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► **To cite this version:**

Eric Goubault, Sylvie Putot. Guaranteed approximations of arbitrarily quantified reachability problems. 2023. hal-04336386

HAL Id: hal-04336386

<https://polytechnique.hal.science/hal-04336386>

Preprint submitted on 11 Dec 2023

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Guaranteed approximations of arbitrarily quantified reachability problems

Eric Goubault and Sylvie Putot

LIX, Ecole Polytechnique, CNRS and IP-Paris, 91128 Palaiseau Cedex, France

Abstract. We propose an approach to compute inner and outer-approximations of the sets of values satisfying constraints expressed as arbitrarily quantified formulas. Such formulas arise for instance when specifying important problems in control such as robustness, motion planning or controllers comparison. We propose an interval-based method which allows for tractable but tight approximations. We demonstrate its applicability through a series of examples and benchmarks using a prototype implementation.

Keywords: Reachability, Quantified problems, Inner-approximations

1 Introduction

We consider the problem of computing inner and outer approximations of sets of reachable states constrained by arbitrarily quantified formulas. Although this can be applied to a number of computer science and verification problems, we focus here on quantified formulas that arise in control and validation. Controlled systems are usually subject to disturbances, and are defined by the flow $\varphi(t; x_0, u, w)$ at time t , for any initial state x_0 , control u and disturbance w . Robust reachability in the sense of [20] is defined as computing, for time $t \in [0, T]$, a set such as $R_{\forall\exists}(\varphi)(t)$:

$$R_{\forall\exists}(\varphi)(t) = \{z \mid \forall w \in \mathbb{W}, \exists x_0 \in \mathbb{X}_0, \exists u \in \mathbb{U}, z = \varphi(t; x_0, u, w)\}$$

and solves the problem of knowing whether a controller can compensate disturbances or change of values of parameters that are known to the controller. This is an example of the quantified reachability problems targeted in this work.

In classical robust control, the problem can be different and consider the existence of a controller leading to a target robustly whatever the disturbances in a given set. In this case, we may need to relax the problem to find a non-empty solution, for instance by some tolerance in time or space on reaching the target. This leads to more complex quantified problems of the form, for example here with a relaxation in time:

$$R_{\exists\forall\exists}(\varphi) = \{z \in \mathbb{R}^m \mid \exists u \in \mathbb{U}, \exists x_0 \in \mathbb{X}_0, \forall w \in \mathbb{W}, \exists s \in [0, T] \\ z = \varphi(s; x_0, u, w)\} \quad (1)$$

This generalization is one of the motivations of the work described hereafter, that considers arbitrary alternations of quantifiers. We discuss in Section 1.2 other problems in control requiring such alternations, among which motion planning problems and problems specified by hyperproperties such as robustness or comparisons of controllers.

1.1 Problem statement

Let f be a function from \mathbb{R}^p to \mathbb{R}^m , which can be a flow function φ as above, a discrete-time dynamical system etc. We suppose the p arguments of f are partitioned into consecutive j_i arguments $i = 1, \dots, 2n$ corresponding to the alternations of quantifiers, with $p = \sum_{i=1}^{2n} j_i$. This partition, identified with the sequence (j_1, \dots, j_{2n}) is denoted by \mathbf{p} . For simplicity's sake, we will note $\mathbf{x}_i = (x_{k_i+1}, \dots, x_{k_{i+1}})$ where k_i stands for $\sum_{l=1}^{i-1} j_l$, $i = 1, \dots, 2n+1$, and $f(x_1, x_2, \dots, x_{k_{2n}}) = f(\mathbf{x}_1, \dots, \mathbf{x}_{2n})$.

We consider the general quantified problems, with n alternations of quantifiers $\forall\exists$, of finding $R_{\mathbf{p}}(f)$ defined as:

$$R_{\mathbf{p}}(f) = \left\{ z \in \mathbb{R}^m \mid \forall \mathbf{x}_1 \in [-1, 1]^{j_1}, \exists \mathbf{x}_2 \in [-1, 1]^{j_2}, \dots, \forall \mathbf{x}_{2n-1} \in [-1, 1]^{j_{2n-1}}, \exists \mathbf{x}_{2n} \in [-1, 1]^{j_{2n}}, z = f(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{2n}) \right\} \quad (2)$$

Remark 1. Note that this formulation does not prevent us from considering a formula starting with an existential quantifier (nor one finishing with a universal quantifier): formally this can be done by adding a universal quantifier at the start of the sequence of quantifiers, quantifying over a dummy variable.

When only few quantifier alternations are involved, we will use the notations $R_{\forall}(f)$, $R_{\exists}(f)$, $R_{\forall\exists}(f)$, $R_{\exists\forall}(f)$ etc. instead of $R_{\mathbf{p}}(f)$, for brevity.

Remark 2. Problem (2) naturally also includes, up to reparametrization, quantified problems with other boxes than $[-1, 1]^{j_i}$. It is also possible to consider more general sets over which to quantify variables x_i . As shown in Proposition 2, any outer-approximation (resp. inner-approximation) of the set of values for universally quantified variables x_{2i-1} and inner-approximation of the set for existentially quantified variables x_{2i} , by boxes, provides with our method an inner-approximation (resp. outer-approximation) of $R_{\mathbf{p}}(f)$.

Remark 3. In control applications, control u and disturbance w are generally functions of time. We are not quantifying over functions here, which would be a much more intricate problem to solve, but, as in e.g. [21], we are considering that control and disturbances are discretized, hence constant, over small time intervals: they thus are identified with a finite set of parameters, over a bounded time horizon.

Computing reachable sets $R_{\mathbf{p}}(f)$ being intractable in general, as it includes in particular the computation of the range of a function, we focus on computing tight inner- and outer-approximations.

Running example In the sequel, we will illustrate our approach on a simple Dubbins vehicle model described below, where function f in (4) is the flow function φ of the system. In general, the flow function does not admit closed forms, but our method will still be applicable in that context as it will only require outer-approximations of its values and of its Jacobian. Still, when comparing with quantifier elimination methods, we will need to give polynomial approximations for φ , which will be developed in Example 3.

Example 1 (Dubbins vehicle [21]). We simplify the model from [21] to consider only uncertainties on the x axis: $\dot{x} = v\cos(\theta) + b_1$, $\dot{y} = v\sin(\theta)$, $\dot{\theta} = a$. We suppose that the speed v is equal to 1 and we have a control period of $t = 0.5$. The initial conditions are uncertain given in $\mathbb{X}_0 = \{(x, y, \theta) \mid x \in [-0.1, 0.1], y \in [-0.1, 0.1], \theta \in [-0.01, 0.01]\}$, the control a can take values in $\mathbb{U} = [-0.01, 0.01]$ and disturbance b_1 can take values in $\mathbb{W} = [-0.01, 0.01]$. Both control and disturbance are supposed to be constant over the control period $[0, 0.5]$. This could naturally be extended for any number of control periods, with piecewise constant control and disturbances. We are interested in computing approximations of reachable sets of the form (4) where φ is the solution flow of the system.

1.2 Quantified reachability problems

Quantified reachability problems are central in control and hybrid systems, we detail below a few examples.

General robust reachability A classical robust reachability problem consists in computing the states reachable at some time $T \geq 0$ for some control, independently of disturbances which can even be adversarial with respect to the control and initial state:

$$R_{\exists\forall}(\varphi) = \{z \in \mathbb{R}^m \mid \exists u \in \mathbb{U}, \exists x_0 \in \mathbb{X}_0, \forall w \in \mathbb{W}, z = \varphi(T; x_0, u, w)\} \quad (3)$$

However, requiring to reach a given target point $z \in R_{\exists\forall}(\varphi)$ at time T independently of the disturbance is most often a too constrained problem. A better quantified problem is the relaxation to whether we can reach this point within time $[0, T]$ instead of at fixed time:

$$R_{\exists\forall\exists}(\varphi) = \{z \in \mathbb{R}^m \mid \exists u \in \mathbb{U}, \exists x_0 \in \mathbb{X}_0, \forall w \in \mathbb{W}, \exists s \in [0, T] \\ z = \varphi(s; x_0, u, w)\} \quad (4)$$

Example 2 (Dubbins vehicle (continued)). We want the robust reachable set within one time period, i.e. until time $t = 0.5$, with one control value a_0 applied between times 0 and 0.5. This corresponds to $R_{\exists\forall\exists}(\varphi)$ of Equation (4) with sets \mathbb{U} , \mathbb{X}_0 , \mathbb{W} defined in Example 1 and $T = 0.5$.

Another possible relaxation of Equation (3) is to consider the set of states that can be reached up to δ . This corresponds to the quantified problems:

$$R_{\exists\forall\exists}(\varphi) = \{z \in \mathbb{R}^m \mid \exists u \in \mathbb{U}, \exists x_0 \in \mathbb{X}_0, \forall w \in \mathbb{W}, \\ \exists \delta \in [-\delta, \delta]^m, z = \varphi(T; x_0, u, w) + \delta\} \quad (5)$$

Time and space tolerances can also be combined. We will for instance do so on the running example, by considering the quantified problem of Equation (17).

Finally, even more complicated quantified problems are of interest in robust control, such as:

$$R_{\forall\exists\forall\exists}(\varphi) = \{z \in \mathbb{R}^m \mid \forall x_0 \in \mathbb{X}_0, \exists u \in \mathbb{U}, \forall w \in \mathbb{W}, \exists s \in [0, T], \\ z = \varphi(s; x_0, u, w)\} \quad (6)$$

where the control u can observe and react to the initial conditions x_0 , but not to the disturbance w .

Motion planning Motion planning problems are typically described by quantified formulas, for instance when prescribing waypoints or regions along with specific time intervals through which a controller should steer a dynamical system.

As an example, suppose we want to go through regions S_j between times T_{j-1} and T_j , for $j = 1, \dots, k$, and characterize the set of final states or locations z_k , this implies finding the following set $R_{\exists\forall\dots\forall\exists}(\varphi)$:

$$\{z_k \in \mathbb{R}^m \mid \exists u_1 \in \mathbb{U}, \forall x_0 \in \mathbb{X}_0, \forall w_1 \in \mathbb{W}, \exists t_1 \in [0, T_1], \exists z_1 \in S_1, \exists u_2 \in \mathbb{U}, \\ \forall w_2 \in \mathbb{W}, \exists t_2 \in [T_1, T_2], \exists z_2 \in S_2, \dots, \exists u_k \in \mathbb{U}, \forall w_k \in \mathbb{W}, \exists t_k \in [T_{k-1}, T], \\ \begin{pmatrix} z_1 \\ \dots \\ z_k \end{pmatrix} = \begin{pmatrix} \varphi(t_1; u_1, x_0, w_1) \\ \dots \\ \varphi(t_k - t_{k-1}; u_k, z_{k-1}, w_k) \end{pmatrix}\} \quad (7)$$

Temporal logic properties Temporal logics such as Metric Interval Temporal Logic (MITL) and Signal Temporal Logic (STL) have been successful in specifying numerous properties of interest for control systems, see e.g. [11]. Such formulas naturally produce complex quantified formulas since the semantics of "always Φ between times a and b " (resp. "eventually Φ between times a and b ") in terms of ordinary first-order propositional formulas is $\forall t \in [a, b], \Phi$ (resp. $\exists t \in [a, b], \Phi$).

It is not the subject here to discuss the class of temporal logic formulas that we can interpret through Equation (2), but rather to exemplify the potential for our approach. It is important though to note that not only we can interpret the standard boolean semantics of a class of such temporal formulas, but also their robust semantics [12]. Moreover, formulas such as Equation (2) allow for quantifying over any parameters of the dynamics of a control system, hence to express quantifications over trajectories, making it possible to compare trajectories such as in hyperproperties, see e.g. [34]. For instance, if we consider the behavioural robustness of a system, which specifies that small differences in system inputs result in small differences in system outputs, this can be measured by different quantified expressions such as:

$$R_{\exists\forall\exists\forall\exists}(\varphi) = \{z \mid \exists x_0 \in \mathbb{X}_0, \exists \delta \in [-\epsilon, \epsilon]^i, \\ \forall u \in \mathbb{U}, \exists u' \in \mathbb{U}, \forall w \in \mathbb{W}, \exists t \in [T_1, T_2], z = \|\varphi(t; x_0, u, w) - \varphi(t; x_0 + \delta, u', w)\|\}$$

which measures the distance between two trajectories of the same system when starting with close enough initial conditions, under any disturbance but taken equal for the two trajectories.

