant; at this point, it is not 102necessary to define them more specifically. To (5), we have to associate boundary 103 conditions. At each boundary between two different media, the continuity of the dis-104 placement u^{ε} and of the normal stress $\sigma^{\varepsilon} \cdot \mathbf{n}$ apply (with \mathbf{n} the vector normal to the 105boundary); this applies at the boundaries between two layers within the stratified 106 medium and at the boundaries between the layers and the substrate at $x_1 = 0$ (Fig. 107 108 2). Finally, once the wave source is defined, the conditions satisfied by $(u^{\varepsilon}, \sigma^{\varepsilon})$ for $x_2 = \pm kH/2$ and for $|x_1| \to +\infty$, often referred as to radiation conditions, can be 109 defined; for the time being, we do not need to specify them. 110

2.1. The asymptotic analysis. The idea is to define three regions where dif-111 ferent asymptotic expansions will be used, Eqs. (7). The inner region contains the 112boundary between the stratified medium and the substrate. Roughly speaking, it is 113the region where the boundary layer effects are significant; in terms of wavefield, this 114 means that the inner region contains the so-called evanescent field vanishing far from 115the boundary. The two outer regions for $x_1 > 0$ and $x_1 < 0$ are the regions far enough 116from the interface, where the evanescent field can be neglected. Next, the inner region 117 and the outer regions are connected using so -called matching conditions, which will 118 constitute the boundary conditions for the outer solutions. 119



FIGURE 3. On the left, configuration in \mathbf{x} coordinate; the periodicity along x_2 is $\varepsilon \equiv kh$; the inner region corresponds to the neighborhood of the boundary between the stratified medium $(x_1 < 0)$ and the substrate being a homogeneous medium $(x_1 > 0)$. On the right, the unit cell (inner region) in \mathbf{y} coordinate, with $\mathbf{y} = \mathbf{x}/\varepsilon$, and $\mathbf{y} \in \mathbb{R} \times Y$, with Y = (-1/2, 1/2).

2.1.1. The outer and inner expansions. As in the classical homogenization, the asymptotic expansions are thought with spatial dependences on a *macroscopic* coordinate **x** associated with slow variations of the fields (with the typical scale 1/kof the wave) and a *microscopic* coordinate **y**, associated with rapid variations (the typical scale h of the layers), and in each region, we keep the coordinates that are relevant to describe the variations of the field. To do so, we define $\mathbf{y} \equiv \mathbf{x}/\varepsilon$ and we

 $127 \quad (7) \qquad \begin{cases} \text{outer region } x_1 > 0, \quad u^{\varepsilon} = u^0(\mathbf{x}) + \varepsilon u^1(\mathbf{x}) + \dots, \\ \boldsymbol{\sigma}^{\varepsilon} = \boldsymbol{\sigma}^0(\mathbf{x}) + \varepsilon \boldsymbol{\sigma}^1(\mathbf{x}) + \dots, \\ \text{outer region } x_1 < 0, \quad u^{\varepsilon} = u^0(\mathbf{x}, y_2) + \varepsilon u^1(\mathbf{x}, y_2) + \dots, \\ \boldsymbol{\sigma}^{\varepsilon} = \boldsymbol{\sigma}^0(\mathbf{x}, y_2) + \varepsilon \boldsymbol{\sigma}^1(\mathbf{x}, y_2) + \dots, \\ \text{inner region,} \qquad u^{\varepsilon} = v^0(x_2, \mathbf{y}) + \varepsilon v^1(x_2, \mathbf{y}) + \dots, \\ \boldsymbol{\sigma}^{\varepsilon} = \boldsymbol{\tau}^0(x_2, \mathbf{y}) + \varepsilon \boldsymbol{\tau}^1(x_2, \mathbf{y}) + \dots, \end{cases}$

with the outer terms (u^n, σ^n) for $x_1 < 0$ and the inner terms (v^n, τ^n) being Y-periodic with Y = (-1/2, 1/2). Thus, we shall consider $y_2 \in Y$ and in the inner region $y_1 \in \mathbb{R}$; next, in the three regions $x_2 \in (-kH/2, kH/2)$ and in the outer regions $x_1 \in (-\infty, 0)$ and $x_1 \in (0, +\infty)$ respectively.

132 The differential operator reads, in the different regions, as

133 (8)
$$\begin{cases} \text{ in the outer region, } \nabla \to \nabla_{\mathbf{x}}, & x_1 > 0, \\ \nabla \to \nabla_{\mathbf{x}} + \frac{1}{\varepsilon} \frac{\partial}{\partial y_2} \mathbf{e}_2, & x_1 < 0, \\ \text{ in the inner region, } \nabla \to \frac{\partial}{\partial x_2} \mathbf{e}_2 + \frac{1}{\varepsilon} \nabla_{\mathbf{y}}, \end{cases}$$

where $\nabla_{\mathbf{x}}$ means gradient w.r.t. \mathbf{x} and $\nabla_{\mathbf{y}}$ means gradient w.r.t. \mathbf{y} . Let us comment 134the dependences in \mathbf{x} and \mathbf{y} of the fields in each region, (7). The inner solution is 135characterized by rapid variations of the evanescent field, and these variations are 136naturally described by $\mathbf{y} (|\nabla U| \sim U/h$ gives $|\nabla_{\mathbf{y}} u| \sim u$). But the inner solution 137138 contains also the propagating field associated with slow variations along x_2 , typically the phase variation along the boundary at $x_1 = 0$; this is taken into account by 139keeping x_2 as an additional coordinate. In the outer region $x_1 > 0$, there is no rapid 140 variations due to the evanescent field (this latter being confined in the inner region, 141by definition); thus, we need only the coordinates \mathbf{x} which describe the propagating 142 field with $|\nabla U| \sim kU$ and thus $\nabla_{\mathbf{x}} u \sim u$. The story is different for $x_1 < 0$; there, the 143144 field has still slow variations, but it also experiences rapid variations across the layers and this is accounted for by keeping the coordinate y_2 . 145

146 Finally, from (6), $(a^{\varepsilon}, b^{\varepsilon})$ can be specified with in the outer regions

147 (9)
$$\begin{cases} \text{outer region } x_1 > 0, \quad a^{\varepsilon}(\mathbf{x}) = 1, \qquad b^{\varepsilon}(\mathbf{x}) = 1, \\ \text{outer region } x_1 < 0, \quad a^{\varepsilon}(\mathbf{x}) = a\left(\frac{x_2}{\varepsilon}\right), \quad b^{\varepsilon}(\mathbf{x}) = b\left(\frac{x_2}{\varepsilon}\right) \end{cases}$$

148 and in the inner region $a^{\varepsilon}(\mathbf{x}) = \tilde{a}(\mathbf{x}/\varepsilon)$ and $b^{\varepsilon}(\mathbf{x}) = \tilde{b}(\mathbf{x}/\varepsilon)$ with

149 (10)
$$\tilde{a}(\mathbf{y}) = \begin{cases} a(y_2), & y_1 < 0, \\ 1, & y_1 > 0, \end{cases} \quad \tilde{b}(\mathbf{y}) = \begin{cases} b(y_2), & y_1 < 0, \\ 1, & y_1 > 0, \end{cases}$$

150 with $a(y_2)$, $b(y_2)$ 1-periodic and piecewise constant.

151 **2.1.2. The boundary conditions and the matching conditions.** The inner 152 and outer problems have to be associated with boundary conditions or radiation 153 conditions which ensure that the problems are well-posed. For the inner solution, the 154 continuities of the displacement and of the normal stress apply at the boundaries 155 between two layers within the stratified medium and at the boundaries between the 156 layers and the substrate at $y_1 = 0$, whence

157 (11) $v^n, \boldsymbol{\tau}^n \cdot \mathbf{n}$ are continuous everywhere, n = 0, 1...,

but the conditions at infinity are unknown *a priori*. Reversely, since the outer expansions hold true only far away from the interface, the outer terms satisfy the radiation condition (once defined) but they do not have to satisfy the continuity conditions at $x_1 = 0$; only the conditions of continuity of u^n and $\sigma^n \cdot \mathbf{n}$ at the boundaries between the layers within the stratified medium apply for $x_1 < 0$ (thus, with $\mathbf{n} = \mathbf{e}_2$).

