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Complete Mueller matrix from a partial polarimetry experiment: the nine-element case

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Abstract: We show that an incomplete, nine-element Mueller matrix with a row and a column missing, obtained in a partial polarimetry experiment can be completed to a full, sixteen-element Mueller matrix, provided depolarization is absent experimentally. There exist exactly two solutions for the missing row and column, differing from one another only by the signs of the respective row and column elements. To select the correct solution, additional information on the sample properties, such as weak anisotropy or special symmetry, is needed. We provide analytical and numerical procedures for completing the partial Mueller matrix for the cases of practical interest and illustrate the approach on an experimental example.

OCIS codes: (260.5430) Polarization; (120.2130) Ellipsometry and polarimetry; (120.5410) Polarimetry; (050.1950) Diffraction gratings; (160.1190) Anisotropic optical materials

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1. Introduction

The 4×4 Mueller matrix \mathbf{M} is the most general descriptor of a linear medium or optical system in the context of its interaction (reflection, transmission or scattering) with polarized light [1,2]. A Mueller matrix polarimeter (shorter, Mueller polarimeter) contains a polarization state generator (PSG) and a polarization state analyzer (PSA) in the path of its excitation and detection light beams respectively (the excitation beam is the one probing the sample whereas the detection one is the one collected after interaction with the sample) [2]. The role of the PSG (PSA) is to produce (analyze) polarized light, formally described by a four-component Stokes vector. Combining four (or more) input Stokes vectors generated by the PSG with four (or more) output ones analyzed by the PSA results in a matrix of sixteen measured light intensities that permit the determination of the sixteen elements of the Mueller matrix of the sample.

As briefly sketched above, the principle of Mueller polarimetry is actually a generalization of that of ellipsometry [2-6] whereby the PSA analyzes four (or more) output Stokes vectors whereas the PSG generates just a single input one (or vice versa). Furthermore, in ellipsometry the PSA is typically incomplete in the sense that it analyzes only three out of the four Stokes vector components [2,4]. The PSG is likewise incomplete since at least one (two, usually) of the components of the single Stokes vector it generates are identically zero. Eventually, an ellipsometer measures a vector of three light intensities containing linear combinations of up to three (two, usually) Mueller matrix columns from which up to three matrix elements can be determined. Nevertheless, this partial polarimetry experiment is sufficient to characterize completely an optically isotropic, nondepolarizing sample in terms of the two ellipsometric angles ψ and Δ . Typical ellipsometer designs with incomplete PSAs are the rotating analyzer ellipsometer (RAE) [2-6] and the photoelastic modulator-based phase modulation ellipsometer (PME) [5-7].

Historically, the development of ellipsometers preceded that of Mueller polarimeters and there are still many RAE and PME instruments, either homemade or commercial, in use. These instruments can be readily extended to measure nine, instead of only three, matrix elements [4]. Alternatively stated, RAE and PME designs are potentially capable of performing nine-element partial Mueller polarimetry. A third, more sophisticated design, the two-modulator generalized ellipsometer (2-MGE), also belongs to this class of partial Mueller polarimeters [4,7-9]. Other designs allow for twelve-element partial polarimetry where only a single column (row) of the Mueller matrix is missing [4,10]. All these designs can be considered to be special cases of the so-called channeled partial Mueller polarimetry whereby (at least) nine “channels” (linear combinations of Mueller matrix elements) are measured [11].

The question therefore arises whether it is possible to recover the complete Mueller matrix of the sample knowing only nine of its elements, determined in a partial polarimetry experiment. Such a recovery is clearly of interest if the sample is anisotropic, i.e. if it converts p - to s -polarization and vice versa (p - or s -polarization refer to the electric field vector being parallel or normal to the incidence plane defined by the incident beam and the sample normal). Stated equivalently, an anisotropic sample is described by a non-diagonal Jones matrix and a complete Mueller matrix (except for some trivial orientations of the anisotropy) [3]. Its experimental characterization is often referred to as generalized ellipsometry (GE) [4], as opposed to conventional ellipsometry, commonly performed on samples whose Jones matrix off-diagonal elements are zero (resulting in a special, block-diagonal Mueller matrix). Another important motivation is that knowing the complete Mueller matrix is a formal prerequisite for the application of the powerful algebraic decomposition methods making possible the phenomenological interpretation of complex media and systems in physical terms [12-14]. The purpose of the present work is to show under which conditions the recovery of the complete Mueller matrix is possible, as well as to provide and illustrate the explicit procedures for achieving it.

2. Nine-element partial polarimetry

Denote the partial Mueller polarimeter as $\text{PSG}(m, k)/\text{PSA}(n, l)$ where n (m) is the minimum number of Stokes vectors analyzed (generated) by the PSA (PSG), and l (k) is the subscript (running from 2 to 4) of the inaccessible, i.e. unanalyzed (zero) component in a Stokes vector; if all components are analyzed (non-zero) then $l = 0$ ($k = 0$). A generic complete Mueller polarimeter is then written as $\text{PSG}(4, 0)/\text{PSA}(4, 0)$. It measures a matrix of $m \times n = 16$ light intensities from which the Mueller matrix of the sample is obtained. The historically very common RAE design is therefore written as $\text{FP}(1, 4)/\text{RA}(4, 4)$ where FP and RA stand for fixed polarizer and rotating analyzer, respectively. Indeed, an incomplete PSG based on a fixed polarizer produces a single ($m = 1$) Stokes vector with zero circular ($k = 4$) component out of the unpolarized light of the source. A PSA based on a rotating analyzer analyzes at least four (actually, a continuous infinity of) Stokes vectors (so that $n = 4$) and, as with the fixed polarizer, is incapable of analyzing the circular component ($l = 4$). RAE measures a light intensity vector containing $m \times (n - 1) = 3$ components. To extend the RAE design to a partial polarimeter one, one needs to rotate the polarizer too, resulting in the $\text{RP}(4, 4)/\text{RA}(4, 4)$ design with a measured light intensity matrix of $(m - 1) \times (n - 1) = 9$ elements [2,4,15-17]. This intensity matrix allows for the determination of the Mueller matrix except for the $k = 4^{\text{th}}$ column and $l = 4^{\text{th}}$ row. We shall denote the extended RAE design by RPAE (rotating polarizer – analyzer ellipsometer).

Similarly, the two typical PME design configurations are $\text{FP}(1, 4)/\text{PM}(4, 2)$ or $\text{FP}(1, 4)/\text{PM}(4, 3)$ depending on whether the photoelastic modulator (PM) makes the azimuth $\theta_m = 0^\circ$ (90°) or $\pm 45^\circ$ with the incidence plane. When extended to partial polarimetry by rotating the polarizer, these two ellipsometric configurations become $\text{RP}(4, 4)/\text{PM}(4, 2)$ or $\text{RP}(4,$

4)/PM(4, 3) measuring $(m - 1) \times (n - 1) = 9$ light intensities and therefore allowing for the determination of the Mueller matrix except for the $k = 4^{\text{th}}$ column and $l = 2^{\text{nd}}$ or 3^{rd} row. We shall designate this design as extended PME. Note that if the excitation and detection heads of the extended PME design are exchanged or, equivalently, if the light propagation direction is reversed, then the two configurations become PM(4, 2)/RA(4, 4) or PM(4, 3)/RA(4, 4) yielding partial Mueller matrices with missing $k = 2^{\text{nd}}$ or 3^{rd} column and $l = 4^{\text{th}}$ row respectively. We shall refer to these two configurations as being dual to the initial ones.

Finally, the 2-MGE design is commonly operated in one of the two configurations, PM(4, 3)/PM(4, 2) or PM(4, 2)/PM(4, 3) depending on the azimuths, $\theta_{m1}/\theta_{m2} = \pm 45^\circ/0^\circ$ (90°) or $\theta_{m1}/\theta_{m2} = 0^\circ$ (90°)/ $\pm 45^\circ$, of the PMs in the excitation/detection beams respectively [8]. The two configurations are dual to one another; both measure inherently $(m - 1) \times (n - 1) = 9$ light intensities from which the Mueller matrix can be derived except for the $k = 3^{\text{rd}}$ or 2^{nd} column and $l = 2^{\text{nd}}$ or 3^{rd} row. Unlike the RPAE and extended PME designs, the 2-MGE one performs partial polarimetry without any rotating parts.

In summary, the RPAE and the extended PME designs, as well as the 2-MGE one, represent partial Mueller polarimeters yielding the Mueller matrix of the sample with one row and one column missing. Table 1 puts into correspondence the various instrument designs with the explicit partial Mueller matrices they measure.

Table 1. Instruments and nine-element partial Mueller matrices they measure.
(Bullets denote missing matrix elements.)