1.3 Related work

Set-based methods for reachability analysis Our approach is related to outer-approximations of non-linear continuous and controlled systems: outer-approximations of the reachable set of such systems is a particular case of our approach and we rely on such outer-approximations to compute outer- and inner-approximations of quantified problems. Many methods for outer-approximating reachable sets for continuous systems have been developed. For linear systems, direct set-based methods have been designed for estimating the exponential of a matrix, or of Peano-Baker series for uncertain systems [3], using support functions, [23], zonotopes [15], ellipsoids [27], for efficient representations of sets of states. For non-linear continuous systems, similar set-based techniques have been applied using polytopes [35] or generalized polytopes such as polynomial zonotopes [2]. Authors have also been considering a variety of linearization, hybridization or polynomialization techniques such as in e.g. [2,4]. Instead of directly propagating tractable sets through the dynamics, Taylor methods [30] have been applied extensively by a number of authors, e.g. [7], for computing polynomial approximations of solutions of ODEs (flowpipes), whose image can then be approximated using any of the tractable set representation we mentioned above. Another approach for reachability is through Hamilton-Jacobi techniques, see e.g. [5], that express functions whose zero sub-level set give the reachable sets as solutions to a Hamilton-Jacobi PDE.

There are far less methods for inner-approximating images of functions or sets of reachable states. Interval-based methods, relying on space discretization, have been used for inner-approximating the image of functions [17]. They were also used to outer and inner approximate solutions of differential systems with uncertain initial conditions [31]. An interesting recent work [26] calculates the inner-approximation by scaling down an outer-approximation, until a suitable criterion (involving the boundary of the reachable set of states) is met. A similar criterion is used in [40], with polytopic approximations. An important body of the inner-approximation literature uses either Hamilton-Jacobi methods see [32] and [29] or set-based approximate backwards reachability, i.e. through the inverted dynamics see e.g. [8] and [40].

Our approach is directly linked to previous work on modal intervals and mean-value theorems [16,19] but extends it considerably as we are not bound to consider only $\forall\exists$ statements. It also includes the approximations of robust reachable sets with time-varying inputs and disturbances as defined in [20,21,22].

Quantifier elimination Many verification and synthesis problems in computer science and control theory can be represented by the first order formula

$$\Phi(p_1, \dots, p_m) \equiv Q_1 x_1, \dots Q_n x_n, P(p_1, \dots, p_m, x_1, \dots, x_n) \quad (8)$$

where $Q_i \in \{\forall, \exists\}$ are either universal or existential quantifiers, p_1, \dots, p_m are free variables and P is a quantifier-free formula constructed by conjunction, disjunction and negation of atomic formulas of the form $f \text{ op } 0$ where $\text{op} \in \{=, \neq, <, \leq\}$ is a relational operator and f is a polynomial.

The first quantifier elimination algorithm is due to Tarski [39] for the first order theory of real numbers. Because of its high computational complexity, this algorithm is not used in practice. The first practical algorithm is due to Collins [9], and is based on cylindrical algebraic decomposition. Still, applications of this algorithm are limited because its complexity is doubly exponential in $n + m$.

The applications of quantifier elimination to control design [1] are numerous: output feedback stabilization, simultaneous stabilization, robust stabilization, frequency domain multiobjective design. In [1], they are mostly exemplified on linear systems. The work of [1] has been extended to non-linear systems in e.g. [25], including also some trajectory tracking properties. Reachability is not in general solvable by algebraic methods. The reason is that the solution set of a system of differential equations is not algebraic in general. However, [25] considers a more restricted form of reachability, along prescribed types of trajectories, that can be investigated using semi-algebraic tools. Further generalizations are exemplified in [38], with controller synthesis, stability, and collision avoidance problems. Quantifier elimination techniques have also been applied to model predictive control, see e.g. [37]. Finally, application of quantifier elimination to robot motion planning, similar to the one considered in Section 1.2, has been considered in e.g. [28], for instance for the classical piano mover's problem [36].

Our quantified problem of Equation (2) is an instance of general quantifier elimination, although we do not impose that functions f we consider are polynomial. We compare our method with quantifier elimination techniques in the sequel, although our method is specifically designed to give tight inner and outer approximations in a fast manner whereas quantifier elimination aims at finding exact solution sets along with algebraic conditions under which they exist, at the expense of time complexity. As quantifier elimination needs to consider polynomials, we compare our method with quantifier elimination on approximations of the flow function φ given by e.g. Taylor expansions [33], see Example 3.

Satisfiability modulo theory (SMT) Some SMT solvers interpret quantified formulas over theories. Still, it has long been known that there is no sound and complete procedure already for first-order logic formulas of linear arithmetic with uninterpreted function symbols [14], meaning that the corresponding SMT solvers generally rely on heuristics to deal with quantifiers. The closer SMT solver to our approach is dReal/dReach [13] which has support for some quantified SMT modulo the theory of real numbers and modulo ODEs. Such SMT solvers do not synthesize the set of states that verifies some quantified formula as we do, but can be used for checking this set is correct, up to some "resolution". The time complexity of such methods is also much higher than what we are proposing, and dReal/dReach is limited to the exists-forall fragment.

Example 3 (Dubbins vehicle (continued)). We want to compute the robust reachable set of Example 2 using quantifier elimination. As we do not have the exact flow $\varphi(t; x_0, u, w)$, we use approximations by Taylor expansions, see e.g. [21] where a similar example was discussed. With the initial conditions and parameters values of Example 1, we get the following Taylor expansion in time with zonotopic coefficients (which gives some Taylor model of the solution flow):

$$P(t) : x = 0.1\epsilon_1 + (1 + 0.01\epsilon_2)t + 1.31 \cdot 10^{-7} \epsilon_3 t^2 \wedge y = 0.1\epsilon_4 + (0.01\epsilon_6 + 0.01\epsilon_7 t)t \\ + (0.005\epsilon_5)t^2 \wedge \theta(t) = 0.01\epsilon_6 + 0.01\epsilon_7 t \quad (9)$$

with $\epsilon_i \in [-1, 1]$ for $i = 1, \dots, 7$. and $x_0 = 0.1\epsilon_1$, $b_1 = 0.01\epsilon_2$, $y_0 = 0.1\epsilon_4$, $a = 0.01\epsilon_7$ and $\theta_0 = 0.01\epsilon_6$. These were obtained by a linearization of the cosinus and sinus and simple estimates of remainders, which could be improved but were kept simple for the sake of readability.

We interpret the $R_{\exists\forall\exists}(\varphi)$ formula of Equation (4) by quantifying over the symbolic variables ϵ_1 to ϵ_7 . We have a correspondence between initial states and inputs of the problem and the ϵ_i , except for ϵ_3 and ϵ_5 that abstract the remainder term of the Taylor approximation of the solution. Hence an over-approximation of $R_{\exists\forall\exists}(\varphi)$ can be obtained by quantifier elimination on the formula:

$$\exists\epsilon_7 \in [-1, 1], \exists\epsilon_1 \in [-1, 1], \exists\epsilon_4 \in [-1, 1], \exists\epsilon_6 \in [-1, 1], \\ \forall\epsilon_2 \in [-1, 1], \exists\epsilon_3 \in [-1, 1], \exists\epsilon_5 \in [-1, 1], \exists t \in [0, 0.5], P(t)$$

where $P(t)$ is defined by Equation (9) and all symbols are existentially quantified except ϵ_2 which corresponds to the disturbance b_1 . There are numerous software implementing some form or another of quantifier elimination, e.g. QEPCAD [6], REDUCE RedLog package [10], and Mathematica [24]. We use in the sequel Mathematica and its operation `Reduce`. We refer the reader to the appendix, Section A, where all queries in Mathematica are provided. Using Mathematica for the problem above times out, but when we make independent queries on x , y and θ , we get $x \in [-0.1, 0.595]$, $y \in [-0.10875, 0.10875]$ and $\theta \in [-0.015, 0.015]$, with a warning about potential inexact coefficients, respectively in about 25, 12 and 0.05 seconds on a MacBook Pro 2.3GHz Intel Core i9 8 cores with 16GB of memory. This gives a correct outer approximation of $R_{\exists\forall\exists}(\varphi)$.

Similarly, for inner-approximation we eliminate the quantifiers in:

$$\exists\epsilon_7 \in [-1, 1], \exists\epsilon_1 \in [-1, 1], \exists\epsilon_4 \in [-1, 1], \exists\epsilon_6 \in [-1, 1], \\ \forall\epsilon_2 \in [-1, 1], \forall\epsilon_3 \in [-1, 1], \forall\epsilon_5 \in [-1, 1], \exists t \in [0, 0.5], P(t)$$

where the uncertainties ϵ_3 and ϵ_5 are now quantified universally, reflecting the fact that inner-approximation corresponds to making no hypothesis on the values of these variables corresponding to approximation errors, apart from knowing bounds. The elimination times out for the full problem and returns the same bounds as before up to 10^{-5} , when solving the problem separately on each variable x , y and θ , in respectively 2.2, 17.1 and 0.06 seconds. However, contrarily to the over-approximation, these independent queries do not allow us to conclude

about an actual inner-approximation for $R_{\exists\forall\exists}(\varphi)$, as the existentially quantified variables may be assigned different values in the 3 independent queries.

1.4 Contributions

We extend the approach of [20], which is restricted to solving problems of the form $R_{\exists}(f)$ or $R_{\forall\exists}(f)$, to deal with arbitrary quantified formulas of the form of $R_{\mathcal{P}}(f)$ of Equation (2). These include the generalized robust reachability problems discussed in Section 1.2. The problem of finding the exact set $R_{\mathcal{P}}(f)$ admits closed formulas for a scalar-valued affine function f , as described in Section 2.1, culminating in Proposition 1. By local linearization techniques, akin to the ones used in [20], we get explicit formulas for inner and outer-approximations of general non-linear scalar-valued functions in Section 2.2, Theorem 1.

We consider the general vector-valued case in Section 3 and Theorem 2. The difficulty lies, as for the $\forall\exists$ case of [20], in the computation of inner-approximations. The solution proposed is to interpret slightly relaxed quantified problems, one dimension at a time, that, altogether, give guaranteed inner-approximations of $R_{\mathcal{P}}(f)$, extending the method of [20]. The combinatorics of variables, quantifiers and components of f make the intuition of the indices used in Theorem 2 difficult to fully apprehend: we thus begin Section 3 by an example.

The general form of quantified problems we are considering here makes solutions that we propose difficult to assess and compare: we are not aware of any existing tool solving similar problems, at the exception of quantifier elimination algorithms, discussed in Section 1.3. We also develop a sampling method, see Remark 4, for checking the tightness of our results.

Finally, we report on our implementation of this method in `Julia` in Section 4. Benchmarks show that this method is tractable, with experiments up to thousands of variables solved in a matter of tens of seconds.

2 Approximations of arbitrary quantified formulas in the case of scalar-valued functions

We first focus in Section 2.1 on the computation of $R_{\mathcal{P}}(f)$ where f is an affine function from \mathbb{R}^p to \mathbb{R} . In this case, we derive exact bounds. We then rely on this result to carry on with the general case in Section 2.2, using a mean-value theorem.

2.1 Exact bounds for scalar affine functions

We consider affine functions, i.e. functions of the form $f(x_1, \dots, x_q) = \delta_0 + \sum_{i=1}^q \delta_i x_i$. For these functions, we consider the general quantified problem defined, for $Q_j = \forall$ or \exists , as:

$$S_q(\delta_0; Q_1, \delta_1; \dots; Q_q, \delta_q) = \{z \in \mathbb{R} \mid Q_1 x_1 \in [-1, 1], \\ Q_2 x_2 \in [-1, 1], \dots, Q_q x_q \in [-1, 1], z = f(x_1, x_2, \dots, x_q)\}$$

We first see that we have:

Lemma 1.

$$S_q(\delta_0; Q_1, \delta_1; \dots; Q_q, \delta_q) = \begin{cases} \bigcap_{x_1 \in [-1,1]} S_{q-1}(\delta_0 + \delta_1 x_1; Q_2, \delta_2; \dots; Q_q, \delta_q) & \text{if } Q_1 = \forall \\ \bigcup_{x_1 \in [-1,1]} S_{q-1}(\delta_0 + \delta_1 x_1; Q_2, \delta_2; \dots; Q_q, \delta_q) & \text{if } Q_1 = \exists \end{cases} \quad (10)$$

The proof is given in Section B.