163 The missing conditions for the inner and outer terms are provided simultaneously 164 by the matching conditions (see the discussion on alternative matching in [1]), at 165 leading order

166 (12a) $u^0(0^-, x_2, y_2) = \lim_{y_1 \to -\infty} v^0(x_2, \mathbf{y}),$

167 (12b)
$$u^0(0^+, x_2) = \lim_{y_1 \to +\infty} v^0(x_2, \mathbf{y}),$$

168 (12c)
$$\sigma^{0}(0^{-}, x_{2}, y_{2}) = \lim_{y_{1} \to -\infty} \tau^{0}(x_{2}, \mathbf{y}),$$

169 (12d)
$$\boldsymbol{\sigma}^0(0^+, x_2) = \lim_{y_1 \to +\infty} \boldsymbol{\tau}^0(x_2, \mathbf{y}),$$

171 and at order ε

172 (13a)
$$u^{1}(0^{-}, x_{2}, y_{2}) = \lim_{y_{1} \to -\infty} \left[v^{1}(x_{2}, \mathbf{y}) - y_{1} \frac{\partial u^{0}}{\partial x_{1}} (0^{-}, x_{2}, y_{2}) \right],$$

173 (13b)
$$u^{1}(0^{+}, x_{2}) = \lim_{y_{1} \to +\infty} \left[v^{1}(x_{2}, \mathbf{y}) - y_{1} \frac{\partial u^{0}}{\partial x_{1}}(0^{+}, x_{2}) \right],$$

174 (13c)
$$\boldsymbol{\sigma}^{1}(0^{-}, x_{2}, y_{2}) = \lim_{y_{1} \to -\infty} \left[\boldsymbol{\tau}^{1}(x_{2}, \mathbf{y}) - y_{1} \frac{\partial \boldsymbol{\sigma}^{0}}{\partial x_{1}}(0^{-}, x_{2}, y_{2}) \right],$$

175 (13d)
$$\boldsymbol{\sigma}^{1}(0^{+}, x_{2}) = \lim_{y_{1} \to +\infty} \left[\boldsymbol{\tau}^{1}(x_{2}, \mathbf{y}) - y_{1} \frac{\partial \boldsymbol{\sigma}^{0}}{\partial x_{1}}(0^{+}, x_{2}) \right].$$

177 At order ε , the conditions are obtained using the Taylor expansions of $(u^0, \boldsymbol{\sigma}^0)$, for 178 instance for $x_1 > 0$, $u^0(\mathbf{x}) = u^0(0^+, x_2) + x_1\partial_{x_1}u^0(0^+, x_2) + \cdots = u^0(0^+, x_2) + \cdots$ 179 $\varepsilon y_1\partial_{x_1}u^0(0^+, x_2) + \cdots$

180 **2.2. The homogenized wave equation.** We want to establish the wave equa-181 tion, up to second-order, satisfied by the mean fields $(\overline{u}(\mathbf{x}), \overline{\sigma}(\mathbf{x}))$ with

182 (14)
$$\overline{u} \equiv \langle u^0 \rangle + \varepsilon \langle u^1 \rangle, \quad \overline{\sigma} \equiv \langle \sigma^0 \rangle + \varepsilon \langle \sigma^1 \rangle.$$

183 We have defined the average over $y_2 \in Y$ for any function f

184 (15)
$$\langle f \rangle(\mathbf{x}) \equiv \int_Y \, \mathrm{d}y_2 \, f(\mathbf{x}, y_2),$$

and obviously, if f does not depend on y_2 , $\langle f \rangle = f$.

The homogenized wave equation is sought for $x_1 < 0$ only. For $x_1 > 0$, from (5) along with (9), the wave equation is

188 (16)
$$\operatorname{div}_{\mathbf{x}}\boldsymbol{\sigma}^n + u^n = 0, \quad \boldsymbol{\sigma}^n = \boldsymbol{\nabla}_{\mathbf{x}} u^n, \quad (n = 0, 1), \quad \text{for } x_1 > 0,$$

being the same at each order, and the fields being independent of y_2 equal their averages.

191 **2.2.1. The homogenized wave equation in** $x_1 < 0$ **at first order.** For 192 $x_1 < 0$, the Eqs. (5), at leading order $(1/\varepsilon)$, read $\partial_{y_2} \sigma_2^0 = 0 = \partial_{y_2} u^0$, whence we can 193 note

194 (17)
$$u^0(\mathbf{x}, y_2) = u^0(\mathbf{x}), \quad \sigma_2^0(\mathbf{x}, y_2) = \sigma_2^0(\mathbf{x}),$$

195 and u^0, σ_2^0 equal their averages. Now, we establish the relation between $\langle \sigma^0 \rangle$ and u^0 . 196 The Eqs. (5) at order ε^0 in the outer problems $x_1 < 0$ give

197 (18)
$$\boldsymbol{\sigma}^{0}(\mathbf{x}, y_{2}) = a(y_{2}) \left[\boldsymbol{\nabla}_{\mathbf{x}} u^{0}(\mathbf{x}) + \frac{\partial u^{1}}{\partial y_{2}}(\mathbf{x}, y_{2}) \mathbf{e}_{2} \right].$$

198 and

199 (19)
$$\operatorname{div}_{\mathbf{x}}\boldsymbol{\sigma}^{0}(\mathbf{x}, y_{2}) + \frac{\partial \sigma_{2}^{1}}{\partial y_{2}}(\mathbf{x}, y_{2}) + b(y_{2})u^{0}(\mathbf{x}) = 0.$$

Averaging both equations, with $\sigma^0(\mathbf{x}, y_2) = \sigma_1^0(\mathbf{x}, y_2) \mathbf{e}_1 + \sigma_2^0(\mathbf{x}) \mathbf{e}_2$, and owing to the periodicity of u^1 and of $\sigma_2^1 w.r.t. y_2$ (thus, $\langle \partial_{y_2} u^1 \rangle = 0 = \langle \partial_{y_2} \sigma_2^1 \rangle$), we easily get the wave equation at the first order

203 (20a)
$$\langle \boldsymbol{\sigma}^0 \rangle(\mathbf{x}) = \langle a \rangle \frac{\partial u^0}{\partial x_1}(\mathbf{x}) \, \mathbf{e}_1 + \langle 1/a \rangle^{-1} \frac{\partial u^0}{\partial x_2}(\mathbf{x}) \, \mathbf{e}_2,$$

(20b)
$$\operatorname{div}_{\mathbf{x}} \langle \boldsymbol{\sigma}^0 \rangle + \langle b \rangle u^0 = 0.$$

206 **2.2.2. Second-order - useful relations.** Before going to the next order, we 207 shall establish the relations (22) for u^1 and (24) for σ_2^1 , that we shall use later. Both 208 relations use the same property : consider a piecewise differentiable function g(y), with 209 g'(y) even; then $(g - \langle g \rangle)$ is odd and, for any function f(y) being even, $f(g - \langle g \rangle)$ is 210 odd. Thus, the integral over $y \in Y$ vanishes, from which $\langle fg \rangle = \langle f \rangle \langle g \rangle$. 211 First, from (18), we have

212 (21)
$$\frac{\partial u^1}{\partial y_2}(\mathbf{x}, y_2) = \frac{1/a(y_2) - \langle 1/a \rangle}{\langle 1/a \rangle} \frac{\partial u^0}{\partial x_2}(\mathbf{x}),$$

and we used that $\sigma_2^0(\mathbf{x}) = \langle 1/a \rangle^{-1} \partial_{x_2} u^0(\mathbf{x})$ from (20a). Because $a(y_2)$ is even, $\partial_{y_2} u^1$ is even too, so the property on the average applies and

215 (22)
$$\langle f(\cdot)u^1(\mathbf{x},\cdot)\rangle = \langle f\rangle\langle u^1\rangle(\mathbf{x}), \text{ for any even } f.$$

216 Integrating (21), we also have

217 (23)
$$u^{1}(\mathbf{x}, y_{2}) = A(y_{2}) \frac{\partial u^{0}}{\partial x_{2}}(\mathbf{x}) + \langle u^{1} \rangle(\mathbf{x}), \quad A(y_{2}) \equiv \int_{-1/2}^{y_{2}} \mathrm{d}y \; \frac{1/a(y) - \langle 1/a \rangle}{\langle 1/a \rangle}$$

In the above expression, we have used that $\langle A \rangle = 0$, by construction.

Next, we use that $\sigma_1^0(\mathbf{x}, y_2) = a(y_2)\partial_{x_1}u^0(\mathbf{x})$ from (18), and thus $\langle \sigma_1^0 \rangle(\mathbf{x}) = 220 \langle a \rangle \partial_{x_1} u^0(\mathbf{x})$. Inserting the resulting relation $\sigma_1^0(\mathbf{x}, y_2) = a(y_2)/\langle a \rangle \langle \sigma_1^0 \rangle(\mathbf{x})$ in (19), we get

$$\frac{\partial \sigma_2^1}{\partial y_2}(\mathbf{x}, y_2) = -\frac{a(y_2)}{\langle a \rangle} \frac{\partial \langle \sigma_1^0 \rangle}{\partial x_1}(\mathbf{x}) - \frac{\partial \sigma_2^0}{\partial x_2}(\mathbf{x}) - b(y_2)u^0(\mathbf{x})$$

223 (a,b) being even, $\partial_{y_2}\sigma_2^1$ is even *w.r.t.* y_2 , from which the property on the average 224 applies and

225 (24)
$$\langle f(\cdot)\sigma_2^1(\mathbf{x},\cdot)\rangle = \langle f\rangle\langle\sigma_2^1\rangle(\mathbf{x}),$$
 for any even f .