Instrument	partial Mueller matrix	
RPAE	$\begin{bmatrix} M_{11} & M_{12} & M_{13} & \bullet \\ M_{21} & M_{22} & M_{23} & \bullet \\ M_{31} & M_{32} & M_{33} & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix}$	
Extended PME	PM in PSA	$\begin{array}{cc} \theta_m = 0^\circ (90^\circ) & \theta_m = \pm 45^\circ \\ \begin{bmatrix} M_{11} & M_{12} & M_{13} & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ M_{31} & M_{32} & M_{33} & \bullet \\ M_{41} & M_{42} & M_{43} & \bullet \end{bmatrix} & \begin{bmatrix} M_{11} & M_{12} & M_{13} & \bullet \\ M_{21} & M_{22} & M_{23} & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ M_{41} & M_{42} & M_{43} & \bullet \end{bmatrix} \end{array}$
	PM in PSG	$\begin{array}{cc} \theta_m = 0^\circ (90^\circ) & \theta_m = \pm 45^\circ \\ \begin{bmatrix} M_{11} & \bullet & M_{13} & M_{14} \\ M_{21} & \bullet & M_{23} & M_{24} \\ M_{31} & \bullet & M_{33} & M_{34} \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix} & \begin{bmatrix} M_{11} & M_{12} & \bullet & M_{14} \\ M_{21} & M_{22} & \bullet & M_{24} \\ M_{31} & M_{32} & \bullet & M_{34} \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix} \end{array}$

2-MGE		$\theta_{m1} = 0^\circ (90^\circ); \theta_{m2} = \pm 45^\circ$	$\theta_{m1} = \pm 45^\circ; \theta_{m2} = 0^\circ (90^\circ)$
	PM1 in PSG	$\begin{bmatrix} M_{11} & \bullet & M_{13} & M_{14} \\ M_{21} & \bullet & M_{23} & M_{24} \\ \bullet & \bullet & \bullet & \bullet \\ M_{41} & \bullet & M_{43} & M_{44} \end{bmatrix}$	$\begin{bmatrix} M_{11} & M_{12} & \bullet & M_{14} \\ \bullet & \bullet & \bullet & \bullet \\ M_{31} & M_{32} & \bullet & M_{34} \\ M_{41} & M_{42} & \bullet & M_{44} \end{bmatrix}$
	&	$\theta_{m1} = 0^\circ (90^\circ); \theta_{m2} = 0^\circ (90^\circ)$	$\theta_{m1} = \pm 45^\circ; \theta_{m2} = \pm 45^\circ$
	PM2 in PSA	$\begin{bmatrix} M_{11} & \bullet & M_{13} & M_{14} \\ \bullet & \bullet & \bullet & \bullet \\ M_{31} & \bullet & M_{33} & M_{34} \\ M_{41} & \bullet & M_{43} & M_{44} \end{bmatrix}$	$\begin{bmatrix} M_{11} & M_{12} & \bullet & M_{14} \\ M_{21} & M_{22} & \bullet & M_{24} \\ \bullet & \bullet & \bullet & \bullet \\ M_{41} & M_{42} & \bullet & M_{44} \end{bmatrix}$

3. Recovery of the complete Mueller matrix

Given the 2×2 complex Jones matrix \mathbf{J} of a sample (a medium or an optical system),

$$\mathbf{J} = \begin{bmatrix} J_1 & J_3 \\ J_4 & J_2 \end{bmatrix} \quad (1)$$

its corresponding 4×4 real Mueller matrix \mathbf{M} is [3]

$$\mathbf{M} = \begin{bmatrix} \frac{1}{2}(E_1 + E_2 + E_3 + E_4) & \frac{1}{2}(E_1 - E_2 - E_3 + E_4) & F_{13} + F_{42} & -G_{13} - G_{42} \\ \frac{1}{2}(E_1 - E_2 + E_3 - E_4) & \frac{1}{2}(E_1 + E_2 - E_3 - E_4) & F_{13} - F_{42} & -G_{13} + G_{42} \\ F_{14} + F_{32} & F_{14} - F_{32} & F_{12} + F_{34} & -G_{12} + G_{34} \\ G_{14} + G_{32} & G_{14} - G_{32} & G_{12} + G_{34} & F_{12} - F_{34} \end{bmatrix} \quad (2a)$$

where

$$E_i = \langle |J_i|^2 \rangle \quad (2b)$$

$$F_{ij} = F_{ji} = \text{Re} \langle J_i^* J_j \rangle \quad (2c)$$

$$G_{ij} = -G_{ji} = \text{Im} \langle J_i^* J_j \rangle \quad (2d)$$

are the second-order moments of the Jones matrix elements (the brackets $\langle \dots \rangle$ denote spatial or time averaging). If the sample is nondepolarizing, i.e. transforming totally polarized incident light into totally polarized outgoing one, then the averaging process is inactive and the brackets can be omitted. There is equivalence between \mathbf{J} and \mathbf{M} (the one can be obtained from the other, except for the “absolute phase” of \mathbf{J}). If the sample is depolarizing, then the averaging, physically originating from the interaction between the sample and the

measurement system, is essential. Consequently, to the single \mathbf{M} there correspond many spatial or time realizations of \mathbf{J} (\mathbf{J} is then called the Jones generator of \mathbf{M} [18]). To every \mathbf{M} (depolarizing or not) there can be associated a Hermitian matrix \mathbf{H} , called covariance matrix [14,19], given by

$$\mathbf{H} = \frac{1}{4} \sum_{i,j} M_{ij} (\sigma_i \otimes \sigma_j) = \frac{1}{2} \begin{bmatrix} E_1 & F_{13} - iG_{13} & F_{14} - iG_{14} & F_{12} - iG_{12} \\ F_{13} + iG_{13} & E_3 & F_{34} - iG_{34} & F_{32} - iG_{32} \\ F_{14} + iG_{14} & F_{34} + iG_{34} & E_4 & F_{42} - iG_{42} \\ F_{12} + iG_{12} & F_{32} + iG_{32} & F_{42} + iG_{42} & E_2 \end{bmatrix} \quad (3)$$

where σ_i are the Pauli spin matrices (the symbol “ \otimes ” denotes the Kronecker product). The two matrices \mathbf{H} and \mathbf{M} are informationally equivalent. The matrix \mathbf{H} is positive semidefinite, i.e. its four (real) eigenvalues λ_k , $k = 1, 2, 3, 4$, are nonnegative. The rank of \mathbf{H} (denoted by $\text{rank}(\mathbf{H})$), i.e. the number of its non-vanishing eigenvalues, is an important indicator of the depolarization introduced by the averaging process. The rank of \mathbf{H} for a nondepolarizing \mathbf{M} equals one, i.e. \mathbf{H} has a single non-vanishing eigenvalue ($\lambda_1 \neq 0$; $\lambda_2 = \lambda_3 = \lambda_4 = 0$) indicating an inactive averaging process. The rank of \mathbf{H} can be two, three or four for a depolarizing \mathbf{M} (with an active averaging process).

Denote by $\mathbf{M}(r, c)$ a partial Mueller matrix whose r th row and c th column are missing (i.e. are not determined experimentally). As discussed in Section 2 and presented in Table 1, the RPAE and the extended PME designs, as well as the 2-MGE one, provide the partial Mueller matrices $\mathbf{M}(4, 4)$; $\mathbf{M}(2, 4)$ or $\mathbf{M}(3, 4)$, and $\mathbf{M}(2, 3)$, $\mathbf{M}(3, 2)$, $\mathbf{M}(2, 2)$ and $\mathbf{M}(3, 3)$, respectively. To recover the complete Mueller matrix \mathbf{M} given a partial one, i.e. to “fill in” the missing r th row and c th column of $\mathbf{M}(r, c)$, additional assumptions on \mathbf{M} should be made. One “natural” hypothesis is to assume that \mathbf{H} is rank-deficient and that its rank is known. Indeed, if $\text{rank}(\mathbf{H}) = p$ then $4 - p$ eigenvalues of the positive semidefinite \mathbf{H} must vanish which is equivalent to setting to zero all its principal minors of order higher than p . Thus, if $\text{rank}(\mathbf{H}) = 3$ then \mathbf{H} has exactly one vanishing eigenvalue, so that its determinant (i.e. its only minor of order four) must vanish, yielding a single equation. If $\text{rank}(\mathbf{H}) = 2$ then all four principal 3×3 minors of \mathbf{H} , as well as its determinant are zero, providing five equations. Finally, $\text{rank}(\mathbf{H}) = 1$ produces a total of eleven equations (six 2×2 and four 3×3 principal minors, as well as the determinant vanishing) [19]. The number of equations is to be compared to the number of unknown Mueller matrix elements, which, for a row and a column missing, equals $2 \times 4 - 1 = 7$. Clearly, only the case $\text{rank}(\mathbf{H}) = 1$ can be potentially addressed (since it is the only one where the number of equations exceeds the number of unknowns). This restricts nine-element partial Mueller polarimetry to the recovery of nondepolarizing Mueller matrices only.

Even though a solution to the problem of completing the partial $\mathbf{M}(r, c)$ to the full \mathbf{M} potentially exists if \mathbf{M} is nondepolarizing, it may not be unique. It is shown in Appendix A that there exist exactly two solutions for a nondepolarizing \mathbf{M} namely, \mathbf{M} itself, as well as \mathbf{M} having the signs of its r th row and c th column inverted (except for that of its rc th element). In order to select the correct solution, one therefore needs extra information on the sample \mathbf{M} characterizes.