Remark 4. The first consequence of Lemma 1 is that it gives a way to probe quantified formulas of the form of Equation (2), by sampling, as follows. For each quantifier i , in the order of encounter, join samples or ranges for $\exists x_i$, intersect them for $\forall x_i$. For example, for the alternation $\exists x_1, \forall x_2, \exists x_3$, we compute

$$[\min_{x_i} \max_{x_2} \min_{x_3} f(x_1, x_2, x_3), \max_{x_i} \min_{x_2} \max_{x_3} f(x_1, x_2, x_3)]$$

This gives an estimation of the robust range of the function, which is in the general case neither an inner-approximation nor an outer-approximation, as this sampling approach performs inner-approximation with respect to existential quantification and outer-approximation with respect to universal quantification. Note that in some particular cases (e.g. affine) we can design an exact method based on these and on unions and intersections of particular polyhedra.

Using Lemma 1, we see that $S_2(\delta_0; \forall, \delta_1; \exists, \delta_2)$ is empty when the impact on f of the existentially quantified variable is not of large enough magnitude to counteract the effect of the universally quantified variable. The formula given below expresses such conditions for $R_{\mathbf{p}}(f)$ to be non-empty, by imposing a bound on the ℓ_1 norm of the universally quantified variables, by a suitable combination of the ℓ_1 norms, noted $\|\mathbf{x}\|$, of other variables, in particular the existentially quantified ones.

Proposition 1. *Let f be an affine function defined by:*

$$f(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{2n}) = \delta_0 + \langle \Delta_1, \mathbf{x}_1 \rangle + \langle \Delta_2, \mathbf{x}_2 \rangle + \dots + \langle \Delta_{2n}, \mathbf{x}_{2n} \rangle$$

with $\Delta_i = (\delta_{k_i+1}, \dots, \delta_{k_{i+1}}) \in \mathbb{R}^{j_i}$, $i = 1, \dots, 2n$, where $k_i = \sum_{l=1}^{i-1} j_l$, and $\langle \cdot, \cdot \rangle$ denotes the scalar product. Consider $R_{\mathbf{p}}(f)$ for the partition $\mathbf{p} = (j_1, \dots, j_{2n})$ of p , as in Equation (2), then we have:

$$R_{\mathbf{p}}(f) = \delta_0 + \left[\sum_{k=1}^n (\|\Delta_{2k-1}\| - \|\Delta_{2k}\|), \sum_{k=1}^n (\|\Delta_{2k}\| - \|\Delta_{2k-1}\|) \right]$$

if $\|\Delta_{2l-1}\| \leq \|\Delta_{2l}\| + \sum_{k=l+1}^n (\|\Delta_{2k}\| - \|\Delta_{2k-1}\|)$ for $l = 1, \dots, n$, otherwise $R_{\mathbf{p}}(f) = \emptyset$

The proof is given in Appendix C.

Remark 5. In the sequel, when applying Proposition 1, we will use notations Δ_x with x such as control a , disturbance b_1 , angle θ instead of using for indices a potentially less understandable numbering.

2.2 Inner and outer-approximations for non-linear scalar-valued functions

We are now in a position to give inner and outer approximations of $R_{\mathbf{p}}(f)$ for general scalar-valued $f(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{2n})$ from \mathbb{R}^p to \mathbb{R} . The principle is to carefully linearize f , so that inner and outer-approximations of $R_{\mathbf{p}}(f)$ are given by inner and outer-approximations of a similar quantified problem on its linearization, this is Proposition 2. Combining this with e.g. simple mean-value approximations mentioned in Remark 6, we obtain Theorem 1. After exemplifying these formulas on toy examples, we apply it, twice, to the Dubbins vehicle model of Example 1. We first use the Taylor approximation of its dynamics, Example 4. We then show that we do not need to compute such approximations and that our approach can also compute direct inner and outer-approximations of $R_{\mathbf{p}}(f)$ where f is the solution of a differential equation, Example 5.

As before, for a given function $f : \mathbb{R}^p \rightarrow \mathbb{R}$, we denote by $\mathbf{p} = (j_1, \dots, j_{2n})$ a partition of the p arguments of f and $k_l = \sum_{i=1}^{l-1} j_i$, for $l = 1, \dots, 2n + 1$. We suppose we have p intervals A_1, \dots, A_p and we write $\mathbf{A}_i = (A_{k_i+1}, \dots, A_{k_{i+1}})$, $i = 1, \dots, 2n$ the corresponding boxes in \mathbb{R}^{j_i} . We will use the notation:

$$\begin{aligned} \mathcal{C}(\mathbf{A}_1, \dots, \mathbf{A}_{2n}) &= \{z \mid \forall \alpha_1 \in \mathbf{A}_1, \exists \alpha_2 \in \mathbf{A}_2, \dots, \\ &\quad \forall \alpha_{2n-1} \in \mathbf{A}_{2n-1}, \exists \alpha_{2n} \in \mathbf{A}_{2n}, z = \sum_{j=1}^{2n} \alpha_j\}. \end{aligned}$$

Proposition 2. *Given function $f : \mathbb{R}^p \rightarrow \mathbb{R}$ and partition \mathbf{p} as above, define the following families of functions*

$$h^{x_1, \dots, x_{j-1}}(x_j) = f(x_1, \dots, x_{j-1}, x_j, 0, \dots, 0) - f(x_1, \dots, x_{j-1}, 0, \dots, 0)$$

for $j = 1, \dots, p$, and suppose we have the following inner and outer-approximations of their images, independently of x_1, \dots, x_{j-1} , denoted by $\text{range}(\cdot)$: $I_j \subseteq \text{range}(h^{x_1, \dots, x_{j-1}}) \subseteq O_j$ for $j = 1, \dots, p$. Then, writing $\mathbf{I}_i = \prod_{j=k_i+1}^{k_{i+1}} [I_j, \bar{I}_j]$,

$\mathbf{O}_i = \prod_{j=k_i+1}^{k_{i+1}} [\underline{Q}_j, \bar{O}_j]$, $i = 1, \dots, 2n$, we have:

$$\begin{aligned} f(0, \dots, 0) + \mathcal{C}(\mathbf{O}_1, \mathbf{I}_2, \dots, \mathbf{O}_{2n-1}, \mathbf{I}_{2n}) &\subseteq R_{\mathbf{p}}(f) \\ &\subseteq f(0, \dots, 0) + \mathcal{C}(\mathbf{I}_1, \mathbf{O}_2, \dots, \mathbf{I}_{2n-1}, \mathbf{O}_{2n}) \quad (11) \end{aligned}$$

Proof. The proof is based on the fact that f is the sum of all the h_i , $i = 1, \dots, 2n$, and of $f(0, \dots, 0)$, and is proven by induction on the number of quantifier alternations, see Section E.

Remark 6. We do have such approximants of the range as necessary for Proposition 2, thanks to a generalized mean-value theorem [16,21]. If we have, for all $i = 1, \dots, 2n$ and all $j = k_i + 1, \dots, k_{i+1}$, $\nabla_j = [\underline{\nabla}_j, \overline{\nabla}_j]$ such that:

$$\left\{ \left| \frac{\partial f}{\partial x_j}(\mathbf{x}_1, \dots, \mathbf{x}_i, 0, \dots, 0) \right| \mid \mathbf{x}_l \in [-1, 1]^{j_l}, l = 1, \dots, i \right\} \subseteq \nabla_j$$

then we can use, for all $j = 1, \dots, 2n$: $I_i = \underline{\nabla}_j[-1, 1]$, and $O_j = \overline{\nabla}_j[-1, 1]$. We can naturally also use other approximation methods.

We now deduce inner and outer-approximations of $R_{\mathbf{p}}(f)$:

Theorem 1. *With the hypotheses of Proposition 2 on sets I_j and O_j , and denoting $\sum \mathbf{A}$, for \mathbf{A} any vector of reals, the sum of all its components, we have:*

$$f(0, \dots, 0) + \left[\sum_{k=1}^n \sum (\overline{\mathbf{O}}_{2k-1} + \underline{\mathbf{I}}_{2k}), \sum_{k=1}^n \sum (\overline{\mathbf{I}}_{2k} + \underline{\mathbf{O}}_{2k-1}) \right] \subseteq R_{\mathbf{p}}(f)$$

if $\sum \overline{\mathbf{O}}_{2l-1} - \sum \underline{\mathbf{O}}_{2l-1} \leq \sum_{k=l}^n \sum (\overline{\mathbf{I}}_{2k} - \underline{\mathbf{I}}_{2k}) - \sum_{k=l+1}^n \sum (\overline{\mathbf{O}}_{2k-1} - \underline{\mathbf{O}}_{2k-1})$ for $l = 1, \dots, n$, otherwise the inner-approximation is empty, and:

$$R_{\mathbf{p}}(f) \subseteq f(0, \dots, 0) + \left[\sum_{k=1}^n \sum (\overline{\mathbf{I}}_{2k-1} + \underline{\mathbf{O}}_{2k}), \sum_{k=1}^n \sum (\overline{\mathbf{O}}_{2k} + \underline{\mathbf{I}}_{2k-1}) \right]$$

if $\sum \overline{\mathbf{I}}_{2l-1} - \sum \underline{\mathbf{I}}_{2l-1} \leq \sum_{k=l}^n \sum (\overline{\mathbf{O}}_{2k} - \underline{\mathbf{O}}_{2k}) - \sum_{k=l+1}^n \sum (\overline{\mathbf{I}}_{2k-1} - \underline{\mathbf{I}}_{2k-1})$ for $l = 1, \dots, n$, otherwise the outer-approximation is empty.

Proof. The proof uses Proposition 1 on $C(\mathbf{O}_1, \mathbf{I}_2, \dots, \mathbf{O}_{2n-1}, \mathbf{I}_{2n})$ and $C(\mathbf{I}_1, \mathbf{O}_2, \dots, \mathbf{I}_{2n-1}, \mathbf{O}_{2n})$, after rescaling of the interval \mathbf{O}_i and \mathbf{I}_j to $[-1, 1]$, and Proposition 2. It is detailed in Section F.

In Example 4, we now apply Theorem 1 to the x component of the Dubbins vehicle as expressed in Example 3, ignoring any constraint on y and θ .

Example 4 (Dubbins vehicle (continued)). We recall that:

$$x(t) = 0.1\epsilon_1 + (1 + 0.01\epsilon_2)t + (1.31 \cdot 10^{-7}\epsilon_3)t^2$$

The ∇_k , outer-approximations of the absolute value of the partial derivatives of Remark 6, $\frac{\partial x}{\partial \epsilon_k}$ and $\frac{\partial x}{\partial t}$ evaluated between times $t = 0$ and $t = 0.5$, are: $\nabla_{\epsilon_1} = 0.1$, $\nabla_{\epsilon_2} = [0, 0.005]$, $\nabla_{\epsilon_3} = [0, 3.275 \cdot 10^{-8}]$, $\nabla_t = [0.989999869, 1.010000131]$. We thus have $I_{\epsilon_1} = O_{\epsilon_1} = [-0.1, 0.1]$, $I_{\epsilon_2} = 0$, $O_{\epsilon_2} = [-0.005, 0.005]$, and $I_{\epsilon_3} = 0$,

$O_{\epsilon_3} = [-3.275 \cdot 10^{-8}, 3.275 \cdot 10^{-8}]$ by a direct application of Remark 6. Note that for computing I_t and O_t , we use the generalized mean-value theorem of [16] again, but in a slightly different way than in Remark 6, since the point at which we can evaluate the corresponding function is $t = 0$, which is the lower bound of the extent of the values of t ($[0, 5]$) and not its center as for other variables. In that case we can compute the tighter bounds: $I_t = \underline{\nabla}_t[0, 0.5] = [0, 0.4949999345]$ and $O_t = \overline{\nabla}_t[0, 0.5] = [0, 0.5050000655]$.