226 **2.2.3. The homogenized wave equation in** $x_1 < 0$ at second-order. Now, 227 we shall establish the relation between $\langle u^1 \rangle$ and $\langle \sigma^1 \rangle$ (for $x_1 < 0$). Eq. (5b) at order 228 ε reads

229
$$\boldsymbol{\sigma}^{1}(\mathbf{x}, y_{2}) = a(y_{2}) \left[\boldsymbol{\nabla}_{\mathbf{x}} u^{1}(\mathbf{x}, y_{2}) + \frac{\partial u^{2}}{\partial y_{2}}(\mathbf{x}, y_{2}) \mathbf{e}_{2} \right].$$

To average the above equation, it is sufficient to use (22) and (24), with $a(y_2)$ being even. We get

232 (25)
$$\begin{cases} \sigma_1^1(\mathbf{x}, y_2) = a(y_2) \frac{\partial u^1}{\partial x_1}(\mathbf{x}, y_2) & \to \langle \sigma_1^1 \rangle(\mathbf{x}) = \langle a \rangle \frac{\partial \langle u^1 \rangle}{\partial x_1}(\mathbf{x}), \\ \frac{1}{a(y_2)} \sigma_2^1(\mathbf{x}, y_2) = \frac{\partial u^1}{\partial x_2}(\mathbf{x}, y_2) + \frac{\partial u^2}{\partial y_2}(\mathbf{x}, y_2) & \to \langle 1/a \rangle \langle \sigma_2^1 \rangle(\mathbf{x}) = \frac{\partial \langle u^1 \rangle}{\partial x_2}(\mathbf{x}), \end{cases}$$

where the arrow indicates the average process. Next, (5a) at order ε reads

234
$$\operatorname{div}_{\mathbf{x}}\boldsymbol{\sigma}^{1}(\mathbf{x}, y_{2}) + \frac{\partial \sigma_{2}^{2}}{\partial y_{2}}(\mathbf{x}, y_{2}) + b(y_{2}) u^{1}(\mathbf{x}, y_{2}) = 0$$

235 whose average leads to

236 (26)
$$\operatorname{div}_{\mathbf{x}}\langle \boldsymbol{\sigma}^1 \rangle(\mathbf{x}) + \langle b \rangle \langle u^1 \rangle(\mathbf{x}) = 0.$$

and we have used $\langle bu^1 \rangle = \langle b \rangle \langle u^1 \rangle$ from (22) and $\langle \partial_{y_2} \sigma_2^2 \rangle = 0$ because σ_2^2 is periodic w.r.t. y_2 .

239 **2.2.4.** Up to second-order homogenized wave equation. It is now suffi-240 cient to gather (20) and (25)-(26) to get the homogenized wave equation up to second 241 order for $(\overline{u}(\mathbf{x}), \overline{\sigma}(\mathbf{x}))$ in (14)

242 (27)
$$\operatorname{div}\overline{\boldsymbol{\sigma}} + \langle b \rangle \overline{\boldsymbol{u}} = 0, \quad \overline{\boldsymbol{\sigma}} = \begin{pmatrix} \langle a \rangle & 0 \\ 0 & \langle 1/a \rangle^{-1} \end{pmatrix} \boldsymbol{\nabla} \overline{\boldsymbol{u}}, \quad \text{for } x_1 < 0,$$

which coincides, when coming back to the real space, to (3).

244 **2.3. Jump conditions.** To the homogenized wave equation (27), we have to 245 associate jump conditions at the interface $x_1 = 0$. In this section, we will show that 246 the usual continuities of the displacement and of the normal stress are obtained at 247 leading order, while the second order makes discontinuities of these two fields to 248 appear. To that aim, we have to consider the inner solution and its matching with 249 the two outer solutions. We are looking for the jumps of $(\bar{u}, \bar{\sigma_1})$ defined in (14),

250 (28)
$$\llbracket \overline{u} \rrbracket = \llbracket u^0 \rrbracket + \varepsilon \llbracket \langle u^1 \rangle \rrbracket, \quad \llbracket \overline{\sigma_1} \rrbracket = \llbracket \langle \sigma_1^0 \rangle \rrbracket + \varepsilon \llbracket \langle \sigma_1^1 \rangle \rrbracket.$$

and their expressions in terms of \overline{u}^{\pm} and $\overline{\sigma}^{\pm}$, the values of \overline{u} and $\overline{\sigma}$ on both sides of the interface (at this stage, we have to distinguish the values on both sides, the fields being discontinuous).

254 **2.3.1. The jump conditions at the first order.** Eq. (5b) for the inner pro-255 blem at the leading order in $1/\varepsilon$ tells us that $\nabla_{\mathbf{y}} v^0 = 0$ from which v^0 does not depend 256 on \mathbf{y} . From the previous section, we already know that $u^0(\mathbf{x})$ does not depend on y_2 , 257 from which the matching conditions, (12a) and (12b), give

258 (29)
$$u^0(0^-, x_2) = u^0(0^+, x_2) = v^0(x_2), \text{ and } \llbracket u^0 \rrbracket = 0.$$

Next, (5a) in the inner region gives at the leading order $\operatorname{div}_{\mathbf{y}} \boldsymbol{\tau}^0 = 0$; by integrating this equation on $\mathbb{R} \times Y$, we get

$$\int_{Y} dy_2 \left[\tau_1^0(x_2, +\infty, y_2) - \tau_1^0(x_2, -\infty, y_2) \right] = 0,$$

(we have used the periodicity of τ^0 w.r.t. y_2 and the continuity of $\tau^0 \cdot \mathbf{n}$ between the layers along y_2). From the matching conditions (12c)-(12d) integrated over Y, we get

261 (30)
$$\langle \sigma_1^0 \rangle (0^-, x_2) = \sigma_1^0 (0^+, x_2), \text{ and } [\![\langle \sigma_1^0 \rangle]\!] = 0.$$

At first order, the usual continuities of the displacement and of the normal stress are obtained. To capture the effect of the boundary layers in the neighborhood of $x_1 = 0$, we have to go up to the second order.

265 **2.3.2. The elementary problems.** Before going to the second order, we need 266 to inspect further the inner solution. There, the variations of $\tilde{a}(\mathbf{y})$ and $\tilde{b}(\mathbf{y})$ in (5) are 267 more involved than the simple forms $a(y_2)$ and $b(y_2)$ considered until now for $x_1 < 0$, 268 see (10). The difference between $\tilde{a}(\mathbf{y})$ and $a(y_2)$ will constitute the whole story. From 269 (5a) at order ε^{-1} and (5b) at order ε^0 , we have

270 (31)
$$\operatorname{div}_{\mathbf{y}}\boldsymbol{\tau}^{0} = 0, \quad \boldsymbol{\tau}^{0}(x_{2}, \mathbf{y}) = \tilde{a}(\mathbf{y}) \left[\frac{\partial u^{0}}{\partial x_{2}}(0, x_{2}) \mathbf{e}_{2} + \boldsymbol{\nabla}_{\mathbf{y}} v^{1}(x_{2}, \mathbf{y}) \right],$$

where we used that $\partial_{x_2} u^0(\mathbf{x})$ is continuous at $x_1 = 0$ as u^0 does from (29). The matching conditions (12c)-(12d) yield

$$\begin{cases} \boldsymbol{\tau}^{0}(x_{2},-\infty,y_{2}) &= \frac{a(y_{2})}{\langle a \rangle} \langle \sigma_{1}^{0} \rangle(0,x_{2}) \, \mathbf{e}_{1} + \langle 1/a \rangle^{-1} \frac{\partial u^{0}}{\partial x_{2}}(0,x_{2}) \, \mathbf{e}_{2} \\ &= a(y_{2}) \frac{\partial u^{0}}{\partial x_{2}}(0,x_{2}) \, \mathbf{e}_{2} + a(y_{2}) \boldsymbol{\nabla}_{\mathbf{y}} v^{1}(x_{2},-\infty,y_{2}). \\ \boldsymbol{\tau}^{0}(x_{2},+\infty,y_{2}) &= \langle \sigma_{1}^{0} \rangle(0,x_{2}) \, \mathbf{e}_{1} + \frac{\partial u^{0}}{\partial x_{2}}(0,x_{2}) \, \mathbf{e}_{2} \\ &= \frac{\partial u^{0}}{\partial x_{2}}(0,x_{2}) \, \mathbf{e}_{2} + \boldsymbol{\nabla}_{\mathbf{y}} v^{1}(x_{2},+\infty,y_{2}). \end{cases}$$