3.1 Weakly anisotropic Mueller matrix

The large majority of continuous media are only weakly anisotropic, i.e. are described electromagnetically by a dielectric tensor whose associated refractive index ellipsoid is very close to a sphere. Examples are most of the natural crystals (e.g. quartz), as well as biological

and turbid media, such as human (or animal) tissues or their phantoms, etc. Some artificial optical structures, such as diffraction gratings whose period is close to the probing wavelength also belong to the weak anisotropy class. From the viewpoint of Jones-Mueller algebra, the weak anisotropy condition consists in the off-diagonal elements of the Jones matrix \mathbf{J} being much smaller, in absolute value, than the diagonal ones [8] or, formally, $\max(|J_3|, |J_4|) \ll \min(|J_1|, |J_2|)$, in the notations of Eq. (1). The physical meaning of this condition is that the sample exhibits a much weaker cross-polarization (i.e. p -to- s and s -to- p polarization conversion) than co-polarization (i.e. p -to- p and s -to- s polarization conversion). The condition can be equivalently stated as $|r_3|^2 + |r_4|^2 \ll |r_1|^2 + |r_2|^2$ and expressed, as directly follows from Eqs. (2b) and (2a), in terms of Mueller matrix elements,

$$\frac{|r_3|^2 + |r_4|^2}{|r_1|^2 + |r_2|^2} = \frac{E_3^2 + E_4^2}{E_1^2 + E_2^2} = \frac{M_{11} - M_{22}}{M_{11} + M_{22}} \ll 1 \quad (4)$$

(Notice that the last expression holds even if \mathbf{M} is depolarizing, i.e. with the averaging process (...) being active.) The weak anisotropy condition (4) can be used to select the correct solution for the complete \mathbf{M} knowing the experimental partial one, $\mathbf{M}(r, c)$, provided either $r = 2$ or $c = 2$ and $r \neq c$. This covers the cases of partial Mueller matrices $\mathbf{M}(2, 3)$ and $\mathbf{M}(3, 2)$ provided by a 2-MGE equipment, as well as $\mathbf{M}(2, 4)$ (or its dual $\mathbf{M}(4, 2)$) from an extended PME with PM azimuth set to $\theta_m = 0^\circ$ (90°). However, the cases $\mathbf{M}(3, 4)$ (or its dual $\mathbf{M}(4, 3)$), as well as $\mathbf{M}(4, 4)$, arising respectively with an extended PME whose PM azimuth set to $\theta_m = \pm 45^\circ$, and a RPAE, remain unresolved. Notice that if it is known *a priori* that the sample is weakly anisotropic, then it is unnecessary to calculate the ratio in the expression (4), but it is instead sufficient to pick that one of the two possible solutions for \mathbf{M} in which the element M_{22} is positive. (The other possible solution will necessarily feature a negative M_{22} element and the ratio appearing in Eq. (4) will be larger than the unit.)

3.2 Mueller matrix exhibiting symmetries

To address the unresolved case of partial $\mathbf{M}(3, 4)$ (or its dual $\mathbf{M}(4, 3)$) occurring with an extended PME whose PM azimuth is $\theta_m = \pm 45^\circ$ (as well as with the rest of the cases if the weak anisotropy assumption does not hold) one can resort, for certain classes of samples, to symmetry considerations. The Mueller matrix \mathbf{M} is said to exhibit symmetries if its off-diagonal elements obey relations of the kind $M_{ij} = \pm M_{ji}$. These relations stem directly from their Jones matrix counterparts, $J_4 = \pm J_3$, see Eq. (1), which in turn follow physically from the intrinsic symmetry properties of the sample. More specifically, it has been shown [20] that there exist exactly two reflection and one rotation transformations of the sample whose Jones matrices can be derived from the Jones matrix of the sample in its initial position. If the sample image obtained after applying one of these three transformations coincides identically with the sample itself, then a symmetry relation results. Thus, if the sample coincides with its own mirror image with respect to the incidence plane (i.e. the sample is mirror-symmetric with respect to this plane) then $J_4 = J_3 = 0$; the Jones matrix is diagonal and its Mueller matrix \mathbf{M}_{bd} has the well-known special block-diagonal (“psi-delta”, in ellipsometric language) form,

$$\mathbf{M}_{bd} = \begin{bmatrix} M_{11} & M_{12} & 0 & 0 \\ M_{12} & M_{11} & 0 & 0 \\ 0 & 0 & M_{33} & M_{34} \\ 0 & 0 & -M_{34} & M_{33} \end{bmatrix} \quad (5)$$

Second, if the mirror image of the sample with respect to the plane perpendicular to the incidence plane and containing the sample normal (called the bisectrix plane [20]) is identical to the sample itself, then $J_4 = J_3$ (i.e. the Jones matrix is symmetric) and the corresponding Mueller matrix \mathbf{M}_b is

$$\mathbf{M}_b = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{12} & M_{22} & M_{23} & M_{24} \\ M_{13} & M_{23} & M_{33} & M_{34} \\ -M_{14} & -M_{24} & -M_{34} & M_{44} \end{bmatrix} \quad (6)$$

Finally, if a 180°-rotation of the sample about its normal brings it in its initial position, then $J_4 = -J_3$ (i.e. the Jones matrix is antisymmetric) and the resulting \mathbf{M}_r is

$$\mathbf{M}_r = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{12} & M_{22} & M_{23} & M_{24} \\ -M_{13} & -M_{23} & M_{33} & M_{34} \\ M_{14} & M_{24} & -M_{34} & M_{44} \end{bmatrix} \quad (7)$$

Eventually, it is easy to see that if the sample exhibits simultaneously any two of the above three symmetries, this entails the third one and, consequently, its Mueller matrix is the special block-diagonal one, \mathbf{M}_{bd} , given by Eq. (5). This is the most common case of an isotropic medium. Being of general nature, Eqs. (5-7) hold even for depolarizing samples and apply to both anisotropic media (crystals [21] or photonics structures) and optical systems (natural or artificial). For instance, a diffraction grating with a symmetric profile whose grating direction makes an arbitrary azimuth with the incidence plane is invariant with respect to a 180°-rotation about its normal. Consequently, it has an antisymmetric Jones matrix and its Mueller matrix is given by Eq. (7). If the grating direction azimuth assumes a trivial value (either 0° or 90°, i.e. the grating is either parallel or perpendicular to the incidence plane, respectively), then the first symmetry property (mirror symmetry with respect to the incidence plane) is additionally satisfied and the resulting Mueller matrix is of the special block-diagonal form \mathbf{M}_{bd} specified by Eq. (5).

Since Mueller matrix symmetries involve off-diagonal elements of the same row and column, they resolve, when present, all partial $\mathbf{M}(r, c)$ cases where $r \neq c$, i.e. different rows and columns are missing. Indeed, they allow one to select the correct solution out of the two potential ones for the complete \mathbf{M} , determined from the partial $\mathbf{M}(r, c)$. In particular, they can be advantageously used to resolve the case of partial $\mathbf{M}(3, 4)$ (or its dual $\mathbf{M}(4, 3)$) to which the weak anisotropy condition (4) cannot be applied.

Incidentally, one may wonder if, instead for discriminating the correct solution for \mathbf{M} , symmetries could not be used for “filling in” the missing off-diagonal elements in the partial

$\mathbf{M}(r, c)$ prior to determining the only two unknown diagonal elements left, M_{rr} and M_{cc} , knowing the rank of \mathbf{H} . This increases the rank of \mathbf{H} that could potentially be tackled since, if $\text{rank}(\mathbf{H}) = 2$, one then has five equations for only two unknowns. However, the detailed analysis shows that, despite the effective overdetermination, there exist multiple solutions for the two unknown diagonal elements that cannot be further discriminated from one another.

3.3 Partial Mueller matrix with same row and column missing

A partial $\mathbf{M}(r, c)$ with $r = c$ cannot be completed to full \mathbf{M} by using either the weak anisotropy assumption or any symmetry properties. In practice, to this class belong $\mathbf{M}(2, 2)$ and $\mathbf{M}(3, 3)$ obtained in a 2-MGE experiment with both photoelastic modulators set at azimuths $\theta_{m1,2} = 0^\circ$ (90°) and $\theta_{m1,2} = \pm 45^\circ$ respectively [8] and, most notably, $\mathbf{M}(4, 4)$ provided by an RPAE [4]; see Table 1. In the latter case, the ambiguity is between \mathbf{M} and $\mathbf{S}_4\mathbf{M}\mathbf{S}_4$ where $\mathbf{S}_4 = \text{diag}(1 \ 1 \ 1 \ -1)$, as shown in Appendix A. Since to $\mathbf{S}_4\mathbf{M}\mathbf{S}_4$ there corresponds the complex conjugate Jones matrix \mathbf{J}^* as can be easily seen from Eqs. (2), one then cannot discriminate \mathbf{J} from \mathbf{J}^* . This is the formal proof of the well-known fact that the RPAE-based experiment is unable to determine the signs of the phases of the Jones matrix elements [22]. The procedure from Appendix A can be applied to obtain the two possible solutions for the complete Mueller matrix. (Similar procedures can be devised for the $\mathbf{M}(2, 2)$ and $\mathbf{M}(3, 3)$ cases, too.) Nonetheless, additional information on the sample, other than weak anisotropy or presence of symmetries, is needed to select the correct complete Mueller matrix.