The quantified formula of Example 2 has only one \forall, \exists alternation, the condition of Theorem 1 will involve $O_{\epsilon_2} + O_{\epsilon_3} = [-0.005, 0.005] + [-3.275 \cdot 10^{-8}, 3.275 \cdot 10^{-8}]$ and $I_t = [0, 0.4949999345]$. We see that indeed, $(\overline{O}_{\epsilon_2} - \underline{O}_{\epsilon_3}) + (\overline{O}_{\epsilon_3} - \underline{O}_{\epsilon_2}) = 0.010000066 \leq \overline{I}_t - \underline{I}_t = 0.4949999345$, hence we can compute an inner-approximation for the x component of φ . Its lower bound is:

$$\begin{array}{cccccc} x_c & +\underline{I}_{\epsilon_1} & +\overline{O}_{\epsilon_2} & +\overline{O}_{\epsilon_3} & +\underline{I}_t & \\ = 0 & -0.1 & +0.005 & +3.275 \cdot 10^{-8} & +0 & = -0.095 \end{array}$$

and its upper bound:

$$\begin{array}{cccccc} x_c & +\overline{I}_{\epsilon_1} & +\underline{O}_{\epsilon_2} & +\underline{O}_{\epsilon_3} & +\overline{I}_t & \\ = 0 & +0.1 & -0.005 & -3.275 \cdot 10^{-8} & +0.4949999345 & = 0.59. \end{array}$$

Similarly, we compute an outer-approximation and find the following bounds for $R_{\exists\forall\exists}(\varphi_x)$: $[-0.095, 0.590] \subseteq R_{\exists\forall\exists}(\varphi) \subseteq [-0.1, 0.605]$, to be compared with the solution from Mathematica quantifier elimination $-0.1 \leq x \leq 0.595$. Sampling also yields estimate $[-0.1, 0.595]$.

In the former example, and in general for continuous-time controlled systems defined by a flow function $\varphi(t; x_0, u, w)$ solution of an initial value problem, we do not need as with quantifier elimination techniques to first compute polynomial approximations. We only need to compute outer approximations of the flow for one initial condition ("central trajectory") and of the Jacobian of the flow for the set of initial conditions, as exemplified below.

Example 5 (Dubbins vehicle (continued)). We consider again the Dubbins vehicle, but defined as the direct solution of the ODEs of Example 1. We first compute an outer-approximation of a "central trajectory" (x_c, y_c, θ_c) , i.e. of the trajectory starting at $x = 0, y = 0, \theta = 0, b_1 = 0$ and $a = 0$. This gives $x_c = t, y_c = 0$ and $\theta_c = 0$.

We note that $\frac{\partial x}{\partial t} = \cos(\theta) + b_1 \in [0.989999965, 1.01]$ thus, using notations from Remark 6, we have the inner and outer-approximations of the effect of variable t on the value of x , $I_{x,t} = [0, 0.494999982]$, $O_{x,t} = [0, 0.505]$, and similarly for the other variables: $I_{y,t} = 0$, $O_{y,t} = [-\sin(0.015)/2, \sin(0.015)/2] = [-1.309 \cdot 10^{-4}, 1.309 \cdot 10^{-4}]$ and $I_{\theta,t} = 0$, $O_{\theta,t} = [-0.005, 0.005]$.

The Jacobian of φ with respect to x_0, y_0, θ_0, b_1 and a , $J_{i,x_0} = \frac{\partial \varphi_i}{\partial t}$, $J_{i,y_0} = \frac{\partial \varphi_i}{\partial t}$, $J_{i,\theta_0} = \frac{\partial \varphi_i}{\partial t}$, $J_{i,b_1} = \frac{\partial \varphi_i}{\partial t}$ and $J_{i,a} = \frac{\partial \varphi_i}{\partial t}$, for $i = x, y, \theta$ respectively, satisfies a variational equation [20], solved in appendix, Section I. By Remark 6, this gives the following inner and outer approximations for all parameters x_0, y_0, θ_0, a and b_1 , and all components x, y and θ of φ :

- $I_{x,a} = 0$, $O_{x,a} = [-6.545 \cdot 10^{-7}, 6.545 \cdot 10^{-7}]$, $I_{x,x_0} = O_{x,x_0} = [-0.1, 0.1]$,
 $I_{x,\theta_0} = 0$, $O_{x,\theta_0} = [-1.309 \cdot 10^{-6}, 1.309 \cdot 10^{-6}]$, $I_{x,b_1} = 0$, $O_{x,b_1} = [-0.005, 0.005]$,
- $I_{y,a} = 0$, $O_{y,a} = [-0.0025, 0.0025]$, $I_{y,y_0} = O_{y,y_0} = [-0.1, 0.1]$, $I_{y,\theta_0} = 0$,
 $O_{y,\theta_0} = [-0.005, 0.005]$,
- $I_{\theta,\theta_0} = O_{\theta,\theta_0} = [-0.01, 0.01]$, $I_{\theta,a} = 0$, $O_{\theta,a} = [0, 0.005]$,

We now compute the set $R_{\exists\forall\exists}$ consisting of z such that:

$$\begin{aligned} & \exists a \in [-0.01, 0.01], \exists x_0 \in [-0.1, 0.1], \exists y_0 \in [-0.1, 0.1], \\ & \exists \theta_0 \in [-0.01, 0.01], \forall b_1 \in [-0.01, 0.01], \exists t \in [0, 0.5], z = \varphi(t; x_0, y_0, \theta_0, a, b_1) \end{aligned}$$

Applying Theorem 1 we find first an inner-approximation for x (again, ignoring any condition on y and θ) of φ . Its lower bound is:

$$\begin{array}{cccccc} x_c & +\underline{I}_{x,a} & +\underline{I}_{x,x_0} & +\underline{I}_{x,y_0} & +\underline{I}_{x,\theta_0} & +\overline{O}_{x,b_1} & +\underline{I}_{x,t} \\ = 0 & -0 & -0.1 & +0 & -0 & +0.005 & +0 \end{array}$$

which is equal to -0.095, and its upper bound:

$$\begin{array}{cccccc} x_c & +\overline{I}_{x,a} & +\overline{I}_{x,x_0} & +\overline{I}_{x,y_0} & +\overline{I}_{x,\theta_0} & +\underline{O}_{x,b_1} & +\overline{I}_{x,t} \\ 0 & +0 & +0.1 & +0 & +0 & -0.005 & +0.494999982 \end{array}$$

which is equal to 0.589999982. Therefore the inner-approximation for x is equal to $[-0.095, 0.589999982]$, given that the conditions for the inner-approximation to be non-void are met. Similarly, we compute an outer-approximation for the x component of φ and find $[-0.1000019635, 0.6050019635]$.

The approximations for the y and θ components of φ are computed similarly, see Appendix, Section J for detailed computation. We obtain for y the inner-approximation $[-0.1, 0.1]$ and over-approximation $[0.1076309, 0.1076309]$, and for θ the inner-approximation $[-0.01, 0.01]$ and over-approximation $[-0.02, 0.02]$.

All these results are very close to the ones obtained in Section 1.3 with quantifier elimination¹, but are obtained here with a much smaller complexity.

3 Approximations in the case of vector-valued functions

Outer-approximations in the general case when f goes from \mathbb{R}^p to \mathbb{R}^m for any strictly positive value of m are directly obtained by the Cartesian product of the ranges obtained separately by the method of Section 2.2 on each component of f . The case of inner-approximations is more involved, since a Cartesian product of inner-approximations is not in general an inner-approximation.

In this section, we generalize the method of [21] to the case of arbitrary quantified formulas. We begin by a simple example, before stating the result for the general case in Theorem 2.

¹ Note though that the linearization we used for simplifying formulas given to a quantifier elimination tool are slightly over-approximated (especially in the y component).

Example 6. Suppose we want to inner approximate the following set $R_{\forall\exists\forall\exists}(f)$ for a function f with two components f_1 and f_2 :

$$R_{\forall\exists\forall\exists}(f) = \{z \mid \forall x_1, \exists x_2, \exists x_3, \forall x_4, \exists x_5, \exists x_6, z = f(x)\}.$$

The main idea is that we can rely on the conjunction of quantified formulas for each component if no variable is existentially quantified for several components. We thus transform if necessary the quantified formula by strengthening them for that objective, which is sound with respect to computing inner-approximations. For example here, we can interpret, for all z_1 and z_2 :

$$\forall x_1, \forall x_2, \exists x_3, \forall x_4, \forall x_5, \exists x_6, z_1 = f_1(x_1, x_2, x_3, x_4, x_5, x_6) \quad (12)$$

$$\forall x_1, \forall x_3, \exists x_2, \forall x_4, \forall x_6, \exists x_5, z_2 = f_2(x_1, x_2, x_3, x_4, x_5, x_6) \quad (13)$$

Then we get Skolem functions: $x_3(z_1, x_1, x_2)$ and $x_6(z_1, x_1, x_2, x_3, x_4, x_5)$ from Equation (12), such that $z_1 = f_1(x_1, x_2, x_3(z_1, x_1, x_2), x_4, x_5, x_6(z_1, x_1, x_2, x_3, x_4, x_5))$ and $x_2(z_2, x_1, x_3)$ and $x_5(z_2, x_1, x_2, x_3, x_4, x_6)$ from (13), such that $z_2 = f_2(x_1, x_2(z_2, x_1, x_3), x_3, x_4, x_5(z_2, x_1, x_2, x_3, x_4, x_6), x_6)$. Supposing that f_1 and f_2 are elementary functions, these Skolem functions can be chosen to be continuous [16,18]. Consider now functions $g_{z_1, z_2} : \mathbb{R}^6 \rightarrow \mathbb{R}^6$ defined by

$$g(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1, x_2(z_2, x_1, x_3), x_3(z_1, x_1, x_2), x_4, \\ x_5(z_2, x_1, x_2, x_3, x_4, x_6), x_6(z_1, x_1, x_2, x_3, x_4, x_5))$$

for all $(z_1, z_2) \in \mathbf{z}_1 \times \mathbf{z}_2$. This is a continuous function as composition of continuous functions, from $\mathbf{x} = \mathbf{x}_1 \times \mathbf{x}_2 \times \dots \times \mathbf{x}_6$ to itself.

By Brouwer's fixpoint theorem, we have fixpoints $x_3^\infty(z_1, z_2, x_1)$, $x_6^\infty(z_1, z_2, x_1, x_4)$, $x_2^\infty(z_1, z_2, x_1)$ and $x_5^\infty(z_1, z_2, x_1, x_4)$, for all values of z_1, z_2, x_1, x_4 (x_1 and x_4 being the existentially quantified input variables of Equation (6)), such that $x_3^\infty(z_1, z_2, x_1) = x_3(z_1, x_1, x_2^\infty(z_1, z_2, x_1))$, $x_6^\infty(z_1, z_2, x_1, x_4) = x_6(z_1, x_1, x_2^\infty(z_1, z_2, x_1), x_3^\infty(z_1, z_2, x_1), x_4, x_5^\infty(z_1, z_2, x_1, x_4))$, $x_2^\infty(z_1, z_2, x_1) = x_2(z_2, x_1, x_3^\infty(z_1, z_2, x_1))$ and $x_5^\infty(z_1, z_2, x_1, x_4) = x_5(z_2, x_1, x_2^\infty(z_1, z_2, x_1), x_4, x_6^\infty(z_1, z_2, x_1, x_4))$. This implies that for all $(z_1, z_2) \in \mathbf{z}$ and for all x_1, x_4 :

$$z_1 = f_1(x_1, x_2^\infty(z_1, z_2, x_1), x_3^\infty(z_1, x_1, x_4), x_4, x_5^\infty(z_1, z_2, x_1, x_4), x_6^\infty(z_1, z_2, x_1, x_4)) \\ z_2 = f_2(x_1, x_2^\infty(z_1, z_2, x_1), x_3^\infty(z_1, z_2, x_1), x_4, x_5^\infty(z_1, z_2, x_1, x_4), x_6^\infty(z_1, z_2, x_1, x_4))$$

allowing to deduce an inner-approximation of $R_{\forall\exists\forall\exists}(f)$ since:

$$\forall z, \forall x_1, \forall x_4, \exists x_2 = x_2^\infty(z_1, z_2, x_1), \exists x_3 = x_3^\infty(z_1, z_2, x_1), \\ \exists x_5 = x_5^\infty(z_1, z_2, x_1, x_4), \exists x_6 = x_6^\infty(z_1, z_2, x_1, x_4), z = f(x_1, x_2, x_3, x_4, x_5, x_6)$$

is equivalent to: $\forall z, \forall x_1, \exists x_2 = x_2^\infty(z_1, z_2, x_1), \exists x_3 = x_3^\infty(z_1, z_2, x_1), \forall x_4, \\ \exists x_5 = x_5^\infty(z_1, z_2, x_1, x_4), \exists x_6 = x_6^\infty(z_1, z_2, x_1, x_4), z = f(x_1, x_2, x_3, x_4, x_5, x_6)$.

In Theorem 2, we formalize this for any number of quantifier alternations and dimension for \mathbf{z} . The principle is similar to the approach used in [21] for the joint range in the case of $\forall\exists$ formulas.

As previously, we are going to solve the quantified problem $R_{\mathcal{P}}(f)$ where sets $J_A^i = \{k_{2i-1} + 1, \dots, k_{2i}\}$ and $J_E^i = \{k_{2i} + 1, \dots, k_{2i+1}\}$ for $i = 1, \dots, n$ define the n sequences of indices of variables that are universally quantified (J_A^i) and existentially quantified (J_E^i).