273

For both limits above, the first line is given by the matching conditions with σ^0 expressed as a function of $\langle \sigma_1^0 \rangle$ and $\partial_{x_2} u^0$; it is obvious for $x_1 > 0$, and for $x_1 < 0$, we used (18) and (20a) (with $\sigma_2^0 = \langle \sigma_2^0 \rangle$ from (17)). The second line is given by the expression of τ^0 in (31) along with (10). It follows that the system satisfied by 278 $v^1(x_2, \mathbf{y})$ can be written (32)

$$\begin{cases} \operatorname{div}_{\mathbf{y}}\boldsymbol{\tau}^{0} = 0, \text{ with } \boldsymbol{\tau}^{0} = \tilde{a}(\mathbf{y}) \left[\frac{\partial u^{0}}{\partial x_{2}}(0, x_{2}) \, \mathbf{e}_{2} + \boldsymbol{\nabla}_{\mathbf{y}} v^{1}(x_{2}, \mathbf{y}) \right], \\ v^{1} \text{ and } \boldsymbol{\tau}^{0} \cdot \mathbf{n} \text{ continuous,} \\ \lim_{y_{1} \to -\infty} \boldsymbol{\nabla}_{\mathbf{y}} v^{1}(x_{2}, \mathbf{y}) = \langle a \rangle^{-1} \langle \sigma_{1}^{0} \rangle(0, x_{2}) \, \mathbf{e}_{1} + \frac{1/a(y_{2}) - \langle 1/a \rangle}{\langle 1/a \rangle} \frac{\partial u^{0}}{\partial x_{2}} \end{cases}$$

279

$$\lim_{y_1 \to -\infty} \nabla_{\mathbf{y}} v^1(x_2, \mathbf{y}) = \langle a \rangle^{-1} \langle \sigma_1^0 \rangle(0, x_2) \mathbf{e}_1 + \frac{1/a(y_2) - \langle 1/a \rangle}{\langle 1/a \rangle} \frac{\partial u^0}{\partial x_2}(0, x_2) \mathbf{e}_2,$$
$$\lim_{y_1 \to +\infty} \nabla_{\mathbf{y}} v^1(x_2, \mathbf{y}) = \langle \sigma_1^0 \rangle(0, x_2) \mathbf{e}_1,$$

with v^1 and τ^0 periodic w.r.t y_2 , and we have used (11). The system (32) is linear w.r.t $\langle \sigma_1^0 \rangle (0, x_2)$ and $\partial_{x_2} u^0(0, x_2)$. Thus, we define $V^{(1)}(\mathbf{y})$ and $V^{(2)}(\mathbf{y})$ such that

282 (33)
$$\begin{cases} v^{1}(x_{2},\mathbf{y}) = \langle \sigma_{1}^{0} \rangle(0,x_{2}) V^{(1)}(\mathbf{y}) + \frac{\partial u^{0}}{\partial x_{2}}(0,x_{2}) \left[A(y_{2}) + V^{(2)}(\mathbf{y})\right] + \hat{v}(x_{2}), \\ \tau^{0}(x_{2},\mathbf{y}) = \langle \sigma_{1}^{0} \rangle(0,x_{2}) \mathbf{T}^{(1)}(\mathbf{y}) + \frac{\partial u^{0}}{\partial x_{2}}(0,x_{2}) \left[\frac{\tilde{a}(\mathbf{y})/a(y_{2})}{\langle 1/a \rangle} \mathbf{e}_{2} + \mathbf{T}^{(2)}(\mathbf{y})\right], \end{cases}$$

with $\mathbf{T}^{(1)}(\mathbf{y}) \equiv \tilde{a}(\mathbf{y}) \nabla V^{(1)}(\mathbf{y})$ and $\mathbf{T}^{(2)}(\mathbf{y}) \equiv \tilde{a}(\mathbf{y}) \nabla V^{(2)}(\mathbf{y})$ (and $A(y_2)$ defined in (23)). Note that the field v^1 in (32) is defined up to a function of x_2 , and it is denoted $\hat{v}(x_2)$ in (33); we shall see that the determination of $\hat{v}(x_2)$ is not needed. It is easy to see that if $(V^{(i)}, \mathbf{T}^{(i)})$ satisfy the elementary problems

(34)
$$\begin{cases} \operatorname{div} \mathbf{T}^{(1)} = 0, \quad \text{with } \mathbf{T}^{(1)}(\mathbf{y}) = \tilde{a}(\mathbf{y}) \nabla V^{(1)}(\mathbf{y}) \\ V^{(1)} \text{ and } \mathbf{T}^{(1)} \cdot \mathbf{n} \text{ continuous,} \\ V^{(1)}, \mathbf{T}^{(1)} \text{ periodic } w.r.t. \ y_2 \\ \lim_{y_1 \to -\infty} \nabla V^{(1)}(\mathbf{y}) = \frac{\mathbf{e}_1}{\langle a \rangle}, \quad \lim_{y_1 \to +\infty} \nabla V^{(1)}(\mathbf{y}) = \mathbf{e}_1, \end{cases}$$

288 and

$$(35) \quad \begin{cases} \operatorname{div} \left[\mathbf{T}^{(2)} + \frac{\tilde{a}(\mathbf{y})/a(y_2)}{\langle 1/a \rangle} \mathbf{e}_2 \right] = 0, & \operatorname{with} \mathbf{T}^{(2)}(\mathbf{y}) = \tilde{a}(\mathbf{y}) \nabla V^{(2)}(\mathbf{y}), \\ V^{(2)} & \operatorname{and} \left[\mathbf{T}^{(2)} + \frac{\tilde{a}(\mathbf{y})/a(y_2)}{\langle 1/a \rangle} \mathbf{e}_2 \right] \cdot \mathbf{n} \text{ continuous}, \\ V^{(2)}, \mathbf{T}^{(2)} \text{ periodic } w.r.t. \ y_2 \\ \lim_{y_1 \to -\infty} \nabla V^{(2)}(\mathbf{y}) = \mathbf{0}, \quad \lim_{y_1 \to +\infty} \nabla V^{(2)}(\mathbf{y}) = -\frac{1/a(y_2) - \langle 1/a \rangle}{\langle 1/a \rangle} \mathbf{e}_2, \end{cases}$$

290 then
$$v^1(x_2, \mathbf{y})$$
 satisfies (32). The elementary solutions $V^{(1,2)}$ satisfy

291 (36)
$$\begin{cases} \lim_{y_1 \to -\infty} \left[V^{(1)} - \frac{y_1}{\langle a \rangle} \right] = -\mathcal{B}, \\ \lim_{y_1 \to +\infty} \left[V^{(1)} - y_1 \right] = 0, \end{cases} \begin{cases} \lim_{y_1 \to -\infty} V^{(2)} = -\mathcal{B}', \\ \lim_{y_1 \to +\infty} V^{(2)} = -A(y_2). \end{cases}$$

The above limits are obtained by integrating the limits of $\nabla V^{(i)}$, i = 1, 2, thus with unknown constants being *a priori* different at $y_1 \to \pm \infty$. Next, because $V^{(i)}$ are defined

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in (34)-(35) up to a constant, we can set the constant equal to zero at $y_1 \to +\infty$; for $V^{(1)}$, we denote $-\mathcal{B}$ the constant at $y_1 \to -\infty$ (it is the first interface parameter). For $V^{(2)}$, it is denoted $-\mathcal{B}'$; next, $V^{(2)}$ being odd w.r.t. y_2 , we have $\mathcal{B}' = 0$. It is important to stress that the elementary problems, as the unit cell problems in the classical homogenization, can be solved once and for all, being written in the static limit. The relations between the elementary solutions $V^{(1,2)}$ and the evanescent fields in the actual problem (for a given scattering problem) are illustrated in Appendix A.



FIGURE 4. The domain $\mathbf{Y} = \mathbf{Y}^- \cup \mathbf{Y}^+$, with $\mathbf{Y}^- = (-y_1^m, 0) \times Y$, $\mathbf{Y}^+ = (0, +y_1^m) \times Y$. $\tilde{a}(\mathbf{y}) = a(y_2)$ and $\tilde{b}(\mathbf{y}) = b(y_2)$ in \mathbf{Y}^- , and a = 1 = b in \mathbf{Y}^+ .