There is, however, one trivial situation when the solution is unique, namely, if $\mathbf{M} = \mathbf{S}_4\mathbf{M}\mathbf{S}_4$, i.e. the fourth row and column of \mathbf{M} are zero, except possibly for the M_{44} element which can be recovered following the procedure from Appendix A. This situation occurs when $\mathbf{J} = \mathbf{J}^*$, i.e. the Jones matrix of the sample is real, as follows from Eqs. (2). Experimentally, this is the case where one characterizes a perfectly smooth (with respect to the wavelength), transparent semi-infinite medium by using reflection polarimetry. A classic example is provided by the reflection measurement of thick polished (or cleaved) anisotropic crystals.

Before ending the theoretical part and proceeding to its experimental validation, it should be noted that the ultimate objective of partial Mueller polarimetry is not always the recovery of the complete Mueller matrix. There exist alternative approaches, belonging to the so-called adaptive polarimetry [23] whereby only a selection of measurable linear combinations of Mueller matrix elements (termed “channels”) [24], instead of the reconstructed complete Mueller matrix, is applied to the problems of remote sensing or invariant discrimination [25].

4. Experimental examples

4.1 Experimental details

A home-made UV-visible spectroscopic Mueller polarimeter based on four photoelastic modulators (abbreviated 4-PEM) was used to measure the complete Mueller matrix in a reflection configuration at the incidence angle of 65° . A detailed description of the equipment can be found in Ref. 26. The sample was a symmetric profile diffraction grating with a 500-nm period whose structure and elaboration are described in detail elsewhere [27]. The grating direction made the azimuthal angle of 45° with the incidence plane of the instrument, ensuring non-vanishing off-diagonal-block Mueller matrix elements. Being spatially homogeneous and optically thick (i.e. no signal coming from its backside), the grating sample is nondepolarizing (i.e. $\text{rank}(\mathbf{H}) = 1$), thus making possible the recovery of its complete Mueller matrix from a partial, nine-element experimental one.

4.2 Completion of partial $\mathbf{M}(2, 3)$ from a 2-MGE experiment

Figures 1(a, b) present the complete Mueller matrix (solid line) of the grating measured in the 240-nm – 650-nm wavelength range. Assuming that the second row and the third column of \mathbf{M} are missing, i.e. that we are dealing with a partial $\mathbf{M}(2, 3)$ originating from a 2-MGE experiment instead of the complete \mathbf{M} , one can recover \mathbf{M} analytically by following the algebraic procedure from Appendix B. The result of the analytical recovery is reported in red crosses in Fig. 1(a). The elements of the second row and the third column (except for the M_{23} one) change randomly their signs with the change in wavelength, toggling between the actual solution and the opposite one, as discussed in Section 3. One thus gets isolated points or, at best, piecewise continuous spectra. For comparison, a numerical recovery is also presented (green circles). It was obtained by simultaneously minimizing the six 2×2 and the four 3×3 principal minors, as well as the determinant of the covariance matrix to get the seven missing Mueller matrix elements, implemented as adjustable parameters in the algorithm. The agreement between the measured elements and the recovered ones is very good for both analytical and numerical approaches, if abstraction is made of the sign changes (inherent to the existence of two solutions). This means that either approach can be used on experimental data with comparable quality in terms of noise and systematic errors, the analytical one being much faster computationally, while the numerical one is generally more noise-resilient.

To select the correct solution out of the two potential ones (and thus, suppress the spectral sign “jumps”) one can use either the weak anisotropy assumption or the symmetries of the sample. A diffraction grating approximately behaves as an isotropic system at wavelengths larger than its period; the anisotropy in its optical response increases with decreasing the wavelength, so that at wavelengths comparable to its period the grating can be considered as weakly anisotropic in the sense of Eq. (4). The expected behavior is experimentally confirmed by the assumed partial spectroscopic Mueller matrix $\mathbf{M}(2, 3)$ shown in Fig. 1(a). At wavelengths above 500 nm (the grating period) the lower 2×2 off-diagonal block and the M_{14} element of $\mathbf{M}(2, 3)$ are small suggesting that the Mueller matrix is close to the special block-diagonal form \mathbf{M}_{bd} from Eq. (5), characteristic of an isotropic sample. Conversely, below 500 nm neither of the elements of $\mathbf{M}(2, 3)$ vanishes which is a clear manifestation of anisotropy. Since the anisotropy is relatively weak at wavelengths around the grating period then, according to the discussion following Eq. (4), one must choose that one of the two solutions which features a positive M_{22} element. The recovery, shown in Fig. 1(b) for both analytical and numerical approaches, matches perfectly the experimental data (no more sign changes are seen on the second row and third column elements).

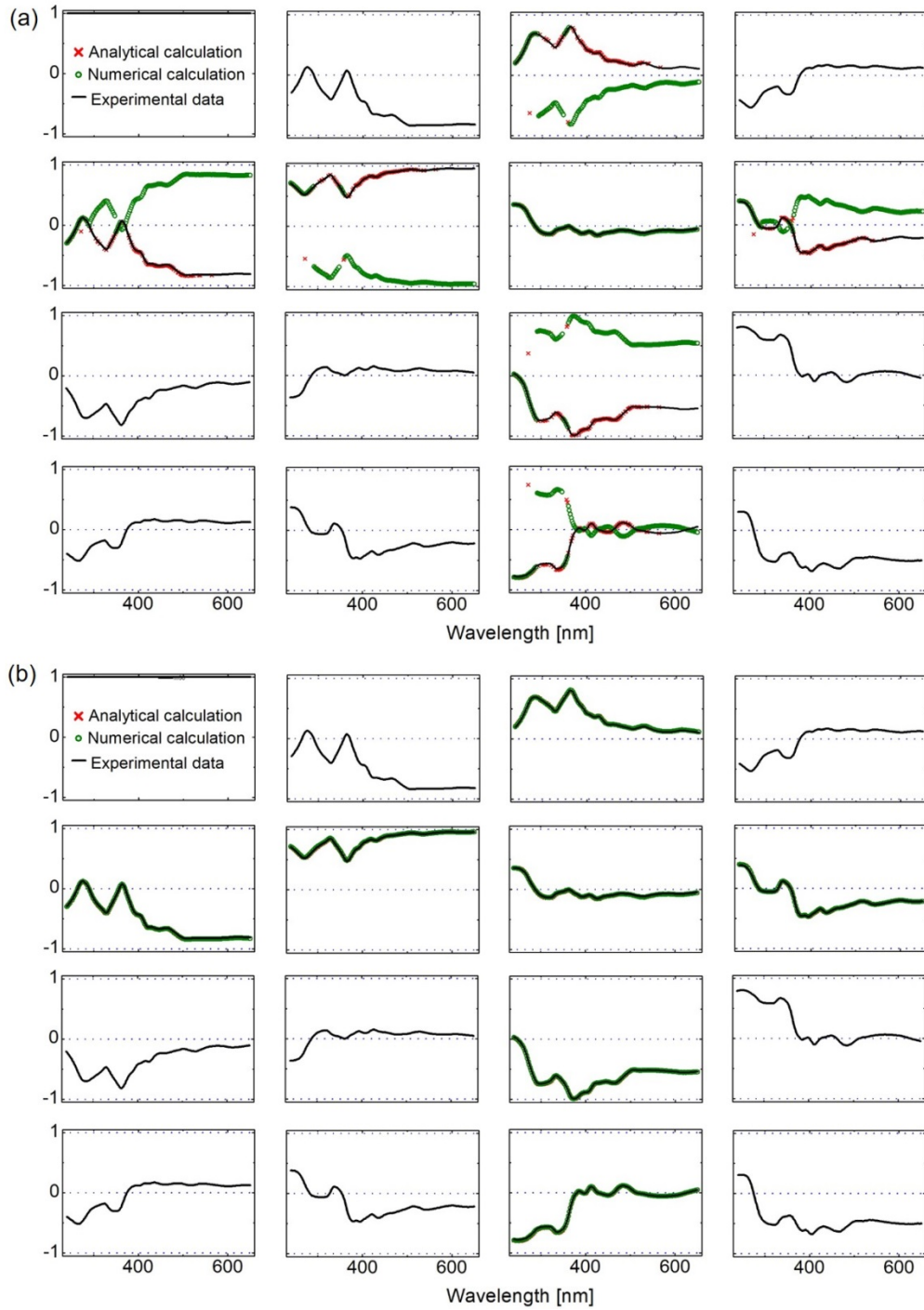


Fig. 1. (a) Complete Mueller matrix of the grating (solid line) and its recovery from a partial $M(2, 3)$ by using the analytical procedure from Appendix B (red crosses) and the numerical approach (green circles). (b) Complete and recovered Mueller matrices assuming weak anisotropy or exploiting symmetry.