The principle is to choose for each existentially quantified variable x_j a unique component of f (among the m ones) that will be used with an existential quantifier as one of the m scalar quantified problems to solve. In the $m - 1$ remaining quantified problem, x_j will be universally quantified. This choice is described by the functions π^i in Theorem 2. There are n such functions, one for each existential block appearing in the quantified problem $R_{\mathcal{P}}(f)$. This is Theorem 3 of [21] generalized to arbitrary alternation of quantifiers $\forall\exists$.

Theorem 2. *Let $f : \mathbb{R}^u \rightarrow \mathbb{R}^m$ be an elementary function and $\pi^i : \{k_{2i} + 1, \dots, k_{2i+1}\} \rightarrow \{1, \dots, m\}$ for $i = 1, \dots, n$. Let us note, for all $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$ $J_{E,z_j}^i = \{l \in \{k_{2i} + 1, \dots, k_{2i+1}\}, \pi^i(l) = j\}$ and $J_{A,z_j}^i = \{k_{2i-1} + 1, \dots, k_{2i}\} \setminus J_{E,z_j}^i$. Consider the following m quantified problems, $j \in \{1, \dots, m\}$:*

$$\begin{aligned} \forall z_j \in z_j, (\forall \mathbf{x}_l \in [-1, 1])_{l \in J_{A,z_j}^1}, (\exists \mathbf{x}_l \in [-1, 1])_{l \in J_{E,z_j}^1}, \dots \\ (\forall \mathbf{x}_l \in [-1, 1])_{l \in J_{A,z_j}^n}, (\exists \mathbf{x}_j \in [-1, 1])_{l \in J_{E,z_j}^n}, z_i = f_i(x_1, \dots, x_{k_{2n}}) \end{aligned}$$

Then $\mathbf{z} = \mathbf{z}_1 \times \mathbf{z}_2 \times \dots \times \mathbf{z}_n$, if non-empty, is an inner-approximation of $R_{\mathcal{P}}(f)$ defined in Equation (2).

The proof is a generalization of the example given in the beginning of this section, and is detailed in Section K.

Remark 7. It is possible to include skewed boxes that can be much tighter than boxes as in Theorem 2, using similar ideas as in [22]. There are also simple heuristics to be used that allows us not to go through the combinatorics of potential choices, for getting the best possible inner-approximation. The sensitivity of the output to variables is computed as part of our algorithm and the best choices of quantifiers are the ones which quantify universally the variables for which there is lower sensitivity, and which quantify existentially the variables for which there is higher sensitivity, giving higher contributions to the inner-approximations.

Example 7. Consider the function $f = (f_1, f_2) : \mathbb{R}^4 \rightarrow \mathbb{R}^2$:

$$\begin{aligned} f_1(x_1, x_2, x_3, x_4) &= 2 + 2x_1 + x_2 + 3x_3 + x_4 \\ f_2(x_1, x_2, x_3, x_4) &= -1 - x_1 - x_2 + x_3 + 5x_4 \end{aligned}$$

We want to find the disturbance set

$$\begin{aligned} R_{\exists\forall\exists}(f) = \{z \in \mathbb{R}^2 \mid \exists x_1 \in [-1, 1], \forall x_2 \in [-1, 1], \exists x_3 \in [-1, 1], \\ \exists x_4 \in [-1, 1], z = f(x_1, x_2, x_3, x_4)\} \quad (14) \end{aligned}$$

An outer-approximation for $R_{\exists\forall\exists}(f)$ is found to be $[-3, 7] \times [-7, 5]$, using a the 1D computation of the previous section, one component at a time.

Now, there are several possible quantified formulas giving a 2D inner-approximation. One of them is,

$$\boxed{\exists x_1}, \forall x_2, \forall x_4, \boxed{\exists x_3}, z_1 = f_1(x_1, x_2, x_3, x_4) \quad (15)$$

$$\forall x_1, \forall x_2, \forall x_3, \boxed{\exists x_4}, z_2 = f_2(x_1, x_2, x_3, x_4) \quad (16)$$

The conditions of Proposition 1 for obtaining a non-empty inner-approximation are met and we get for Equation (26):

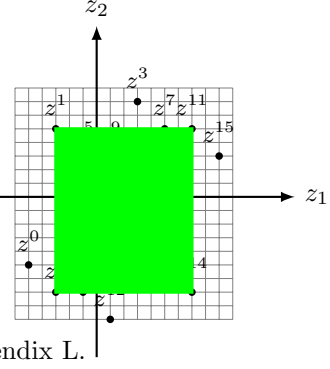
$$\begin{aligned} & [z_1^c - \|\Delta_{x_1}\| + \|\Delta_{x_2, x_4}\| - \|\Delta_{x_3}\|, z_1^c + \|\Delta_{x_1}\| - \|\Delta_{x_2, x_4}\| + \|\Delta_{x_3}\|] \\ & = [2 \quad -2 \quad +1 +1 \quad -3, \quad 2 \quad +2 \quad -1 -1 \quad +3] \end{aligned}$$

which is equal to $[-1, 5]$, and for Equation (27):

$$\begin{aligned} & [z_2^c + \|\Delta_{x_1, x_2, x_4}\| - \|\Delta_{x_3}\|, z_2^c - \|\Delta_{x_1, x_2, x_4}\| + \|\Delta_{x_3}\|] \\ & = [-1 \quad +1 +1 +1 \quad -5, \quad -1 \quad -1 -1 -1 \quad +5] \end{aligned}$$

which is equal to $[-3, 1]$. Hence $[-1, 5] \times [-3, 1]$ is in the set $R_{\exists\forall\exists}(f)$.

These inner and outer-approximations, together with the exact robust joint range, are depicted in the figure on the right-hand side: we represented some particular points of the image by z^1 to z^{13} ; the inner and outer boxes represent the inner and outer-approximations $[-1, 5] \times [-3, 1]$ and $[-3, 7] \times [-7, 5]$; finally the polyhedron lying in between is the exact robust image. Other possibilities are discussed in Appendix L.



Example 8 (Generalized robust reachability for the Dubbins vehicle). We consider the following problem, which is a slight space relaxation of the original problem solved in 1D in Example 5:

$$\begin{aligned} R_{\exists\forall\exists}(\varphi) = \{ (x, y, \theta) \mid & \exists a \in [-0.01, 0.01], \exists x_0 \in [-0.1, 0.1], \exists y_0 \in [-0.1, 0.1], \\ & \exists \theta_0 \in [-0.01, 0.01], \forall b_1 \in [-0.01, 0.01], \exists t \in [0, 0.5], \exists \delta_2 \in [-1.31e^{-4}, 1.31e^{-4}], \\ & \exists \delta_3 \in [-0.005, 0.005], (x, y, \theta) = \varphi(t; x_0, y_0, \theta_0, a, b_1) + (0, \delta_2, \delta_3) \} \quad (17) \end{aligned}$$

where φ is the flow map for the Dubbins vehicle of Example 1. This means we want to characterize precisely which abscissa x can be reached for some control a , whatever the disturbance b_1 . We allow here a relaxation in space and will determine an inner-approximation of the sets of ordinate y and angle θ which can be reached with control a whatever disturbance b_1 , up to a small tolerance of $1.309 \cdot 10^{-4}$ for the ordinate and 0.005 for θ .

The outer-approximation for $R_{\exists\forall\exists}(\varphi)$ is easy to find from the outer-approximations of each component of φ we already computed in Example 5. We just need to add the extra contributions of δ_2 to y and δ_3 to θ , giving

$$R_{\exists\forall\exists}(\varphi) \subseteq [-0.10000196, 0.60500196] \times [0.1077618, 0.1077618] \times [-0.025, 0.025]$$

In order to find an inner-approximation of $R_{\exists\forall\exists}(\varphi)$, we interpret the following quantified formulas (with the same interval bounds as in Equation (17) for the inputs):

$$\begin{aligned} & \forall a, \forall y_0, \forall \theta_0, \boxed{\exists x_0}, \forall b_1, \forall \delta_2, \forall \delta_3, \boxed{\exists t}, x = \varphi_x(t; x_0, y_0, \theta_0, a, b_1) \\ & \forall a, \forall x_0, \forall \theta_0, \boxed{\exists y_0}, \forall b_1, \forall \delta_3, \forall t, \boxed{\exists \delta_2}, y = \varphi_y(t; x_0, y_0, \theta_0, a, b_1) + \delta_2 \\ & \forall x_0, \forall y_0, \boxed{\exists \theta_0}, \boxed{\exists a}, \forall b_1, \forall \delta_2, \forall t, \boxed{\exists \delta_3}, \theta = \varphi_\theta(t; x_0, y_0, \theta_0, a, b_1) + \delta_3 \text{ and find} \\ & [-0.0949993455, 0.5899993275] \times [-0.0925, 0.0925] \times [-0.01, 0.01] \subseteq R_{\exists\forall\exists}(\varphi). \end{aligned}$$

Note that we were not able to obtain an estimate of the solution of this joint quantified problem (translated using the linearisation for φ of Example 3) with Mathematica, it resulted in a timeout.

4 Implementation and benchmarks

We implemented the method, including the non-linear case of Theorem 1 and the vector-valued case of Theorem 2 in Julia, using packages `LazySets` for manipulating boxes (`Hyperrectangles`) and `Symbolics` for automatic differentiation.

Table 1. Benchmark for quantified reachability problems

Benchmark	# vars	dim	# alternations	non-linear	time (s)	inner/sample	outer/sample
Ex11	3	1	2	✓	0.29	0.33	2.12
Ex4	4	1	2	✓	0.32	1	1.03
Ex7	4	2	2		0.21	(0.78,0.40)	(1.30,1.21)
Linear-2	4	1	2		0.43	1	1
Linear-5	10	1	5		0.4	1	1
Linear-10	20	1	10		0.41	1	1
Linear-25	50	1	25		0.47	1	1
Linear-50	100	1	50		0.58	1	1
Linear-100	200	1	100		0.91	1	1
Linear-500	1000	1	500		8.1	1	1
Linear-1000	2000	1	1000		28.25	1	1
Motion-2	7	1	3	✓	0.62	–	–
Motion-5	14	1	6	✓	0.76	–	–
Motion-10	24	1	11	✓	1.06	–	–
Motion-25	54	1	26	✓	9.4	–	–
Motion-50	104	1	51	✓	148.68	–	–

We ran benchmarks reported in Table 1 on a Macbook Pro 2.3GHz Intel core i9 with 8 cores, measuring timings using the `Benchmark Julia` package. The columns # vars, dim, # alternations, non-linear, time, inner/sample, outer/sample denote, respectively, for each benchmark, the number of quantified variables, the dimension of the image of the function considered, the number of alternations \forall/\exists , whether the function considered is non-linear or not, the time the analyzer took to compute both the inner and the outer-approximation

of the quantified reachability problem, the estimated ratio of the width of the inner-approximation, then the outer-approximation, for each component of the function, with respect to the estimate using sampling².

In this table, Ex k correspond to Example k of this paper, Linear- k are random linear functions on $2k$ variables, quantified as \forall, \exists k times, and Motion-2 to 50 are several instances of a motion planning problem of the same type as the one of Equation 7. Motion- k consists of the x component of the same perturbed Dubbins vehicle as modeled in Example 1, see Appendix M, with k control steps, generating $3 + 2k$ variables and $k + 1$ quantifier alternations. These variables are the k (angular) controls a_i , the k perturbations b_i and the two initial conditions on x and θ . The function, from \mathbb{R}^{4+2k} to \mathbb{R} , that expresses the dynamics at the k th control step is a sum of $2k$ sine functions evaluated on sums of 1 to $k + 1$ variables, plus a sum of $k + 1$ variables.

The theoretical complexity of our method, both for inner and for outer-approximation, for a n dimensional vector-valued quantified problem on p quantified variables, is of the order of n times the complexity of a 1D quantified problem on p quantified variables. Each of these 1D problems has a cost of the order of p times the cost of the evaluation of the function on a (center) point plus the cost of evaluation of its Jacobian on an interval. In the Linear- k problem, the cost of evaluation of the function on a point is of the order of k , and for the Jacobian, apart from the cost of the automatic differentiation, it is of the order of k again. The resolution time can slightly decrease for higher-dimensional problems, which is due to the fact that some of these random problems are found to have empty quantified reachable sets already with few quantifiers. In the Motion- k problem, which has always a non-empty quantified reachable set, the cost of evaluation of the function on a point is of the order of k^2 , and for the Jacobian, it is of the order of k^3 without the cost of the automatic differentiation.