2.3.3. Jump conditions at second-order. Once the elementary problems are
 solved, it is possible to determine the jump conditions.

303 Jump of $\langle u^1 \rangle$ – To get the jump of $\langle u^1 \rangle$, it is sufficient to use the matching conditions 304 (13a)-(13b) and we want $v^1(x_2, \pm \infty, y_2)$. From (33) along with (36), we have

$$\begin{cases} v^1(x_2, -\infty, y_2) &= \lim_{y_1 \to -\infty} \left[\left(\frac{y_1}{\langle a \rangle} - \mathcal{B} \right) \langle \sigma_1^0 \rangle(0, x_2) + A(y_2) \frac{\partial u^0}{\partial x_2}(0, x_2) + \hat{v}(x_2) \right], \\ v^1(x_2, +\infty, y_2) &= \lim_{y_1 \to +\infty} \left[y_1 \langle \sigma_1^0 \rangle(0, x_2) + \hat{v}(x_2) \right]. \end{cases}$$

Now, we have $\langle \sigma_1^0 \rangle(0, x_2) = \langle a \rangle \partial_{x_1} u^0(0^-, x_2)$ from (20a) and $\langle \sigma_1^0 \rangle(0, x_2) = \partial_{x_1} u^0(0^+, x_2)$ from (16) and (30). Averaging $[\![u^1]\!] = u^1(0^+, x_2) - u^1(0^-, x_2, y_2)$ over Y and owing to $\langle A \rangle = 0$, we get

309 (37)
$$[\![\langle u^1 \rangle]\!] = \mathcal{B} \langle \sigma_1^0 \rangle (0, x_2).$$

310 $\begin{array}{l} Jump \ of \ \langle \sigma_1^1 \rangle - \text{The derivation of the jump of } \sigma_1^1 \text{ is more tricky, or at least longer. First,} \\ 311 & \overline{\text{we define } \mathbf{Y} \equiv (-y_1^m, y_1^m) \times Y} \text{ (Figure 4) and we shall use the matching conditions} \\ 312 & (13c)-(13d) \text{ integrated over } Y \text{ and written in terms of } y_1^m \end{array}$

313 (38)
$$\begin{cases} \langle \sigma_1^1 \rangle (0^-, x_2) = \lim_{y_1^m \to +\infty} \left[\langle \tau_1^1 \rangle (x_2, -y_1^m) + y_1^m \frac{\partial \langle \sigma_1^0 \rangle}{\partial x_1} (0^-, x_2) \right], \\ \langle \sigma_1^1 \rangle (0^+, x_2) = \lim_{y_1^m \to +\infty} \left[\langle \tau_1^1 \rangle (x_2, y_1^m) - y_1^m \frac{\partial \langle \sigma_1^0 \rangle}{\partial x_1} (0^+, x_2) \right]. \end{cases}$$

Integrating over Y the Eq. (5a) written at order ε^0 for the inner problem, we get

315 (39)
$$\int_{\mathbf{Y}} d\mathbf{y} \left[\operatorname{div}_{\mathbf{y}} \boldsymbol{\tau}^{1}(x_{2}, \mathbf{y}) + \frac{\partial \tau_{2}^{0}}{\partial x_{2}}(x_{2}, \mathbf{y}) + \tilde{b}(\mathbf{y})u^{0}(0, x_{2}) \right] = 0.$$

316 Two of the three integrals above are easily obtained, namely

317 (40)
$$\begin{cases} \int_{\mathbf{Y}} d\mathbf{y} \ \operatorname{div}_{\mathbf{y}} \boldsymbol{\tau}^{1}(x_{2}, \mathbf{y}) = \langle \tau_{1}^{1} \rangle (x_{2}, y_{1}^{m}) - \langle \tau_{1}^{1} \rangle (x_{2}, -y_{1}^{m}), \\ \int_{\mathbf{Y}} d\mathbf{y} \ \tilde{b}(\mathbf{y}) u^{0}(0, x_{2}) = y_{1}^{m} \left[1 + \langle b \rangle \right] u^{0}(0, x_{2}). \end{cases}$$

We used, for the first integral, the continuity of $\tau^1 \cdot \mathbf{n}$ and the periodicity of $\tau^1 w.r.t.$ 319 y_2 . Note that the first integral of (40) corresponds to the first term in $[\langle \sigma_1^1 \rangle]$, from

(38). For the second integral, we used (10).

Now, let us consider the second integral in (39). First, from (33), we have

322 (41)
$$\frac{\partial \tau_2^0}{\partial x_2}(x_2, \mathbf{y}) = \frac{\partial \langle \sigma_1^0 \rangle}{\partial x_2}(0, x_2) T_2^{(1)}(\mathbf{y}) + \frac{\partial^2 u^0}{\partial x_2^2}(0, x_2) \left[\frac{\tilde{a}(\mathbf{y})/a(y_2)}{\langle 1/a \rangle} + T_2^{(2)}(\mathbf{y}) \right].$$

323 Next, defining $\mathsf{Y}^{\text{-}} = (-y_1^m, 0) \times Y$, $\mathsf{Y}^{\text{+}} = (0, +y_1^m) \times Y$, and using (10), we get

324 (42)
$$\begin{cases} \int_{\mathbf{Y}^{-}} \mathrm{d}\mathbf{y} \ \frac{\tilde{a}(\mathbf{y})/a(y_2)}{\langle 1/a \rangle} \frac{\partial^2 u^0}{\partial x_2^2}(0, x_2) = y_1^m \ \langle 1/a \rangle^{-1} \frac{\partial^2 u^0}{\partial x_2^2}(0, x_2), \\ \int_{\mathbf{Y}^{+}} \mathrm{d}\mathbf{y} \ \frac{\tilde{a}(\mathbf{y})/a(y_2)}{\langle 1/a \rangle} \frac{\partial^2 u^0}{\partial x_2^2}(0, x_2) = y_1^m \ \frac{\partial^2 u^0}{\partial x_2^2}(0, x_2), \end{cases}$$

In (42), we want $\partial_{x_1} \langle \sigma_1^0 \rangle$ to appear, in order to absorb the (diverging) terms in y_1^m in the matching condition (38). Do do so, we use (20) for $x_1 < 0$ and (16) for $x_1 > 0$ and we get

328 (43)
$$\begin{cases} -\frac{\partial \langle \sigma_1^0 \rangle}{\partial x_1} (0^-, x_2) = \langle 1/a \rangle^{-1} \frac{\partial^2 u^0}{\partial x_2^2} (0, x_2) + \langle b \rangle u^0 (0, x_2), \\ -\frac{\partial \langle \sigma_1^0 \rangle}{\partial x_1} (0^+, x_2) = \frac{\partial^2 u^0}{\partial x_2^2} (0, x_2) + u^0 (0, x_2), \end{cases}$$

329 whence (44)

330
$$\int_{\mathbf{Y}}^{\mathbf{(H)}} \mathrm{d}\mathbf{y} \left[\frac{\partial^2 u^0}{\partial x_2^2}(0, x_2) \frac{\tilde{a}(\mathbf{y})/a(y_2)}{\langle 1/a \rangle} + \tilde{b}(\mathbf{y}) u^0(0, x_2) \right] = -y_1^m \left[\frac{\partial \langle \sigma_1^0 \rangle}{\partial x_1}(0^+, x_2) + \frac{\partial \langle \sigma_1^0 \rangle}{\partial x_1}(0^-, x_2) \right].$$

331 It is now sufficient to use (40), (41) and (44) in (39), to get the jump condition

332 (45)
$$[\![\langle \sigma_1^1 \rangle]\!] = -\mathcal{C} \frac{\partial^2 u^0}{\partial x_2^2}(0, x_2), \quad \text{with } \mathcal{C} \equiv \int_{\mathbf{Y}} d\mathbf{y} \ T_2^{(2)}(\mathbf{y}),$$

and we used that $\int_{\mathbf{Y}} d\mathbf{y} \ T_2^{(1)}(\mathbf{y}) = 0$ since $V^{(1)}$ is symmetric *w.r.t.* y_2 . The constant \mathcal{C} is the second interface parameters entering in the jump conditions.