Alternatively, the Mueller matrix symmetries can be used to discriminate between the two solutions for the complete \mathbf{M} . Indeed, being a symmetric profile one, the diffraction grating is invariant with respect to a 180° -rotation about its normal and, consequently, its Mueller matrix is given by \mathbf{M}_r from Eq. (7). This is consistent with the assumed experimental partial $\mathbf{M}(2, 3)$ from Fig. 1(a). Indeed, one clearly observes that $M_{41} = M_{14}$ in accordance with Eq. (7). One therefore should select that one of the two possible solutions for \mathbf{M} which satisfies the symmetries $M_{21} = M_{12}$, $M_{23} = -M_{32}$, $M_{24} = M_{42}$, $M_{13} = -M_{31}$ and $M_{34} = -M_{43}$ characteristic of \mathbf{M}_r from Eq. (7). As illustrated by Fig. 1(b) the recovered complete Mueller matrices (the analytical and the numerical ones) superimpose with those obtained from the weak anisotropy assumption; as with the latter, all sign changes are effectively suppressed and one obtains continuous spectra, perfectly matching the experimental ones.

4.3 Completion of partial $M(2, 4)$ from an extended PME experiment

Like in Figs. 1(a, b), Figures 2(a, b) reproduce the complete spectroscopic Mueller matrix of the diffraction grating (solid line). This time we assume we know only the partial $\mathbf{M}(2, 4)$ provided by an extended PME experiment and, like in the previous subsection, compare the recovered \mathbf{M} (more exactly, its second row and fourth column) with the experimentally determined complete one. The procedure from Appendix C produces a recovered \mathbf{M} shown in red crosses in Fig. 2(a). The elements of the second row and the fourth column exhibit multiple sign changes with the wavelength, toggling between the two potential solutions (the correct one and its mirror image). The numerical approach, analogous to that from the previous subsection, performs in a similar fashion (recovery shown in green circles). Except for the sign changes, both recovered \mathbf{M} match equally well the experiment (i.e. virtually coincide either with it or with its mirror image).

To select the correct solution, one can resort either to the weak anisotropy assumption, Eq. (4), or to the Mueller matrix symmetries, Eq. (7). The results are reported in Fig. 2(b). The sign “jumps” are effectively suppressed resulting in continuous curves for all second row and fourth column elements, for both the analytical and the numerical solutions. Eventually, the comparison of Figs. 1(b) and 2(b) shows that both the analytical and the numerical approaches perform equally well whatever the partial \mathbf{M} to be recovered, $\mathbf{M}(2, 3)$ or $\mathbf{M}(2, 4)$.

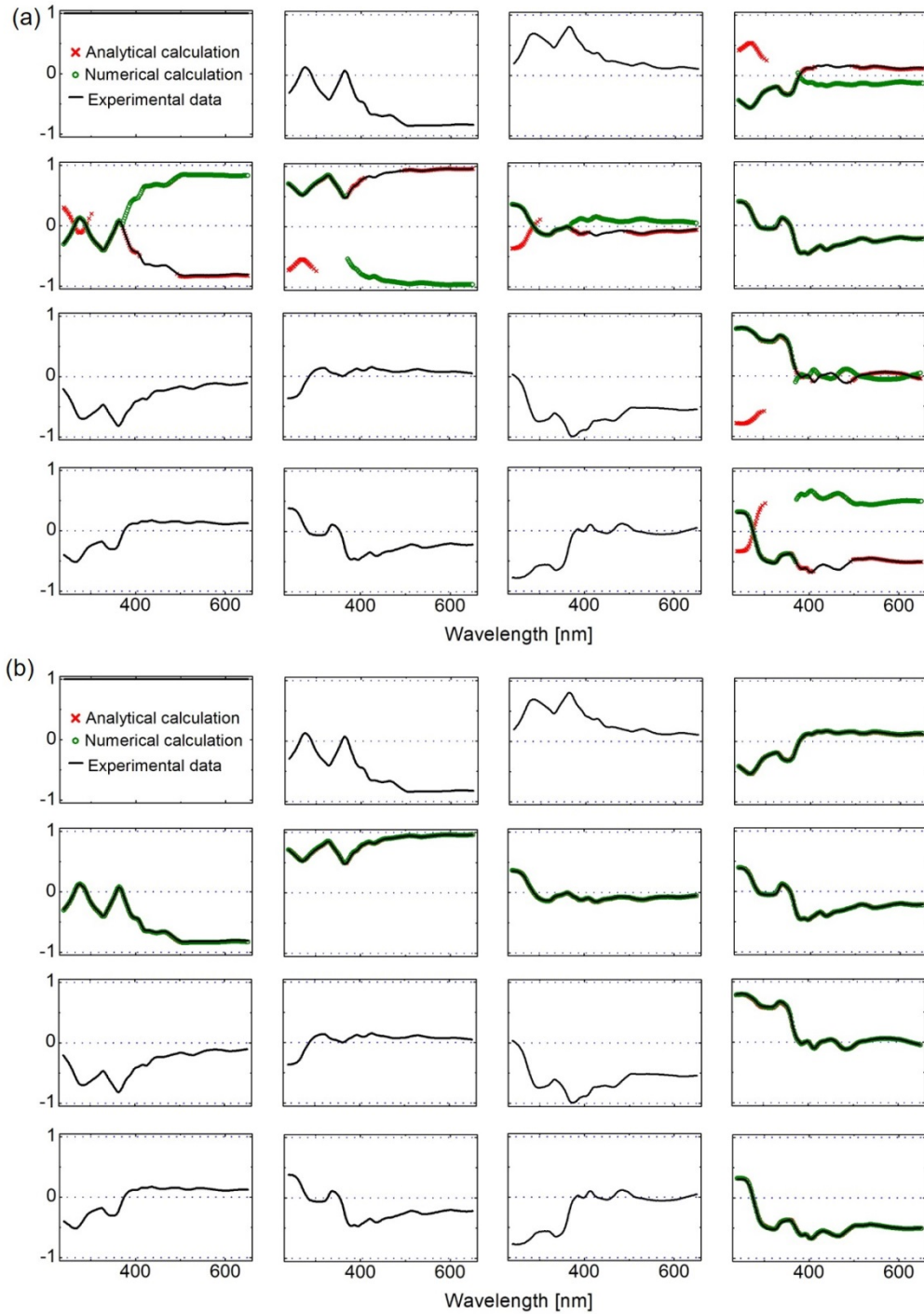


Fig. 2. (a) Complete Mueller matrix of the grating (solid line) and its recovery from a partial $M(2, 4)$ by using the analytical procedure from Appendix C (red crosses) and the numerical approach (green circles). (b) Complete and recovered Mueller matrices assuming weak anisotropy or exploiting symmetry.

4.4 Quantitative performance of the analytical and numerical methods and limit of their practical validity

Although the qualitative agreement between measured and calculated – by both the analytical and numerical approaches – Mueller matrix elements was found to be very good, as seen from Figures 1(b) and 2(b) and discussed in the previous two subsections, there is clearly a need for quantitative measure. Figure 3 shows the rms (root mean square) deviations between the seven experimental Mueller matrix elements from Fig. 2(b), assumed unknown, and their analytically and numerically recovered values. In the visible range (400 nm – 650 nm) the rms deviations for both methods are well below one percent, to be compared to the experimental error of the 4-PEM polarimeter, evaluated at about 0.5 percent [26]. The rms deviations are slightly higher in the UV range (240 nm – 400 nm) but remain still of the order of the experimental error. Clearly, either method (analytical or numerical) can be used for the recovery of the complete Mueller matrix with practically acceptable accuracy.

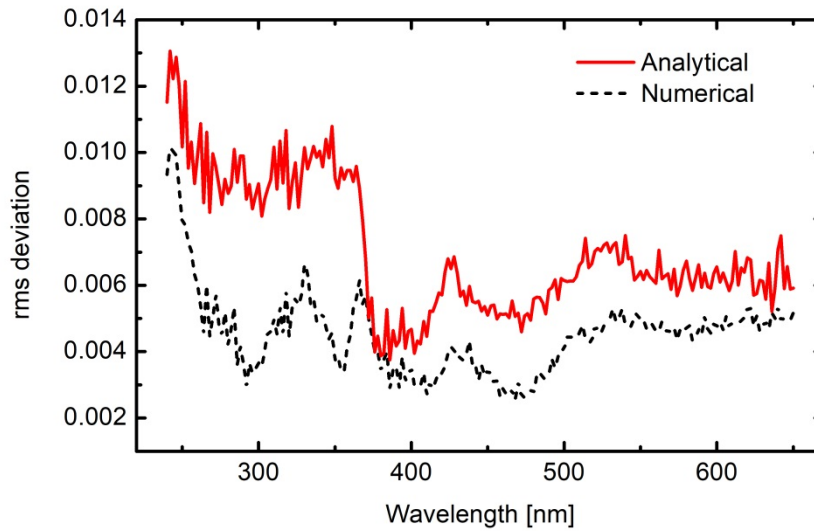


Fig. 3. Root mean square (rms) deviations between the seven unknown Mueller matrix elements from Fig. 2(b) and their analytical (dashed line) and numerical (solid line) estimations.

Figure 3 likewise shows that the numerical method slightly but systematically outperforms the analytical one in terms of accuracy. This is due to the fact that the numerical method is based on the simultaneous minimization of all eleven principal minors of order higher than one of the covariance matrix whereas the analytical method makes use only of the smallest possible number of (principal and not principal) minors; see Appendices B and C. This makes the analytical approach generally both more accurate and noise resilient, albeit at the expense of significantly increased computational time.