5 Conclusion

In this article, we designed a method for inner and outer-approximating general problems, which is essentially an order 0 method, generalizing mean-value theorems. In future work, we are planning on describing higher order methods, generalizing again the higher order methods of [22]. We will also consider preconditioning and quadrature formulas for general quantified formulas.

Finally, we intend to generalize this work to other kinds of quantified problems where the objective is to find a set R such that the quantified predicate is $f(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{2n}) \in R$, and not an equality predicate as in this work. This should be most useful for finding generalized invariant sets, in addition to generalized reachable sets.

² Sampling is too slow and imprecise when the number of variables grows, hence we could not use it in the case of Motion- k , $k > 2$. For Motion-2, it terminates but with at most 30 samples per dimension, and in dimension 7, this is not representative. In the case of Linear- k , the estimate is always one since our method is exact in 1D, for linear functions.

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A Mathematica expressions used in the Examples

The quantified problem for calculating the outer-approximation of Example 3 is expressed in Mathematica as:

$$\begin{aligned}
& \text{Reduce} [\exists\{\epsilon_7, \epsilon_1, \epsilon_4, \epsilon_6\}, \epsilon_7 \geq -1 \wedge \epsilon_7 \leq 1 \wedge \epsilon_1 \geq -1 \\
& \quad \wedge \epsilon_1 \leq 1 \wedge \epsilon_4 \geq -1 \wedge \epsilon_4 \leq 1 \wedge \epsilon_6 \geq -1 \wedge \epsilon_6 \leq 1, \\
& \quad \forall\epsilon_2, \epsilon_2 \geq -1 \wedge \epsilon_2 \leq 1 \\
& \quad \exists\{t, \epsilon_3, \epsilon_5\}, t \geq 0 \wedge t \leq 0.5 \wedge \epsilon_3 \geq -1 \wedge \epsilon_3 \leq 1 \wedge \epsilon_5 \geq -1 \wedge \epsilon_5 \leq 1 \\
& \quad \quad (x = 0.1\epsilon_1 + (1 + 0.01\epsilon_2)t + 1.31 \cdot 10^{-7}\epsilon_3t^2 \\
& \quad \quad \wedge y = 0.1\epsilon_4 + (0.01\epsilon_6 + 0.01\epsilon_7t)t + 0.05\epsilon_5t^2 \\
& \quad \quad \wedge \text{theta} = 0.01\epsilon_6 + 0.01\epsilon_7t), \{y, z, u\}, \mathbb{R}]
\end{aligned}$$

For the outer-approximations of each component of φ , we have the following Mathematica problems, for φ_x first:

$$\begin{aligned}
& \text{Timing}[\text{Reduce}[\text{Exists}[\{e7, e1, e4, e6\}, (e7 \geq -1) \&\& (e7 \leq 1) \\
& \quad \&\& (e1 \geq -1) \&\& (e1 \leq 1) \&\& (e4 \geq -1) \&\& (e4 \leq 1) \\
& \quad \&\& (e6 \geq -1) \&\& (e6 \leq 1), \text{ForAll}[e2, (e2 \geq -1) \&\& (e2 \leq 1), \\
& \quad \quad \text{Exists}[\{t, e3, e5\}, (t \geq 0) \&\& (t \leq 0.5) \\
& \quad \quad \&\& (e3 \geq -1) \&\& (e3 \leq 1) \&\& (e5 \geq -1) \&\& (e5 \leq 1), \\
& \quad \quad x == 0.1e1 + t + 0.01e2t + 0.000000131e3t^2]], \{x\}, \text{Reals}]]
\end{aligned}$$

Then for φ_y :

$$\begin{aligned}
& \text{Timing}[\text{Reduce}[\text{Exists}[\{e7, e1, e4, e6\}, (e7 \geq -1) \&\& (e7 \leq 1) \\
& \quad \&\& (e1 \geq -1) \&\& (e1 \leq 1) \&\& (e4 \geq -1) \&\& (e4 \leq 1) \\
& \quad \&\& (e6 \geq -1) \&\& (e6 \leq 1), \text{ForAll}[e2, (e2 \geq -1) \&\& (e2 \leq 1), \\
& \quad \quad \text{Exists}[\{t, e3, e5\}, (t \geq 0) \&\& (t \leq 0.5) \\
& \quad \quad \&\& (e3 \geq -1) \&\& (e3 \leq 1) \&\& (e5 \geq -1) \&\& (e5 \leq 1), \\
& \quad \quad y == 0.1e4 + (0.01e6 + 0.01e7t)t + (0.005e5)t^2]], \{y\}, \text{Reals}]]
\end{aligned}$$

And finally for φ_θ :

$$\begin{aligned}
& \text{Timing}[\text{Reduce}[\text{Exists}[\{e7, e1, e4, e6\}, (e7 \geq -1) \&\& (e7 \leq 1) \\
& \quad \&\& (e1 \geq -1) \&\& (e1 \leq 1) \&\& (e4 \geq -1) \&\& (e4 \leq 1) \\
& \quad \&\& (e6 \geq -1) \&\& (e6 \leq 1), \text{ForAll}[e2, (e2 \geq -1) \&\& (e2 \leq 1), \\
& \quad \quad \text{Exists}[\{t, e3, e5\}, (t \geq 0) \&\& (t \leq 0.5) \\
& \quad \quad \&\& (e3 \geq -1) \&\& (e3 \leq 1) \&\& (e5 \geq -1) \&\& (e5 \leq 1), \\
& \quad \quad \text{theta} == 0.01e6 + 0.01e7t]], \{\text{theta}\}, \text{Reals}]]
\end{aligned}$$

For the inner-approximations of all components of φ , first for φ_x , we have:

$$\begin{aligned} & \text{Timing}[\text{Reduce}[\text{Exists}[\{e7, e1, e4, e6\}, (e7 \geq -1) \& \& (e7 \leq 1) \\ & \& \& (e1 \geq -1) \& \& (e1 \leq 1) \& \& (e4 \geq -1) \& \& (e4 \leq 1) \\ & \& \& (e6 \geq -1) \& \& (e6 \leq 1), \text{ForAll}[\{e2, e3, e5\}, \\ & (e2 \geq -1) \& \& (e2 \leq 1) \& \& (e3 \geq -1) \& \& (e3 \leq 1) \\ & \& \& (e5 \geq -1) \& \& (e5 \leq 1), \text{Exists}[\{t\}, (t \geq 0) \& \& (t \leq 0.5), \\ & x == 0.1e1 + t + 0.01e2t + 0.000000131e3t^2]], \{x\}, \text{Reals}]] \end{aligned}$$

Then for φ_y :

$$\begin{aligned} & \text{Timing}[\text{Reduce}[\text{Exists}[\{e7, e1, e4, e6\}, (e7 \geq -1) \\ & \& \& (e7 \leq 1) \& \& (e1 \geq -1) \& \& (e1 \leq 1) \& \& (e4 \geq -1) \& \& (e4 \leq 1) \\ & \& \& (e6 \geq -1) \& \& (e6 \leq 1), \text{ForAll}[\{e2, e3, e5\}, \\ & (e2 \geq -1) \& \& (e2 \leq 1) \& \& (e3 \geq -1) \& \& (e3 \leq 1) \\ & \& \& (e5 \geq -1) \& \& (e5 \leq 1), \text{Exists}[\{t\}, (t \geq 0) \& \& (t \leq 0.5), \\ & y == 0.1e4 + (0.01e6 + 0.01e7t)t + (0.005e5)t^2]], \{y\}, \text{Reals}]] \end{aligned}$$

And finally for φ_θ :

$$\begin{aligned} & \text{Timing}[\text{Reduce}[\text{Exists}[\{e7, e1, e4, e6\}, (e7 \geq -1) \& \& (e7 \leq 1) \\ & \& \& (e1 \geq -1) \& \& (e1 \leq 1) \& \& (e4 \geq -1) \& \& (e4 \leq 1) \\ & \& \& (e6 \geq -1) \& \& (e6 \leq 1), \text{ForAll}[\{e2, e3, e5\}, \\ & (e2 \geq -1) \& \& (e2 \leq 1) \& \& (e3 \geq -1) \& \& (e3 \leq 1) \\ & \& \& (e5 \geq -1) \& \& (e5 \leq 1), \text{Exists}[\{t\}, (t \geq 0) \& \& (t \leq 0.5), \\ & theta == 0.01e6 + 0.01e7t]], \{theta\}, \text{Reals}]] \end{aligned}$$

B Proof of Lemma 1

We distinguish two cases:

- If $Q_1 = \forall$, then $z \in S_n(\delta_0; Q_1, \delta_1; \dots; Q_n, \delta_n)$ iff $\forall x_1 \in [-1, 1], Q_2 x_2 \in [-1, 1], \dots, Q_n x_n \in [-1, 1], z = f(x_1, x_2, \dots, x_n)$. This is equivalent to, for all $x_1 \in [-1, 1]: Q_2 x_2 \in [-1, 1], \dots, Q_n x_n \in [-1, 1], z = (\delta_0 + \delta_1 x_1) + \sum_{i=2}^k \delta_i x_i$, hence $z \in S_{n-1}(\delta_0 + \delta_1 x_1; Q_2, \delta_2; \dots; Q_n, \delta_n)$. Therefore, this is equivalent to

$$z \in \bigcap_{x_1 \in [-1, 1]} S_{n-1}(\delta_0 + \delta_1 x_1; Q_2, \delta_2; \dots; Q_n, \delta_n)$$

- If $Q_1 = \exists$, then $z \in S_n(\delta_0; Q_1, \delta_1; \dots; Q_n, \delta_n)$ iff $\exists x_1 \in [-1, 1]$, $Q_2 x_2 \in [-1, 1], \dots, Q_n x_n \in [-1, 1]$, $z = f(x_1, x_2, \dots, x_n)$. This is equivalent to, for some $x_1 \in [-1, 1]$: $Q_2 x_2 \in [-1, 1], \dots, Q_n x_n \in [-1, 1]$, $z = (\delta_0 + \delta_1 x_1) + \sum_{i=2}^k \delta_i x_i$, hence $z \in S_{n-1}(\delta_0 + \delta_1 x_1; Q_2, \delta_2; \dots; Q_n, \delta_n)$ for some $x_1 \in [-1, 1]$. Therefore, this is equivalent to

$$z \in \bigcup_{x_1 \in [-1, 1]} S_{n-1}(\delta_0 + \delta_1 x_1; Q_2, \delta_2; \dots; Q_n, \delta_n)$$

C Proof of Proposition 1

Note first that

$$R_{\mathbf{p}}(f) = S_p(\delta_0; \forall, \delta_{k_1+1}; \dots; \forall, \delta_{k_2}; \exists, \delta_{k_2+1}; \dots; \exists, \delta_{k_3}; \dots; \forall, \delta_{k_{2n-1}+1}; \dots; \forall, \delta_{k_{2n}}; \exists, \delta_{k_{2n}+1}; \dots; \exists, \delta_{k_{2n+1}}) \quad (18)$$

where $k_i = \sum_{l=1}^{i-1} j_l$ for all $i = 1, \dots, 2n+1$.

We use the notation Δ_i to improve readability of formula (18), that we rewrite: $R_{\mathbf{p}}(f) = S_p(\delta_0; \forall, \Delta_1; \exists, \Delta_2; \dots; \forall, \Delta_{2n-1}; \exists, \Delta_{2n})$.

The proof uses the induction relation (10) on S_n . Let us call P_{2n} the property we wish to prove on $R_{\mathbf{p}}(f)$ for any partition $\mathbf{p} = (j_1, \dots, j_{2n})$ of $p = \sum_{i=1}^{2n} j_i$. For all partitions $\mathbf{p}' = (j'_1, \dots, j'_{2n-1})$ of $p' = \sum_{i=1}^{2n-1} j'_i$ and associated functions $f' : \mathbb{R}^{p'} \rightarrow \mathbb{R}^m$ with $f'(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{2n-1}) = \delta'_0 + \langle \Delta'_1, \mathbf{x}_1 \rangle + \langle \Delta'_2, \mathbf{x}_2 \rangle + \dots + \langle \Delta'_{2n-1}, \mathbf{x}_{2n-1} \rangle$, we define:

$$T_{\mathbf{p}'}(f') = S_{p'}(\delta'_0; \exists, \Delta'_1; \forall, \Delta'_2; \exists, \Delta'_3; \dots; \dots; \forall, \Delta'_{2n-2}; \exists, \Delta'_{2n-1}) \quad (19)$$

and we will prove the following property Q_{2n-1} :

$$T_{\mathbf{p}'}(f') = \left[\delta'_0 + \sum_{k=1}^{n-1} (||\Delta'_{2k}|| - ||\Delta'_{2k-1}||) - ||\Delta'_{2n-1}||, \right. \\ \left. \delta'_0 + \sum_{k=1}^{n-1} (||\Delta'_{2k-1}|| - ||\Delta'_{2k}||) + ||\Delta'_{2n-1}|| \right]$$

if $||\Delta'_{2l}|| \leq ||\Delta'_{2l+1}|| + \sum_{k=l+1}^{n-1} (||\Delta'_{2k+1}|| - ||\Delta'_{2k}||)$ for $l = 1, \dots, n-1$, otherwise $T_{\mathbf{p}'}(f) = \emptyset$.