2.3.4. Jump conditions and final homogenized problem. The jump conditions on $(\overline{u}, \overline{\sigma})$ are deduced from (28), with (29),(30) and (37),(45), and read

337 (46)
$$[\![\overline{u}]\!] = \varepsilon \mathcal{B} \langle \sigma_1^0 \rangle (0, x_2), \quad [\![\overline{\sigma}_1]\!] = -\varepsilon \mathcal{C} \ \frac{\partial^2 u^0}{\partial x_2^2} (0, x_2).$$

The above jump conditions define a homogenized problem which can be solved iteratively : first compute $(u^0, \langle \boldsymbol{\sigma}^0 \rangle)$ satisfying (16), (20) with (29) and (30) (compute also

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340 \mathcal{B} and \mathcal{C}) and use the results to get the right hand-side term in (46); then, compute

341 $(\overline{u}, \overline{\sigma})$ which approximate $(u^{\varepsilon}, \sigma^{\varepsilon})$ up to $O(\varepsilon^2)$. As discussed in [4], it is preferable to 342 handle a unique problem and this is done by defining the fields $(\tilde{u}, \tilde{\sigma})$ satisfying the

following homogenized problem and this is done by defining the fields (u, b) satisfying the following homogenized problem

$$\begin{cases} \operatorname{div} \tilde{\boldsymbol{\sigma}} + \langle b \rangle \tilde{u} = 0, \quad \tilde{\boldsymbol{\sigma}} = \begin{pmatrix} \langle a \rangle & 0 \\ 0 & \langle 1/a \rangle^{-1} \end{pmatrix} \boldsymbol{\nabla} \tilde{u}, \quad x_1 < 0 \\ \operatorname{div} \tilde{\boldsymbol{\sigma}} + \tilde{u} = 0, \quad \tilde{\boldsymbol{\sigma}} = \boldsymbol{\nabla} \tilde{u}, \qquad \qquad x_1 > 0 \\ \|\tilde{u}\| = \frac{\varepsilon \mathcal{B}}{2} \left[\tilde{\sigma}_1(0^-, x_2) + \tilde{\sigma}_1(0^+, x_2) \right], \end{cases}$$

$$\begin{bmatrix} \tilde{u} \end{bmatrix} = \frac{\varepsilon \mathcal{B}}{2} \left[\tilde{\sigma}_1(0^-, x_2) + \tilde{\sigma}_1(0^+, x_2) \right], \\ \begin{bmatrix} \tilde{\sigma}_1 \end{bmatrix} = -\frac{\varepsilon \mathcal{C}}{2} \left[\frac{\partial^2 \tilde{u}}{\partial x_2^2}(0^-, x_2) + \frac{\partial^2 \tilde{u}}{\partial x_2^2}(0^+, x_2) \right]$$

and it is easy to see from (16), (27) and (46) that $\tilde{u}, \tilde{\sigma}$ admit the expansions $\langle u^0 \rangle + 346 = \varepsilon \langle u^1 \rangle, \langle \sigma^0 \rangle + \varepsilon \langle \sigma^1 \rangle$ up to $O(\varepsilon^2)$, thus the same expansion as $(u^{\varepsilon}, \sigma^{\varepsilon})$ up to $O(\varepsilon^2)$. Finally, coming back to the real space, in $\mathbf{X} = \mathbf{x}/k$ coordinate and with $U(\mathbf{X}) = \tilde{u}(\mathbf{x})$, $\mathbf{\Sigma}(\mathbf{X}) = k \tilde{\sigma}(\mathbf{x}), (47)$ take the form

$$\begin{cases} \operatorname{div} \boldsymbol{\Sigma} + \langle b \rangle k^2 U = 0, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \langle a \rangle & 0 \\ 0 & \langle 1/a \rangle^{-1} \end{pmatrix} \boldsymbol{\nabla} U, \quad X_1 < 0 \\ \operatorname{div} \boldsymbol{\Sigma} + k^2 U = 0, \quad \boldsymbol{\Sigma} = \boldsymbol{\nabla} U, \qquad \qquad X_1 > 0 \end{cases}$$

349 (48)

$$\llbracket U \rrbracket = \frac{h\mathcal{B}}{2} \left[\Sigma_1(0^-, X_2) + \Sigma_1(0^+, X_2) \right],$$
$$\llbracket \Sigma_1 \rrbracket = -\frac{h\mathcal{C}}{2} \left[\frac{\partial^2 U}{\partial X_2^2}(0^-, X_2) + \frac{\partial^2 U}{\partial X_2^2}(0^+, X_2) \right].$$

The above problem, written for a single interface at $X_1 = 0$, correspond to the system (3)-(4) when two interfaces at $X_1 = \pm e/2$ are considered.

352 **3.** Scattering by an array of rectangular voids. In this section, we apply 353 the previous analysis to rectangular voids or cracks, free of stresses (with Neumann 354 conditions on their boundaries), periodically spaced in a homogeneous matrix being 355 composed of the same elastic material than the substrate. In electromagnetism, this 356 corresponds to a (perfect conducting) metallic array in a dielectric or in the air; in 357 acoustics to an array of sound hard material in a fluid. We consider the scattering of 358 an incident plane wave U^{inc} arriving from $X_1 < 0$ and hitting the array at oblique 359 incidence θ (Figure 5), whence

360 (49)
$$U^{\mathrm{inc}}(\mathbf{X}) = e^{\mathrm{i}k(X_1 + e/2)\cos\theta + \mathrm{i}kX_2\sin\theta}.$$

In the actual problem, the solution is sought in the substrate where the Helmholtz equation apply with Neumann boundary conditions on the boundaries of the voids occupying the subdomains Ω_i , $i = 1 \dots N_v$ and $\Omega_v = \bigcup_i \Omega_i$. The problem is solved in



FIGURE 5. Actual problem of the scattering of a plane wave at oblique incidence θ on an array of rectangular voids; the problem is solved numerically. The homogenized problem involves a slab of same thickness e filled with a homogeneous anisotropic material (the wave equation being (51)); jump conditions (4) apply at $X_1 = \pm e/2$.

364 $\Omega \setminus \Omega_v$ with $\Omega = \{(X_1, X_2) \in \mathbb{R} \times (-H/2, H/2)\}$ and reads

$$365 \quad (50) \quad \begin{cases} \Delta U + k^2 U = 0, & \text{in } \Omega \setminus \Omega_v, \\ \nabla U \cdot \mathbf{n} = 0, & \text{on } \partial \Omega_i, i = 1 \dots N_v \\\\ \lim_{X_1 \to \pm \infty} \left[\frac{\partial}{\partial X_1} (U - U^{\text{inc}}) \mp ik \cos \theta (U - U^{\text{inc}}) \right] = 0, \\\\ U \left(X_1, \frac{H}{2} \right) = e^{ikH \sin \theta} U \left(X_1, -\frac{H}{2} \right), & X_1 \in \mathbb{R}, \\\\ \frac{\partial U}{\partial X_2} \left(X_1, \frac{H}{2} \right) = e^{ikH \sin \theta} \frac{\partial U}{\partial X_2} \left(X_1, -\frac{H}{2} \right), & X_1 \in \mathbb{R}. \end{cases}$$

The conditions at $X_1 \to \pm \infty$ are the radiation conditions required to select an out-366 going scattered waves $(U - U^{inc})$ in the low frequency regime, namely for $k < 2\pi/h$ (if 367 not the case, the radiation condition should to be modified, see [2]). In the case where 368 H = nh, with n an integer (and we shall consider that this is the case), the last condi-369 tion is referred to as the condition of pseudo-periodicity or the Floquet condition, 370 which applies for the incident wave and for the total field [13]. The actual problem is 371 solved numerically using a multimodal method which reduces to the determination of 372 a set of scalar coefficients for $|X_1| < e/2$ and for $|X_1| > e/2$ (see subsection S2.2.1). 373 374 In the following, the computed solution U^{num} is the reference solution.