To explore the limit of applicability of the recovery methods, Table 2 reports the Gil-Bernabeu depolarization index (DI) values [2,12,14] of both experimental and recovered Mueller matrices, together with the second largest eigenvalue λ_2 of the recovered covariance matrices of the diffraction grating sample measured at different wavelengths. The last column of the Table contains the rms deviations of the seven recovered matrix elements, assumed to belong to the second row and the fourth column of the Mueller matrix, like in subsection 4.3.

For simplicity, only the numerical recovery method was applied. The sample was measured so as the probing light impinged not only on the grating pattern but also partially on the substrate surrounding it, thus introducing weak but spectrally dependent depolarization through the incoherent addition of two different polarimetric responses [14]. The amount of depolarization present (depending on the measured wavelength) increases from the top to the bottom of the Table: the DI values decrease whereas those of λ_2 increase. (Recall that $1 \geq \text{DI} \geq 0$ and $\lambda_2 \geq 0$, and that, for a nondepolarizing Mueller matrix, $\text{DI} = 1$ and $\lambda_2 = 0$.)

Table 2. Depolarization indices (DI) of both experimental and recovered Mueller matrices, second largest eigenvalue λ_2 of the recovered coherency matrices and rms deviation values of the seven recovered Mueller matrix elements of the diffraction grating sample measured at different wavelengths

DI experimental	DI recovered	λ_2 recovered	rms deviation
0.9955	0.9997	0.0004	0.0052
0.9936	0.9981	0.0017	0.0083
0.9866	0.9970	0.0027	0.0142
0.9826	0.9960	0.0039	0.0179
0.9775	0.9950	0.0051	0.0225
0.9712	0.9938	0.0065	0.0282
0.9605	0.9921	0.0081	0.0379

Close inspection of Table 2 reveals a clear correlation between the depolarization index of the experimental Mueller matrix (first column) and the rms deviation of its seven recovered elements (last column): the larger the amount of depolarization, the lower the accuracy of the recovery approach. This behavior is the expected one since the recovery methods are designed for nondepolarizing Mueller matrices only. Further, the depolarization index of the recovered Mueller matrix (second column), while remaining very close to the unit as expected, decreases slightly with the increase in depolarization. Similarly, the second largest eigenvalue of the recovered covariance matrix (third column) monotonously increases with the depolarization. These observations allow for the establishment of a criterion on the practical applicability of the recovery methods as a function of the largest tolerable rms deviation of the recovered matrix elements. For instance, if the rms deviation of the recovered matrix elements should not exceed 1.8 percent, then the DI value of the recovered Mueller matrix or the λ_2 eigenvalue of the recovered covariance matrix should be respectively higher than 0.996 or lower than 0.0039 (fourth numerical row of Table 2). Note that, instead of DI and λ_2 one can alternatively use the Lorentz depolarization indices [28], the Noble's depolarization parameters [29,30] or the elements of the depolarizing Mueller matrix residual [31].

5. Summary

We have shown that a Mueller matrix with a row and a column missing, originating from a partial polarimetry experiment, can be completed to a full Mueller matrix if the sample is nondepolarizing. The problem has exactly two solutions, the only difference between them consisting in the opposite signs of the missing row and column elements (except for their

common matrix element). The two potential solutions can be obtained either analytically or numerically, as demonstrated on experimental data. The correct solution can be identified if the sample is weakly anisotropic or exhibits symmetries, and if a row and a column having different indices are missing in the partial Mueller matrix. Unique recovery is generally impossible if a row and a column of the same index are missing. Authors intend the above results to experimentalists using RPAE, extended PME or 2-MGE instruments and willing to recover the complete Mueller matrix of the sample from the experimentally determined partial, nine-element one.

Appendix A: Analytical procedure for completing a partial $\mathbf{M}(4, 4)$

Assume we are dealing with a partially known \mathbf{M} that has its fourth row and column missing, i.e. we want to recover the complete \mathbf{M} knowing the partial $\mathbf{M}(4, 4)$. (This is experimentally the case with the RPAE design.) Close inspection of Eq. (2a) for \mathbf{M} then shows that the problem is equivalent to not knowing the imaginary parts G_{ij} of the off-diagonal elements of the covariance matrix \mathbf{H} given by Eq. (3), as well as the two real parts F_{12} and F_{34} subject to the constraint $F_{12} + F_{34} = M_{33}$. The covariance matrix \mathbf{H} can be decomposed into its symmetric, \mathbf{S} , and antisymmetric, \mathbf{A} , parts,

$$\mathbf{H} = \mathbf{S} + i\mathbf{A} = \frac{1}{2} \begin{bmatrix} E_1 & F_{13} & F_{14} & F_{12} \\ F_{13} & E_3 & F_{34} & F_{32} \\ F_{14} & F_{34} & E_4 & F_{42} \\ F_{12} & F_{32} & F_{42} & E_2 \end{bmatrix} + i \frac{1}{2} \begin{bmatrix} 0 & G_{13} & G_{14} & G_{12} \\ -G_{13} & 0 & G_{34} & G_{32} \\ -G_{14} & -G_{34} & 0 & G_{42} \\ -G_{12} & -G_{32} & -G_{42} & 0 \end{bmatrix} \quad (\text{A1})$$

and, being of unit rank by assumption (\mathbf{M} is nondepolarizing), \mathbf{H} can be likewise expressed as the projector,

$$\mathbf{H} = (\mathbf{a} + i\mathbf{b})(\mathbf{a} + i\mathbf{b})^+ = (\mathbf{a} + i\mathbf{b})(\mathbf{a}^T - i\mathbf{b}^T) = \mathbf{a}\mathbf{a}^T + \mathbf{b}\mathbf{b}^T + i(\mathbf{a}\mathbf{b}^T - \mathbf{b}\mathbf{a}^T) = \mathbf{S} + i\mathbf{A} \quad (\text{A2})$$

in which \mathbf{a} and \mathbf{b} are two real four-component vectors (the superscripts “ T ” and “ $+$ ” stand for real and complex conjugate transpose respectively). Since the two vectors \mathbf{a} and \mathbf{b} are generally both different from zero, the symmetric part \mathbf{S} , being a sum of two projectors according to Eq. (A2), is of rank two, i.e. it has $4 - 2 = 2$ vanishing eigenvalues. The matrix \mathbf{S} is, furthermore, positive semidefinite since $\mathbf{x}^T \mathbf{S} \mathbf{x} = \mathbf{x}^T \mathbf{a} \mathbf{a}^T \mathbf{x} + \mathbf{x}^T \mathbf{b} \mathbf{b}^T \mathbf{x} = (\mathbf{a}^T \mathbf{x})^2 + (\mathbf{b}^T \mathbf{x})^2 \geq 0$ where \mathbf{x} is an arbitrary vector. Consequently, all four 3×3 principal minors $S^{(m)}$, obtained by striking out the m th row and column of \mathbf{S} , must vanish. In particular,

$$S^{(1)} = \frac{1}{2} \begin{vmatrix} E_3 & F_{34} & F_{32} \\ F_{34} & E_4 & F_{42} \\ F_{32} & F_{42} & E_2 \end{vmatrix} = \frac{1}{2} (E_2 E_3 E_4 + 2F_{32} F_{42} F_{34} - E_4 F_{32}^2 - E_3 F_{42}^2 - E_2 F_{34}^2) = 0 \quad (\text{A3})$$

and

$$S^{(4)} = \frac{1}{2} \begin{vmatrix} E_1 & F_{13} & F_{14} \\ F_{13} & E_3 & F_{34} \\ F_{14} & F_{34} & E_4 \end{vmatrix} = \frac{1}{2} (E_1 E_3 E_4 + 2F_{13} F_{14} F_{34} - E_3 F_{14}^2 - E_4 F_{13}^2 - E_1 F_{34}^2) = 0 \quad (\text{A4})$$

so that

$$\begin{aligned} & 2(E_1 S^{(1)} - E_2 S^{(4)}) \\ & = 2(E_1 F_{34} F_{42} - E_2 F_{13} F_{14}) F_{34} - E_1 E_4 F_{32}^2 - E_1 E_3 F_{42}^2 + E_2 E_3 F_{14}^2 + E_2 E_4 F_{13}^2 = 0 \end{aligned} \quad (\text{A5a})$$