We first have the base case P_0 and Q_{-1} , which are both equal to $[\delta_0, \delta_0]$.

We now suppose Q_{2n-1} and P_{2n} and prove Q_{2n+1} and P_{2n+2} . Consider first the case of Q_{2n+1} and, for a function $f' : \mathbb{R}^{p'} \rightarrow \mathbb{R}^m$, $f'(\mathbf{x}_1, \dots, \mathbf{x}_{2n+1}) =$

$\delta'_0 + \sum_{i=1}^{2n+1} \langle \Delta'_i, \mathbf{x}_i \rangle$ with a partition $\mathbf{p}' = (j'_1, \dots, j'_{2n+1})$ of the arguments of f' , consider the set:

$$T_{\mathbf{p}'}(f') = S_{\mathbf{p}'}(\delta'_0; \exists, \Delta'_1; \forall, \Delta'_2; \exists, \Delta'_3; \dots; \forall, \Delta'_{2n}; \exists, \Delta'_{2n+1}).$$

By the universal quantifier case of Lemma 1 applied j_1 times, this is equal to:

$$\bigcup_{\mathbf{x}_1 \in [-1, 1]^{j_1}} S_{\mathbf{p}''}(\delta'_0 + \langle \Delta'_1, \mathbf{x}_1 \rangle; \forall, \Delta'_2; \exists, \Delta'_3; \dots; \forall, \Delta'_{2n}; \exists, \Delta'_{2n+1})$$

where $S_{\mathbf{p}''}(\dots) = R_{\mathbf{p}''}(f'')$ with partition $\mathbf{p}'' = (j'_2, \dots, j'_{2n+1})$, $\mathbf{p}'' = \sum_{i=2}^{2n+1} j'_i$, and $f''(\mathbf{x}_2, \dots, \mathbf{x}_{2n+1}) = f'(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{2n+1})$.

Then by the induction hypothesis P_{2n} applied to partition \mathbf{p}'' , each $S_{\mathbf{p}''}(\dots)$ is equal to:

$$\delta'_0 + \langle \Delta'_1, \mathbf{x}_1 \rangle + \left[\sum_{k=1}^n (|\Delta_{2k-1}| - |\Delta_{2k}|), \sum_{k=1}^n (|\Delta_{2k}| - |\Delta_{2k-1}|) \right]$$

if $|\Delta_{2l-1}| \leq |\Delta_{2l}| + \sum_{k=l+1}^n (|\Delta_{2k}| - |\Delta_{2k-1}|)$ for $l = 1, \dots, n$, with $\Delta_i = \Delta'_{i+1}$, $i = 1, \dots, 2n$, otherwise is equal to the empty set. Substituting Δ_i by Δ'_{i+1} , we obtain for each $S_{\mathbf{p}''}(\dots)$:

$$\delta'_0 + \langle \Delta'_1, \mathbf{x}_1 \rangle + \left[\sum_{k=1}^n (|\Delta'_{2k}| - |\Delta'_{2k+1}|), \sum_{k=1}^n (|\Delta'_{2k+1}| - |\Delta'_{2k}|) \right]$$

if $|\Delta'_{2l}| \leq |\Delta'_{2l+1}| + \sum_{k=l+1}^n (|\Delta'_{2k+1}| - |\Delta'_{2k}|)$ for $l = 1, \dots, n$, otherwise it is empty. Finally,

$$\delta'_0 - \|\Delta'_1\| \leq \delta'_0 + \langle \Delta'_1, \mathbf{x}_1 \rangle \leq \delta'_0 + \|\Delta'_1\|$$

each bound being reached by some \mathbf{x}_1 . Hence,

$$T_{\mathbf{p}'}(f') = \left[\delta'_0 - \|\Delta'_1\| + \sum_{k=1}^n (|\Delta'_{2k}| - |\Delta'_{2k+1}|), \right. \\ \left. \delta'_0 + \|\Delta'_1\| + \sum_{k=1}^n (|\Delta'_{2k+1}| - |\Delta'_{2k}|) \right]$$

if $|\Delta'_{2l}| \leq |\Delta'_{2l+1}| + \sum_{k=l+1}^n (|\Delta'_{2k+1}| - |\Delta'_{2k}|)$ for $l = 1, \dots, n$, otherwise

$T_{\mathbf{p}'}(f') = \emptyset$. This is precisely Q_{2n+1} since $-\|\Delta'_1\| + \sum_{k=1}^n (|\Delta'_{2k}| - |\Delta'_{2k+1}|) =$

$$\sum_{k=1}^n (\|\Delta'_{2k}\| - \|\Delta'_{2k-1}\|) - \|\Delta'\|_{2n+1} \text{ and } \|\Delta'_1\| + \sum_{k=1}^n (\|\Delta'_{2k+1}\| - \|\Delta'_{2k}\|) = \sum_{k=1}^n (\|\Delta'_{2k-1}\| - \|\Delta'_{2k}\|) + \|\Delta'\|_{2n+1}.$$

The case of P_{2n+2} is similar, using Lemma 1 and property Q_{2n+1} : suppose again that the induction hypothesis holds, i.e. we suppose P_{2n} and Q_{2n-1} , and we prove Q_{2n+1} and P_{2n+2} . We proved Q_{2n+1} and we now examine the case of P_{2n+2} and consider the following set, starting with a function $f : \mathbb{R}^p \rightarrow \mathbb{R}^m$, $f(\mathbf{x}_1, \dots, \mathbf{x}_{2n+2}) = \delta_0 + \sum_{i=1}^{2n+2} \langle \Delta_i, \mathbf{x}_i \rangle$ with a partition $\mathbf{p} = (j_1, \dots, j_{2n+2})$ of the $p = \sum_{i=1}^{2n+2} j_i$ arguments of f :

$$R_{\mathbf{p}}(f) = S_p(\delta_0; \forall, \Delta_1; \exists, \Delta_2; \dots; \forall, \Delta_{2n+1}; \exists, \Delta_{2n+2}) \quad (20)$$

By the existential quantifier case of Lemma 1 applied j_1 times, this is equal to (we set $\mathbf{p}' = \sum_{i=2}^{2n+2} j_i$):

$$\bigcap_{\mathbf{x}_1 \in [-1, 1]^{j_1}} S_{\mathbf{p}'}(\delta_0 + \langle \Delta_1, \mathbf{x}_1 \rangle; \exists, \Delta_2; \forall, \Delta_3; \dots; \forall, \Delta_{2n+1}; \exists, \Delta_{2n+2}) \quad (21)$$

As $S_{\mathbf{p}'}(\delta_0 + \langle \Delta_1, \mathbf{x}_1 \rangle; \exists, \Delta_2; \forall, \Delta_3; \dots; \forall, \Delta_{2n+1}; \exists, \Delta_{2n+2})$ is equal to $T_{\mathbf{p}'}(f')$, by Equation (19), with $f'(\mathbf{x}_2, \dots, \mathbf{x}_{2n+2}) = f(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{2n+2})$ and $\mathbf{p}' = (j_2, \dots, j_{2n+2})$ is a partition of $\mathbf{p}' = \sum_{i=2}^{2n+2} j_i$, by the induction hypothesis Q_{2n+1} applied to partition \mathbf{p}' , each $S_{\mathbf{p}'}(\delta_0 + \langle \Delta_1, \mathbf{x}_1 \rangle; \exists, \Delta_2; \forall, \Delta_3; \dots; \forall, \Delta_{2n+1}; \exists, \Delta_{2n+2})$ is equal to:

$$\left[\delta_0 + \langle \Delta_1, \mathbf{x}_1 \rangle + \sum_{k=1}^n (\|\Delta'_{2k}\| - \|\Delta'_{2k-1}\|) - \|\Delta'\|_{2n+1}, \right. \\ \left. \delta_0 + \langle \Delta_1, \mathbf{x}_1 \rangle + \sum_{k=1}^n (\|\Delta'_{2k-1}\| - \|\Delta'_{2k}\|) + \|\Delta'\|_{2n+1} \right]$$

if $\|\Delta'_{2l}\| \leq \|\Delta'_{2l+1}\| + \sum_{k=l+1}^{n-1} (\|\Delta'_{2k+1}\| - \|\Delta'_{2k}\|)$ for $l = 1, \dots, n-1$ (otherwise, is equal to the empty set), with $\Delta'_i = \Delta_{i+1}$, for all $i = 1, \dots, 2n+1$, $\Delta'_1 = \Delta_1$ and $\delta'_0 = \delta_0$.

Replacing Δ'_i by Δ_{i+1} , Δ'_1 by Δ_1 and δ'_0 by δ_0 , we obtain the value:

$$\left[\delta_0 + \langle \Delta_1, \mathbf{x}_1 \rangle + \sum_{k=1}^n (\|\Delta_{2k+1}\| - \|\Delta_{2k}\|) - \|\Delta\|_{2n+2}, \right. \\ \left. \delta_0 + \langle \Delta_1, \mathbf{x}_1 \rangle + \sum_{k=1}^n (\|\Delta_{2k}\| - \|\Delta_{2k+1}\|) + \|\Delta\|_{2n+2} \right] \quad (22)$$

if $\|\Delta_{2l+1}\| \leq \|\Delta_{2l+2}\| + \sum_{k=l+1}^{n-1} (\|\Delta_{2k+2}\| - \|\Delta_{2k+1}\|)$ for $l = 1, \dots, n-1$, otherwise is empty.

Now,

$$\delta_0 - \|\Delta_1\| \leq \delta_0 + \langle \Delta_1, \mathbf{x}_1 \rangle \leq \delta_0 + \|\Delta_1\|$$

each bound being reached by some \mathbf{x}_1 , and the intersection of the intervals of Equation (22), where \mathbf{x}_1 varies over $[-1, 1]^{j_1}$ is $R_{\mathcal{P}}(f)$ by Equation (21):

$$R_{\mathcal{P}}(f) = \left[\delta_0 + \|\Delta_1\| + \sum_{k=1}^n (\|\Delta_{2k+1}\| - \|\Delta_{2k}\|) - \|\Delta\|_{2n+2}, \right. \\ \left. \delta_0 - \|\Delta_1\| + \sum_{k=1}^n (\|\Delta_{2k}\| - \|\Delta_{2k+1}\|) + \|\Delta\|_{2n+2} \right]$$

when $\|\Delta_{2l+1}\| \leq \|\Delta_{2l+2}\| + \sum_{k=l+1}^{n-1} (\|\Delta_{2k+2}\| - \|\Delta_{2k+1}\|)$ for $l = 1, \dots, n-1$, which is exactly P_{2n+2} since:

$$\delta_0 + \|\Delta_1\| + \sum_{k=1}^n (\|\Delta_{2k+1}\| - \|\Delta_{2k}\|) - \|\Delta\|_{2n+2} = \delta_0 + \sum_{k=1}^{n+1} (\|\Delta_{2k-1}\| - \|\Delta_{2k}\|)$$

and,

$$\delta_0 - \|\Delta_1\| + \sum_{k=1}^n (\|\Delta_{2k}\| - \|\Delta_{2k+1}\|) + \|\Delta\|_{2n+2} = \sum_{k=1}^{n+1} (\|\Delta_{2k}\| - \|\Delta_{2k-1}\|)$$

Finally, note, that $R_{\mathcal{P}}(f)$ is not empty if and only if $\|\Delta_{2l+1}\| \leq \|\Delta_{2l+2}\| + \sum_{k=l+1}^{n-1} (\|\Delta_{2k+2}\| - \|\Delta_{2k+1}\|)$ for $l = 1, \dots, n-1$ as before, which is equivalent

to: $\|\Delta_{2l-1}\| \leq \|\Delta_{2l}\| + \sum_{k=l}^n (\|\Delta_{2k+2}\| - \|\Delta_{2k+1}\|)$ for $l = 2, \dots, n-1$ and if the radius of $R_{\mathcal{P}}(f)$ is positive i.e.

$$\|\Delta_1\| \leq \sum_{k=1}^n (\|\Delta_{2k}\| - \|\Delta_{2k+1}\|) + \|\Delta\|_{2n+2}$$

which is equivalent to:

$$\|\Delta_1\| \leq \|\Delta_2\| + \sum_{k=1}^n (\|\Delta_{2k+2}\| - \|\Delta_{2k+1}\|)$$

which is $\|\Delta_{2l-1}\| \leq \|\Delta_{2l}\| + \sum_{k=l}^n (\|\Delta_{2k+2}\| - \|\Delta_{2k+1}\|)$ for $l = 1$. Overall, the non-vacuity condition for $R_{\mathcal{P}}(f)$ amounts to:

$$\|\Delta_{2l-1}\| \leq \|\Delta_{2l}\| + \sum_{k=l}^n (\|\Delta_{2k+2}\| - \|\Delta_{2k+1}\|)$$

for $l = 1, \dots, n$.