375 **3.1.** Solutions of the homogenized problems at the first and at the 376 second orders. We shall see that the homogenized problems at the first two orders 377 can be solved exactly. Voids or cracks correspond to the limiting case a = 0 = b378 (leading to Neumann boundary condition at the boundary with any other material). 379 Next, with a = 1 = b in the substrate, and with φ the filling fraction of the substrate in 380 the layers, the equivalent medium has bulk parameters $\langle a \rangle = \langle b \rangle = \varphi$ and $\langle 1/a \rangle^{-1} = 0$, 381 whence the homogenized wave equation (3) becomes

382 (51)
$$\operatorname{div}\boldsymbol{\Sigma} + \varphi k^2 U = 0, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \varphi & 0 \\ 0 & 0 \end{pmatrix} \boldsymbol{\nabla} U.$$

$$\begin{aligned} & \frac{\partial^2 U}{\partial X_1^2} + k^2 U = 0, & \text{for } |X_1| < \frac{e}{2} \\ & \Delta U + k^2 U = 0, & \text{for } |X_1| > \frac{e}{2} \\ & \text{Jump conditions } (4), & \text{at } X_1 = \pm \frac{e}{2} \\ & \lim_{X_1 \to \pm \infty} \left[\frac{\partial}{\partial X_1} (U - U^{\text{inc}}) \mp ik \cos \theta (U - U^{\text{inc}}) \right] = 0, \\ & U \left(X_1, \frac{H}{2} \right) = e^{ikH \sin \theta} \left(X_1, -\frac{H}{2} \right), & X_1 \in \mathbb{R}, \end{aligned}$$

$$\left(\begin{array}{c} \frac{\partial U}{\partial X_2} \left(X_1, \frac{H}{2} \right) = e^{ikH\sin\theta} \frac{\partial U}{\partial X_2} \left(X_1, -\frac{H}{2} \right), \quad X_1 \in \mathbb{R}. \end{aligned} \right)$$

385In the following, we refer to the homogenized problems at first and at second orders for

386 the values of $(\mathcal{B}, \mathcal{C})$ entering in the jump conditions (4); at first order, $(\mathcal{B}, \mathcal{C}) = (0, 0)$ and at second order, 387

(53)
$$\mathcal{B} = -\frac{1}{\pi} \log \sin\left(\frac{\pi\varphi}{2}\right), \quad \mathcal{C} \simeq \frac{\pi}{16}\varphi^2,$$

(see subsection S2.1). 389

384 (52)

388

The solution of (52) with (49) is of the form 390

391 (54)
$$\begin{cases} X_1 < -\frac{e}{2}, \quad U(\mathbf{X}) = \left[e^{ik(X_1 + e/2)\cos\theta} + Re^{-ik(X_1 + e/2)\cos\theta}\right] e^{ikX_2\sin\theta}, \\ |X_1| < \frac{e}{2}, \quad U(\mathbf{X}) = \left[ae^{ikX_1} + be^{-ikX_1}\right] e^{ikX_2\sin\theta}, \\ X_1 > \frac{e}{2}, \quad U(\mathbf{X}) = Te^{ik(X_1 - e/2)\cos\theta + ikX_2\sin\theta}, \end{cases}$$

with (R, T, a, b) given by the jump conditions (4). Indeed, applying (4) to (54), we 392 393 find, for any (k, θ) values characterizing the incident wave, a set (R, T, a, b) such that U is solution of the homogenized problems. In particular, the scattering coefficients 394 (R,T) read 395

396 (55)
$$\begin{cases} R = -\frac{z_1^* z_2^* e^{ike} - z_1 z_2 e^{-ike}}{z_1^{*2} e^{ike} - z_2^2 e^{-ike}}, \\ T = \frac{|z_1|^2 - |z_2|^2}{z_1^{*2} e^{ike} - z_2^2 e^{-ike}}, \end{cases}$$

397

398

(56)

$$\begin{cases} z_1 \equiv \left(1 - \frac{\cos\theta}{\varphi}\right) + ikh\left(\mathcal{B}\cos\theta + \mathcal{C}\frac{\sin^2\theta}{\varphi}\right) - (kh)^2 \sin^2\theta \frac{\mathcal{B}\mathcal{C}}{4}\left(1 + \frac{\cos\theta}{\varphi}\right), \\ z_2 \equiv \left(1 + \frac{\cos\theta}{\varphi}\right) - ikh\left(\mathcal{B}\cos\theta - \mathcal{C}\frac{\sin^2\theta}{\varphi}\right) + (kh)^2 \sin^2\theta \frac{\mathcal{B}\mathcal{C}}{4}\left(1 + \frac{\cos\theta}{\varphi}\right). \end{cases}$$

Obviously, for $\mathcal{B} = \mathcal{C} = 0$, the jump conditions (4) simplify to the continuities of U 399 and $\Sigma \cdot \mathbf{n}$ and we recover the usual expressions of the scattering coefficients given by 400 the homogenization at the first order, with $\cos\theta/\varphi$ the effective impedance mismatch 401 between the two media [12]. Finally, the derivation of (a, b) is straightforward and 402 403 together with (R, T) can be used in (54) to calculate the homogenized wavefield $U(\mathbf{X})$.

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3.2. Numerical validation of the homogenized solutions. We shall inspect 404 the validity of the homogenized solution up to the first and second orders for an 405incident plane wave at normal and at oblique incidences. To begin with, we report the 406 fields U^{num} calculated numerically and the fields U of the homogenized solutions, (54), 407with (55)-(56), for $\varphi = 0.5$ (Figure 6) and $\varphi = 0.9$ (Figure 7); in both cases, kh = 1, 408 e/h = 10 and $\theta = \pi/3$. The improvement in the solutions given by the homogenization 409 at the second order, compared to the ones given by the first order, is visible to the 410 naked eye. Defining $\Delta U \equiv |U - U^{\text{num}}|/|U^{\text{num}}|$ (for $|X_1| > e/2$ and with |.| the L^2 411 norm), we get a discrepancy of 10% ($\varphi = 0.5$) and 20% ($\varphi = 0.9$) for the first order 412homogenization, and of 0.7%-0.6% for the second order homogenization. It is worth 413 noting that a very small error is found even at this relatively large kh = 1 value.



FIGURE 6. Scattering of a plane wave at incidence $\theta = \pi/3$ and kh = 1 on a array made of rectangular voids (e/h = 10 and $\varphi = 0.5$). The solution U^{num} in the actual problem is calculated numerically; on the right, the fields U given by the homogenization at the first and at the second orders.



FIGURE 7. Same representation as in Figure 6 with $\varphi = 0.9$.

414

3.2.1. Waves at normal incidence. We first inspect the case $\theta = 0$, for which 415 the homogenizations, at first and second orders, impose $[\Sigma] \cdot \mathbf{n} = 0$ in (4) (because from 416 $(52), U(X_1, X_2) = U(X_1, 0)$ does not depend on X_2 , thus only the influence of \mathcal{B} in the 417 jump condition for the displacement is regarded (see also (55) with (56)). We report 418 in Figure 8 the spectra of the transmission as a function of kh and e/h (with $\varphi = 0.1$ 419and $\theta = 0$ and in Figure 9 the corresponding errors $\Delta T = |T^{\text{num}} - T|/|T^{\text{num}}|$ (for the 420 homogenizations at the first and at the second orders). We considered $kh \in [0, 2\pi]$, 421 $kh = 2\pi$ corresponding to the cut-off frequency above which a second propagating 422 mode exists in the actual problem. 423

424 In Figure 9, errors smaller than 1% appear in dark blue and errors grater than

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FIGURE 8. Transmission coefficient as a function of e/h and of the frequency kh; $\varphi = 0.1$ and $\theta = 0$ have been considered. On the left, in the actual problem, $|T^{num}|$ calculated numerically, and on the right, |T| given by the first and second order homogenizations, (55), with (56).

100% in dark red. The range of validity of the homogenized solution is significantly 425increased when using the second order homogenization; on the one hand, intermediate 426 frequencies become accessible (the error is smaller than 4% for $kh < \pi$ in the whole 427 range of e/h at second order, it is of 50% on average at first order); on the other hand, 428 the first order homogenization predicts erroneously perfect transmissions for vanishing 429thicknesses e/h, while going up to the second order restores the actual scattering 430 properties of an array of flat voids. This is related to the discussion presented in Ref. 431[7] in the context of electromagnetic waves. In this reference, an effective medium 432 approach at second order is presented assuming the continuities of the displacement 433 434 and of the normal stress (instead of our jump conditions); the analysis concludes that the effective bulk parameter a has to be dependent on the thickness (in this reference, 435a denotes the effective permittivity for electromagnetic waves).



FIGURE 9. Errors ΔT on the transmission coefficient in the homogenization problems (at first and at second orders) as a function of e/h and kh ($\varphi = 0.1$ and $\theta = 0$). Errors below 1% appear in dark blue and errors larger than 100% appear in dark red.

437 More specifically, we inspect (i) the profiles of $|T^{\text{num}}|$ (blue symbols) and its ho-438 mogenized counterparts |T| at first order (grey lines) and at second order (black lines) 439 as a function of kh for e/h = 0.05 and e/h = 4 (Figure 10). For a small thickness 440 (C₁ profile from Figure 8), the homogenization at first order largely overestimates 441 the transmission while including the jump conditions (4) at second order recovers

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442 the actual transmission of the array; for larger array (C_2 profile) the first order ho-

443 mogenization is valid for small kh and going up to the second order allows us to 444 increase the range of validity of the homogenized solution. This is because T^{num} is

445 not periodic w.r.t. kh (the kh-distance between two perfect transmissions decreases

446 as kh increases); from (55), this tendency is correctly captured by T at second order,

while T at first order appears to be erroneously periodic, with $2\pi h/e$ period. Finally,

448 as expected, the errors ΔT vary as kh at the first order and as $(kh)^2$ at the second order.