Eq. (A5a) is a linear equation, with a unique solution, for the only unknown F_{34} . The second unknown real part F_{12} is found from the constraint $F_{12} + F_{34} = M_{33}$. Alternatively, instead of F_{34} , one can first determine F_{12} from

$$\begin{aligned} & 2(E_3 S^{(2)} - E_4 S^{(3)}) \\ & = 2(E_3 F_{14} F_{42} - E_4 F_{13} F_{32}) F_{12} + E_1 E_4 F_{32}^2 - E_1 E_3 F_{42}^2 - E_2 E_3 F_{14}^2 + E_2 E_4 F_{13}^2 = 0 \end{aligned} \quad (\text{A5b})$$

and then obtain F_{34} from the constraint. Yet another equivalent, however generally more noise-resilient, way of determining F_{34} and F_{34} consists in solving each of the two quadratic equations involving them,

$$\begin{aligned} 2(S^{(1)} + S^{(4)}) & = E_3 E_4 (E_1 + E_2) - E_3 (F_{42}^2 + F_{14}^2) - E_4 (F_{32}^2 + F_{13}^2) \\ & \quad + 2(F_{32} F_{42} + F_{13} F_{14}) F_{34} - (E_1 + E_2) F_{34}^2 = 0 \end{aligned} \quad (\text{A6a})$$

$$\begin{aligned} 2(S^{(2)} + S^{(3)}) & = E_1 E_2 (E_3 + E_4) - E_1 (F_{42}^2 + F_{32}^2) - E_2 (F_{14}^2 + F_{13}^2) \\ & \quad + 2(F_{14} F_{42} + F_{13} F_{32}) F_{12} - (E_3 + E_4) F_{12}^2 = 0 \end{aligned} \quad (\text{A6b})$$

and selecting the pair of solutions that satisfies the constraint $F_{12} + F_{34} = M_{33}$. Whatever the way, the symmetric part \mathbf{S} of \mathbf{H} is uniquely determined. To get the antisymmetric part \mathbf{A} , explicitly diagonalize \mathbf{S} to obtain

$$\begin{aligned} \mathbf{S} = \mathbf{U} \mathbf{D} \mathbf{U}^T & = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3 \quad \mathbf{u}_4] \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \mathbf{u}_3^T \\ \mathbf{u}_4^T \end{bmatrix} = d_1 \mathbf{u}_1 \mathbf{u}_1^T + d_2 \mathbf{u}_2 \mathbf{u}_2^T \\ & = (\sqrt{d_1} \mathbf{u}_1) (\sqrt{d_1} \mathbf{u}_1)^T + (\sqrt{d_2} \mathbf{u}_2) (\sqrt{d_2} \mathbf{u}_2)^T = \mathbf{a} \mathbf{a}^T + \mathbf{b} \mathbf{b}^T \end{aligned} \quad (\text{A7})$$

where \mathbf{U} is an orthogonal matrix whose column vectors are \mathbf{u}_i , whereas d_1 and d_2 are the two non-negative eigenvalues of the rank-two, positive semidefinite \mathbf{S} . The decomposition (A7) allows one to determine the two vectors \mathbf{a} and \mathbf{b} , $\mathbf{a} = \pm \sqrt{d_1} \mathbf{u}_1$ and $\mathbf{b} = \pm \sqrt{d_2} \mathbf{u}_2$, constituting

the covariance matrix \mathbf{H} in accordance with Eq. (A2). One finally gets for the antisymmetric part \mathbf{A} of \mathbf{H} ,

$$\mathbf{A} = \mathbf{a}\mathbf{b}^T - \mathbf{b}\mathbf{a}^T = \pm\sqrt{d_1d_2}(\mathbf{u}_1\mathbf{u}_2^T - \mathbf{u}_2\mathbf{u}_1^T) \quad (\text{A8})$$

Unlike the symmetric part \mathbf{S} of \mathbf{H} , the antisymmetric one, \mathbf{A} , is clearly not unique, but rather admits two opposite solutions, \mathbf{A} and $-\mathbf{A}$. Likewise, there exist two solutions, \mathbf{H} and $\mathbf{H}^T = \mathbf{H}^*$ (the asterisk denotes the complex conjugate) for the covariance matrix \mathbf{H} , as follows from Eq. (3). In accordance with Eq. (2a), the corresponding solutions for the Mueller matrix are \mathbf{M} and $\mathbf{S}_4\mathbf{M}\mathbf{S}_4$ where $\mathbf{S}_4 = \text{diag}(1 \ 1 \ 1 \ -1)$, i.e. \mathbf{M} with the signs of its fourth row and column elements inverted (except for the M_{44} element). Eventually, Eqs. (1) and (2) show that the Jones matrices corresponding to \mathbf{M} and $\mathbf{S}_4\mathbf{M}\mathbf{S}_4$ are \mathbf{J} and \mathbf{J}^* respectively. Note that the solution for \mathbf{M} will be unique in the trivial case where $\mathbf{b} = \mathbf{0}$ and therefore, $\mathbf{A} = \mathbf{0}$ from Eq. (A2), so that $\mathbf{H} = \mathbf{S}$ is real symmetric; $\mathbf{M} = \mathbf{S}_4\mathbf{M}\mathbf{S}_4$, so that \mathbf{M} has its fourth row and column are zero (except for its M_{44} element, possibly), and finally $\mathbf{J} = \mathbf{J}^*$ from Eqs. (2), so that \mathbf{J} is real.

If, more generally, the experimentally determined partial Mueller matrix is $\mathbf{M}(r, c)$, $r, c = 2, 3, 4$, rather than $\mathbf{M}(4, 4)$, then it can be easily shown than the first one can be reduced to the second one by a Mueller matrix transformation. Consider the two permutation matrices

$$\mathbf{P}_{24} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (\text{A9a})$$

and

$$\mathbf{P}_{34} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (\text{A9b})$$

which respectively permute the 2nd and the 4th, and the 3rd and the 4th row (column) of \mathbf{M} when left- (right-) multiplying it. One can thus write the series of identities

$$\mathbf{M}(4, 4) = \mathbf{P}_{24}\mathbf{M}(2, 4) = \mathbf{P}_{34}\mathbf{M}(3, 4) = \mathbf{P}_{24}\mathbf{M}(2, 3)\mathbf{P}_{34} \quad (\text{A10})$$

showing that $\mathbf{M}(2, 4)$, $\mathbf{M}(3, 4)$ and $\mathbf{M}(2, 3)$ can be transformed through suitable permutations to $\mathbf{M}(4, 4)$. Transposition of Eq. (A9) yields a similar series for the duals $\mathbf{M}(4, 2)$, $\mathbf{M}(4, 3)$ and $\mathbf{M}(3, 2)$. Since there are exactly two solutions for the complete \mathbf{M} knowing $\mathbf{M}(4, 4)$, so will be the case with $\mathbf{M}(2, 4)$, $\mathbf{M}(3, 4)$ and $\mathbf{M}(2, 3)$ or, most generally, with any other partial Mueller matrix $\mathbf{M}(r, c)$. If one of the two solutions is \mathbf{M} , the other one is then $\mathbf{S}_r\mathbf{M}\mathbf{S}_c$ where \mathbf{S}_r (\mathbf{S}_c) is a diagonal matrix with ones on its diagonal and zeroes elsewhere, except for its S_{rr} (S_{cc}) element which equals -1 . In different words, the second solution for a complete \mathbf{M}

knowing $\mathbf{M}(r, c)$ is obtained from the first one by inverting the signs of the r th row and the c th column elements (except for the M_{rc} element) and vice versa.

In principle, the transformation (A10) can be used to recover the complete \mathbf{M} from any partial $\mathbf{M}(r, c)$ by first reducing it to $\mathbf{M}(4, 4)$, performing the recovery procedure on the latter and finally, transforming back the result. However, although being formally equivalent, the specific procedures for partial $\mathbf{M}(2, 3)$ and $\mathbf{M}(2, 4)$ presented in Appendices B and C respectively generally feature more robust performance on experimental data.

Appendix B: Analytical procedure for completing a partial $\mathbf{M}(2, 3)$

The recovery of the complete \mathbf{M} knowing $\mathbf{M}(2, 3)$ is based on the recovery of its complete covariance matrix \mathbf{H} . Two of the (ten independent) elements of \mathbf{H} are obtained immediately with reference to Eqs. (2a) and (3),

$$H_{13} = \frac{1}{2}(F_{14} - iG_{14}) = \frac{1}{4}[(M_{31} + M_{32}) - i(M_{41} + M_{42})] \quad (\text{B1a})$$

$$H_{24} = \frac{1}{2}(F_{32} - iG_{32}) = \frac{1}{4}[(M_{31} - M_{32}) - i(M_{41} - M_{42})] \quad (\text{B1b})$$

Next, one has for the difference $H_{14} - H_{23}$ of the two unknown elements H_{14} and H_{23} ,

$$\begin{aligned} H_{14} - H_{23} &= \frac{1}{2}(F_{12} - iG_{12}) - \frac{1}{2}(F_{34} - iG_{34}) \\ &= \frac{1}{2}[(F_{12} - F_{34}) + i(G_{34} - G_{12})] = \frac{1}{2}(M_{44} + iM_{34}) \end{aligned} \quad (\text{B2})$$

On the other hand, from the 2×2 minor $H_{(34)}^{(12)}$ of \mathbf{H} obtained by striking out the 3rd and 4th row, together with the 1st and 2nd column, one gets a second equation for H_{14} and H_{23} ,

$$H_{(34)}^{(12)} = \begin{vmatrix} H_{13} & H_{14} \\ H_{23} & H_{24} \end{vmatrix} = H_{13}H_{24} - H_{14}H_{23} = 0 \quad (\text{B3})$$

since $\text{rank}(\mathbf{H}) = 1$ by assumption and, consequently, all 2×2 minors of \mathbf{H} must vanish. By combining Eqs. (B2) and (B3) one gets the following quadratic equation for H_{14} ,

$$H_{14}^2 - \frac{1}{2}(M_{44} + iM_{34})H_{14} - H_{13}H_{24} = 0 \quad (\text{B4a})$$

or, alternatively, for H_{23} ,

$$H_{23}^2 + \frac{1}{2}(M_{44} + iM_{34})H_{23} - H_{13}H_{24} = 0 \quad (\text{B4b})$$

The solution of the two Eqs. (B4), after the substitution of Eqs. (B1), produces two potential solutions for the two unknowns H_{14} and H_{23} . (Alternatively, one may solve just one of Eqs. (B4) for one of the unknowns and use Eq. (B2) to get the other unknown).

A similar approach is used to determine the diagonal elements of \mathbf{H} . From Eqs. (2a) and (3) it follows immediately that

$$\begin{aligned}
H_{11} + H_{33} &= \frac{1}{2}E_1 + \frac{1}{2}E_4 \\
&= \frac{1}{2} \left[\frac{1}{2}(E_1 + E_2 + E_3 + E_4) + \frac{1}{2}(E_1 - E_2 - E_3 + E_4) \right] = \frac{1}{2}(M_{11} + M_{12})
\end{aligned} \tag{B5a}$$

$$\begin{aligned}
H_{22} + H_{44} &= \frac{1}{2}E_3 + \frac{1}{2}E_2 \\
&= \frac{1}{2} \left[\frac{1}{2}(E_1 + E_2 + E_3 + E_4) - \frac{1}{2}(E_1 - E_2 - E_3 + E_4) \right] = \frac{1}{2}(M_{11} - M_{12})
\end{aligned} \tag{B5b}$$

while the two vanishing principal minors $H^{(24)}$ and $H^{(13)}$ yield

$$H^{(24)} = \begin{vmatrix} H_{11} & H_{13} \\ H_{31} & H_{33} \end{vmatrix} = H_{11}H_{33} - H_{13}H_{31} = H_{11}H_{33} - |H_{13}|^2 = 0 \tag{B6a}$$

$$H^{(13)} = \begin{vmatrix} H_{22} & H_{24} \\ H_{42} & H_{44} \end{vmatrix} = H_{22}H_{44} - H_{24}H_{42} = H_{22}H_{44} - |H_{24}|^2 = 0 \tag{B6b}$$

The coupling of Eqs. (B5) and (B6) produces two quadratic equations: one for H_{11} and H_{33} ,

$$H_{11,33}^2 - \frac{1}{2}(M_{11} + M_{22})H_{11,33} + |H_{13}|^2 = 0 \tag{B7a}$$

and another one for H_{22} and H_{44} ,

$$H_{22,44}^2 - \frac{1}{2}(M_{11} - M_{22})H_{22,44} + |H_{24}|^2 = 0 \tag{B7b}$$

By solving each one of Eqs. (B7), after having substituted Eqs. (B1), one gets two potential solutions for H_{11} , H_{33} and H_{22} , H_{44} .

To find the actual solutions for the elements of \mathbf{H} , use should be made of the constraint $G_{13} + G_{42} = -M_{14}$ following from the inspection of Eq. (2a). From the two vanishing 2×2 minors $H_{(23)}^{(13)} = H_{12}H_{44} - H_{14}H_{24}^* = 0$ and $H_{(24)}^{(23)} = H_{11}H_{34} - H_{14}H_{13}^* = 0$ one gets the following explicit expressions for the last pair of unknown elements H_{12} and H_{34} of \mathbf{H} ,

$$H_{12} = \frac{H_{14}H_{24}^*}{H_{44}} \tag{B8a}$$

and

$$H_{34} = \frac{H_{14}H_{13}^*}{H_{11}} \tag{B8b}$$

Since $G_{13} = -2\text{Im}(H_{12})$ and $G_{42} = -2\text{Im}(H_{34})$, see Eq. (3), the constraint $G_{13} + G_{42} = -M_{14}$ becomes, after the substitution of Eqs. (B8),

$$\text{Im}(H_{12}) + \text{Im}(H_{34}) = \text{Im}\left(\frac{H_{14}H_{24}^*}{H_{44}}\right) + \text{Im}\left(\frac{H_{14}H_{13}^*}{H_{11}}\right) = \frac{1}{2}M_{14} \quad (\text{B9})$$

Eq. (B9) is used as a condition to be satisfied in order to discriminate the actual solutions from the previously determined set of potential solutions of the elements H_{14} , H_{44} and H_{11} . Since each one of the elements H_{14} , H_{44} and H_{11} takes two values, eight combinations have to be checked. Once H_{14} , H_{44} and H_{11} are determined, the actual solutions for the elements H_{23} , H_{22} and H_{33} are identified automatically. This completes the determination of the elements of covariance matrix \mathbf{H} . Knowing \mathbf{H} , the complete Mueller matrix \mathbf{M} is obtained from Eq. (2a) with the help of Eq. (3). As shown in Appendix A, there are two possible solutions for the Mueller matrix: either \mathbf{M} as obtained or \mathbf{M} with the signs of its 2nd row and 3rd column inverted (except for the M_{23} element). Eventually, if it is not the partial $\mathbf{M}(2, 3)$, but rather its dual, $\mathbf{M}(3, 2)$, that is effectively measured, one should then apply the above procedure on $\mathbf{M}(3, 2)^T$ and re-transpose the resulting complete \mathbf{M} .

Appendix C: Analytical procedure for completing a partial $\mathbf{M}(2, 4)$

To recover the complete \mathbf{M} knowing the partial $\mathbf{M}(2, 4)$ one follows a procedure analogous in many points to that from Appendix B, based on the recovery of the covariance matrix \mathbf{H} . Thus, two of the elements of \mathbf{H} , H_{13} and H_{24} , are evaluated directly from Eqs. (B1).

The sum $H_{14} + H_{23}$ of the two unknown elements H_{14} and H_{23} is readily identified from Eqs. (2a) and (3) to be

$$\begin{aligned} H_{14} + H_{23} &= \frac{1}{2}(F_{12} - iG_{12}) + \frac{1}{2}(F_{34} - iG_{34}) \\ &= \frac{1}{2}[(F_{12} + F_{34}) - i(G_{34} + G_{12})] = \frac{1}{2}(M_{33} - iM_{43}) \end{aligned} \quad (\text{C1})$$

Coupling Eq. (C1) with Eq. (B3) for the product $H_{14}H_{23}$ results in the following quadratic equation for H_{14} and H_{23} ,

$$H_{14,23}^2 - \frac{1}{2}(M_{33} - iM_{43})H_{14,23} + H_{13}H_{24} = 0 \quad (\text{C2})$$

The solution of Eqs. (C2), after the substitution of Eqs. (B1), yields two potential solutions for the two unknowns H_{14} and H_{23} .

The determination of the diagonal elements of \mathbf{H} is identical to that from Appendix B. Elements H_{11} and H_{33} are determined from Eq. (B7a) whereas H_{22} and H_{44} are obtained from Eq. (B7b). There are two potential solutions for H_{11} , H_{33} and H_{22} , H_{44} .

To find the actual solutions for the elements of \mathbf{H} , the constraint $F_{13} + F_{42} = M_{13}$ following from the inspection of Eq. (2a), combined with the explicit expressions (B8) for the last pair of unknown elements of \mathbf{H} , H_{12} and H_{34} , is used. Since $F_{13} = 2\text{Re}(H_{12})$ and $F_{42} = 2\text{Re}(H_{34})$, see Eq. (3), the constraint $F_{13} + F_{42} = M_{13}$ takes the following form after the substitution of Eqs. (B8),

$$\text{Re}(H_{12}) + \text{Re}(H_{34}) = \text{Re}\left(\frac{H_{14}H_{24}^*}{H_{44}}\right) + \text{Re}\left(\frac{H_{14}H_{13}^*}{H_{11}}\right) = \frac{1}{2}M_{13} \quad (\text{C3})$$

Like Eq. (B9), Eq. (C3) is used to identify the actual solutions from the previously determined set of potential solutions for the elements H_{14} , H_{44} and H_{11} . Eight options have to be checked. Once H_{14} , H_{44} and H_{11} are determined, the actual solutions for the elements H_{23} , H_{22} and H_{33} are identified automatically. This completes the determination of the elements of covariance matrix \mathbf{H} . Knowing \mathbf{H} , the complete Mueller matrix \mathbf{M} is obtained from Eq. (2a) with the help of Eq. (3). As shown in Appendix A, there are two possible solutions for the Mueller matrix: either \mathbf{M} as obtained or \mathbf{M} with the signs of its 2nd row and 4th column inverted (except for the M_{24} element). Eventually, if it is not the partial $\mathbf{M}(2, 4)$, but rather its dual, $\mathbf{M}(4, 2)$, that is effectively measured, one should then apply the above procedure on $\mathbf{M}(4, 2)^T$ and re-transpose the resulting complete \mathbf{M} .

Although not discussed here, the algebraic problem of the recovery of the complete \mathbf{M} from the partial $\mathbf{M}(3, 4)$ can be solved in a manner similar to the previous two, involving $\mathbf{M}(2, 3)$ and $\mathbf{M}(2, 4)$. By following the same logic as in the procedures from Appendix B and the present one, the reader should be able to obtain its solution.

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Disclosures

The authors declare that there are no conflicts of interest related to this article.

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