FIGURE 10. Transmission coefficients $|T^{num}|$ and |T| as a function of kh and corresponding error ΔT of the homogenized predictions. C_1 profile for e/h = 0.05 and C_2 for e/h = 4 ($|T^{num}|$: blue symbols and |T|: grey lines at first order and black lines at second order). ΔT are shown in %, dotted lines are guidelines with slopes kh (grey) and (kh)² (black).

449



FIGURE 11. Transmission coefficient and errors as a function of e/h for $kh = \pi$ (C₃ profile from Figure 8). Same representation as in Figure 10.

(ii) the variations of $|T^{num}|$ and |T| (and the corresponding errors ΔT) as a function of e/h for $kh = \pi$ are reported in Figure 11 (C₃ profile from Figure 8). Amusingly, the large error in the first order homogenization appears to be a direct consequence of the "initial" error for vanishing e/h; there, |T| = 1 is erroneously found at the first order, and the periodicity of T w.r.t e/h imposes the same error to occur for any e/h (the periodicity of T w.r.t. e/h is, from (55), $2\pi/kh$, thus equals 2 in the present case).

3.2.2. The case of oblique incidence. In the case of oblique incidence ($\theta =$ 457 458 $\pi/3$ is considered in this section), the difference between the homogenized solutions at first and second orders involves the two interface parameters $(\mathcal{B}, \mathcal{C})$. We report 459the same sequence of figures as for the normal incidence; with $kh \in [0, 2\pi]$, the 460frequency range includes $kh > 2\pi/(1+\sin\theta) \simeq 1.07\pi$ where two modes are propagating 461 (complete extinction of the first mode is observed at this cut-off frequency, Figure 12 462 and Figure 14, corresponding to the Wood anomaly [10]). This range is outside the 463 range of validity of any homogenization approach since mode coupling is not possible 464

at an equivalent flat boundary [10].



FIGURE 12. Same representation as Figure 8 for $\theta = \pi/3$.



FIGURE 13. Same representation as Figure 9 for $\theta = \pi/3$.

465Inspecting Figure 12 to Figure 15, we recover in general the same tendencies 466 as those observed at normal incidence. On average, for $kh < \pi/2$, the error in the 467 transmission coefficient is smaller than 1% in the whole range of e/h at second order, it 468469is of 40% on average at first order. Also, the first order homogenization underestimates the scattering properties of the structure when e/h goes to zero (C₁ profile in Figure 14 470471 and Figure 15). Reversely for larger e/h, and this is more surprising, it overestimates the scattering properties (with |T| significantly smaller than $|T^{num}|$) even for small 472kh value (from Figure 12 and the C_2 profile in Figure 14). Thus, in general, the 473 improvement in the second order homogenized solution is even more significant at an 474475 oblique incidence.



FIGURE 14. Same representation as Figure 10 for $\theta = \pi/3$.



FIGURE 15. Transmission coefficient and errors as a function of e/h for $kh = 0.6\pi$ (C₃ profile from Figure 12). Same representation as in Figure 10.

4. Concluding remarks. The homogenization that we have presented expli-476 citly accounts for the finite size e/h of stratified structures. At first and at second 477 orders, this makes effective bulk parameters to appear which are simply averages of 478 the bulk parameters in each layer and these effective bulk parameters enter in the 479homogenized wave equation. Note the simplicity happening in the case of stratified 480media: (i) the bulk parameters are found without solving cell problems, and this is 481 well known, (ii) the same effective wave equation is found at first and at second or-482 ders, and this is less known. Note also that this simplicity would be lost in the case of 483 periodic media with a more complex unit cell. Finally, the most significant improve-484 485ment in the presented approach is the derivation of jump conditions involving effective interface parameters when accounting for the finite thickness of the structure. More 486 usually, the continuity of the displacement and of the normal stress are assumed, and 487 this is what we recover at first order. Going up to second order allows us to establish 488 jump conditions for these fields and the improvement in the homogenized result has 489 been exemplified. 490

491 **Acknowledgments.** J.-J.M. thanks the support of the French Agence Natio-492 nale de la Recherche (ANR), under grant Aramis (ANR-12-BS01-0021) "Analysis 493 of Robust Asymptotic Methods In numerical Simulation in mechanics". A.M. thank 494 the support of the LABEX WIFI (Laboratory of Excellence within the French Pro-495 gram "Investments for the Future") under references ANR-10-LABX-24 and ANR- 496 10-IDEX-0001-02 PSL*.

The authors thank the anonymous reviewers for their insightful comments and suggestions.

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532 Annexe A. The case of vanishing slab thicknesses. The validity of the 533homogenization of stratified media with finite size has been exemplified in this paper for varying e/h. In fact, the case of vanishing thicknesses has to be inspected more 534carefully. Indeed, we end with jump conditions reflecting the behavior of the evanescent field at each boundary of the equivalent slab, and these boundary layers have 536been assumed to be independent. In fact, for very small thicknesses, the two boundary 537 layers may interact. In this case, the bulk problem is meaningless and the elementary 538 problems have to include the whole structure; doing so, jump conditions across an 539 equivalent interface are found (and the bulk behavior is disregarded), see [9, 11]. We 540541 obtained in this case

542 (57)
$$\begin{cases} R^{\text{int}} = -\frac{1}{2} \left[\frac{Z_1^*}{Z_1} + \frac{Z_2^*}{Z_2} \right], \\ T^{\text{int}} = \frac{1}{2} \left[\frac{Z_1^*}{Z_1} - \frac{Z_2^*}{Z_2} \right], \end{cases}$$

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544 (58)
$$\begin{cases} Z_1 = 1 - ikh\cos\theta \left(\mathcal{B} + \frac{e}{2\varphi}\right), \\ Z_2 = -i\cos\theta - kh\left(\frac{e\varphi}{2h} + \mathcal{C}\sin^2\theta\right) \end{cases}$$

with $(\mathcal{B}, \mathcal{C})$ in (53). Here, we illustrate the limitation of the present homogenization in the light of the size of the boundary layers in the actual problem.



FIGURE 16. Boundary layers in the static elementary problems and evanescent fields in the actual problems for $\varphi = 0.5$. In the actual problem, we considered kh = 1 and $\theta = \pi/4$.

In Figure 16, we reported (i) the elementary solutions $V^{(1,2)}$ of (S9) and (S17) for $\varphi = 0.5$ (for $V^{(1)}$, the external load is (y_1) and it has been removed to exhibit the resulting boundary layer), (ii) the evanescent fields in the actual problem for kh = 1and varying e/h = 1, 0.5 and 0.25 (the total wavefields are continuous but the evanescent fields are not). Qualitatively, for e/h = 1, it is visible that the evanescent

⁵⁴⁶

fields in the actual problem ressemble (by parts) to the fields in the elementary problems. This is quite natural if one remembers that the evanescent field in the actual problem can be decomposed in modes with y_1 dependance being of the form $e^{\pm ik_n y_1}$, and $k_n = \sqrt{k^2 - (n\pi/h)^2} \sim i(n\pi/h)$ for $y_1 > 0$, $k_n = \sqrt{k^2 - (n\pi/h\varphi)^2} \simeq i(n\pi/h\varphi)$ for $y_1 < 0$, thus involving at the dominant order the same y_1 dependences as in the static problems (see subsection S2.2.1). Decreasing *e* result in the interaction of the evanescent fields within the array, or in other words the whole array is concerned by the boundary layer effects.

Quantitatively, Figure 17 shows T^{num} and T (at second order) and the resulting 560error ΔT as a function of e/h in logscale up to $e/h = 10^{-2}$ (and $\varphi = 0.1, 0.5$ and 5610.9). It is visible that the error significantly increases for e/h < 1, and the increase is 562563more significant for larger φ ; to estimate more carefully the critical thickness below which the present homogenization is not valid anymore (or say less accurate), one 564should consider the influence of φ (φ being a measure of the size of boundary layer, 565to be compared to e/h the array thickness in **y**-coordinates). We report in dotted 566lines the transmission coefficient T^{int} given by the interface homogenization (57)-(58) 567 (from [9]. As expected, this homogenization which accounts for the boundary layers 568 569 in the whole array, is more accurate for vanishing thickness.



FIGURE 17. Transmission coefficient $|T^{\text{num}}|$ (blue symbols) and |T| at second order (black lines) as a function of e/h for $\varphi = 0.9$ and 0.5, and corresponding errors ΔT . The errors increase significantly for roughly e/h < 1. Dotted orange lines show the transmission coefficient $|T^{\text{int}}|$ obtained using interface homogenization (57)-(58) (from [9]).