D Difference between \forall, \exists and \exists, \forall in the linear case

Example 9 (Difference between \forall, \exists and \exists, \forall). Consider, for a function f from \mathbb{R}^2 to \mathbb{R} the sets $R_{\forall\exists}(f) = \{z \mid \forall x_2, \exists x_1, z = f(x_1, x_2)\}$ and $R_{\exists\forall}(f) = \{z \mid \exists x_1, \forall x_2, z = f(x_1, x_2)\}$. In any case, $R_{\exists\forall}(f) \subseteq R_{\forall\exists}(f)$. We consider the affine function $f(x_1, x_2) = a + bx_1 + cx_2$. It is easy to see that

- $R_{\exists\forall}(f) = \emptyset$ if $c \neq 0$, since $z \in R_{\exists\forall}(f)$ implies $\exists x_1$ such that e.g. for $x_2 = 0$, $f(x_1, 0) = z$, and for e.g. $x_2 = 1$, $f(x_1, 1) = z$. Hence $a + bx_1 = z = a + c + bx_1$ implying $c = 0$. Conversely, if $c = 0$, $R_{\exists\forall}(f) = [a - |b|, a + |b|]$,
- $R_{\forall\exists}(f) = [a + |c| - |b|, a - |c| + |b|]$ if $|c| \leq |b|$, otherwise it is empty.

Hence in the linear case, either $c \neq 0$ and $R_{\exists\forall}(f) = \emptyset \neq R_{\forall\exists}(f)$ or $R_{\exists\forall}(f) = R_{\forall\exists}(f)$.

E Proof of Proposition 2

Note first that, trivially,

$$f(x_1, \dots, x_p) = f(0, \dots, 0) + \sum_{j=1}^p h^{x_1, \dots, x_{j-1}}(x_j)$$

Consider now $z \in \mathcal{C}(\mathbf{O}_1, \mathbf{I}_2, \dots, 0_{2n-1}, \mathbf{I}_{2n})$. Take any \mathbf{x}_1 in $[-1, 1]^{j_1}$. As \mathbf{O}_1 is an outer-approximation of $\text{range}(h) \times \dots \times \text{range}(h^{x_1, \dots, x_{k_2}})$, $h(\mathbf{x}_1) = \prod_{j=1}^{k_2} h^{x_1, \dots, x_{j-1}}(\mathbf{x}_1) \in \mathbf{O}_1$, and there exists $\alpha_2 \in \mathbf{I}_2$, such that $\forall \alpha_3 \in \mathbf{O}_3, \dots, \exists \alpha_{2n} \in \mathbf{I}_{2n}$ with $z = h(\mathbf{x}_1) + \alpha_2 + \dots + \alpha_{2n}$.

But each component of \mathbf{I}_2 is an inner-approximation of range of $h^{x_{k_2+1}, \dots, x_{j-1}}$ for some $j \in [k_2 + 1, \dots, k_3 - 1]$, so there exists $\mathbf{x}_2 \in [-1, 1]^{j_2}$ such that α_2 is the image of \mathbf{x}_2 by $h^{\mathbf{x}_1} = \prod_{j=k_2+1}^{k_3-1} h^{x_{k_2+1}, \dots, x_j}$. Therefore we have so far $\forall \mathbf{x}_1, \exists \mathbf{x}_2$ such that $\forall \alpha_3 \in \mathbf{O}_3, \dots, \exists \alpha_{2n} \in \mathbf{I}_{2n}$ with $z = h(\mathbf{x}_1) + h^{\mathbf{x}_1}(\mathbf{x}_2) + \alpha_3 + \dots + \alpha_{2n}$.

We carry on inductively to find that $z \in \mathcal{C}(\mathbf{O}_1, \mathbf{I}_2, \dots, 0_{2n-1}, \mathbf{I}_{2n})$ is such that $\forall \mathbf{x}_1, \exists \mathbf{x}_2, \dots, \exists \mathbf{x}_{2n}, z = h(\mathbf{x}_1) + \dots + h^{\mathbf{x}_1, \dots, \mathbf{x}_{2n-1}}(\mathbf{x}_{2n})$, i.e. $z = f(\mathbf{x}_1, \dots, \mathbf{x}_{2n}) - f(0, \dots, 0)$. Thus $z + f(0, \dots, 0) \in R_{\mathbf{p}}(f)$.

The case of outer-approximations of $R_{\mathbf{p}}(f)$ is similar.

F Proof of Theorem 1

Writing $A_l = [\underline{A}_l, \overline{A}_l]$ for $l = 1, \dots, p$, and $\mathbf{A}_i = (A_{k_i+1}, \dots, A_{k_{i+1}})$, for $i = 1, \dots, 2n$, we have

$$\mathcal{C}(\mathbf{A}_1, \dots, \mathbf{A}_{2n}) = S_p \left(\sum_{j=1}^p \frac{\overline{A}_j + \underline{A}_j}{2}; \forall, \frac{\overline{A}_1 - \underline{A}_1}{2}; \right. \\ \left. \exists, \frac{\overline{A}_2 - \underline{A}_2}{2}; \dots; \exists, \frac{\overline{A}_{2n} - \underline{A}_{2n}}{2} \right)$$

where the arithmetic operations on vectors \mathbf{A}_j are taken componentwise. Thus, by Equation (18), $\mathcal{C}(\mathbf{A}_1, \dots, \mathbf{A}_{2n}) = R\mathbf{p}(f)$ with $f(\mathbf{x}_1, \dots, \mathbf{x}_{2n}) = \sum_{j=1}^p \frac{\overline{A}_j + \underline{A}_j}{2} + \left\langle \frac{\overline{A}_1 - \underline{A}_1}{2}, \mathbf{x}_1 \right\rangle + \dots + \left\langle \frac{\overline{A}_{2n} - \underline{A}_{2n}}{2}, \mathbf{x}_{2n} \right\rangle$.

Therefore, by Proposition 1, $\mathcal{C}(\mathbf{O}_1, \mathbf{I}_2, \dots, \mathbf{O}_{2n-1}, \mathbf{I}_{2n})$ is equal to:

$$\sum_{l=1}^n \left(\sum_{j=k_{2l-1}+1}^{k_{2l}} \frac{\overline{O}_j + \underline{O}_j}{2} + \sum_{j=k_{2l}+1}^{k_{2l+1}} \frac{\overline{I}_j + \underline{I}_j}{2} \right) + \\ \frac{1}{2} \left[\sum_{k=1}^n (\|\overline{\mathbf{O}}_{2k-1} - \underline{\mathbf{O}}_{2k-1}\| - \|\overline{\mathbf{I}}_{2k} - \underline{\mathbf{I}}_{2k}\|), \right. \\ \left. \sum_{k=1}^n (\|\overline{\mathbf{I}}_{2k} - \underline{\mathbf{I}}_{2k}\| - \|\overline{\mathbf{O}}_{2k-1} - \underline{\mathbf{O}}_{2k-1}\|) \right] \quad (23)$$

if $\|\overline{\mathbf{O}}_{2l-1} - \underline{\mathbf{O}}_{2l-1}\| \leq \|\overline{\mathbf{I}}_{2l} - \underline{\mathbf{I}}_{2l}\| + \sum_{k=l+1}^n (\|\overline{\mathbf{I}}_{2k} - \underline{\mathbf{I}}_{2k}\| - \|\overline{\mathbf{O}}_{2k-1} - \underline{\mathbf{O}}_{2k-1}\|)$ for $l = 1, \dots, n$, otherwise, is empty. This is equivalent to $\sum \overline{\mathbf{O}}_{2l-1} - \sum \underline{\mathbf{O}}_{2l-1} \leq \sum_{k=l}^n \sum (\overline{\mathbf{I}}_{2k} - \underline{\mathbf{I}}_{2k}) - \sum_{k=l+1}^n \sum (\overline{\mathbf{O}}_{2k-1} - \underline{\mathbf{O}}_{2k-1})$ for $l = 1, \dots, n$.

The left bound of the interval in Equation (23) above is equal to:

$$\sum_{l=1}^n \left(\sum_{j=k_{2l-1}+1}^{k_{2l}} \frac{\overline{O}_j + \underline{O}_j}{2} + \sum_{j=k_{2l}+1}^{k_{2l+1}} \frac{\overline{I}_j + \underline{I}_j}{2} \right) + \\ \frac{1}{2} \sum_{k=1}^n (\|\overline{\mathbf{O}}_{2k-1} - \underline{\mathbf{O}}_{2k-1}\| - \|\overline{\mathbf{I}}_{2k} - \underline{\mathbf{I}}_{2k}\|)$$

which is equal to:

$$\begin{aligned}
 & \sum_{l=1}^n \left(\sum_{j=k_{2l-1}+1}^{k_{2l}} \frac{\bar{O}_j + \underline{O}_j}{2} + \sum_{j=k_{2l}+1}^{k_{2l+1}} \frac{\bar{I}_j + \underline{I}_j}{2} \right) + \\
 & \sum_{l=1}^n \left(\sum_{j=k_{2l-1}+1}^{k_{2l}} \frac{1}{2} \|\bar{O}_j - \underline{O}_j\| - \sum_{j=k_{2l}+1}^{k_{2l+1}} \frac{1}{2} \|\bar{I}_j - \underline{I}_j\| \right) \\
 & = \sum_{l=1}^n \left(\sum_{j=k_{2l-1}+1}^{k_{2l}} \left(\frac{\bar{O}_j + \underline{O}_j}{2} + \frac{1}{2} \|\bar{O}_j - \underline{O}_j\| \right) \right. \\
 & \qquad \qquad \qquad \left. + \sum_{j=k_{2l}+1}^{k_{2l+1}} \left(\frac{\bar{I}_j + \underline{I}_j}{2} - \frac{1}{2} \|\bar{I}_j - \underline{I}_j\| \right) \right)
 \end{aligned}$$

but:

$$\begin{aligned}
 \frac{\bar{O}_j + \underline{O}_j}{2} + \frac{1}{2} \|\bar{O}_j - \underline{O}_j\| &= \bar{O}_j \\
 \frac{\bar{I}_j + \underline{I}_j}{2} - \frac{1}{2} \|\bar{I}_j - \underline{I}_j\| &= \underline{I}_j
 \end{aligned}$$

Therefore, the left bound of the interval of Equation (23) is:

$$\sum_{l=1}^n \left(\sum_{j=k_{2l-1}+1}^{k_{2l}} \bar{O}_j + \sum_{j=k_{2l}+1}^{k_{2l+1}} \underline{I}_j \right) = \sum_{l=1}^n \left(\sum \bar{O}_{2l-1} + \sum \underline{I}_{2l} \right)$$

The right bound of the interval of Equation (23) is treated similarly, and we find:

$$\sum_{l=1}^n \left(\sum_{j=k_{2l-1}+1}^{k_{2l}} \underline{O}_j + \sum_{j=k_{2l}+1}^{k_{2l+1}} \bar{I}_j \right) = \sum_{l=1}^n \left(\sum \underline{O}_{2l-1} + \sum \bar{I}_{2l} \right)$$

Similarly, $\mathcal{C}(\mathbf{O}_1, \mathbf{I}_2, \dots, \mathbf{O}_{2n-1}, \mathbf{I}_{2n})$ is found, by echanging the roles of \mathbf{O}_k with \mathbf{I}_k , to be equal to:

$$\left[\sum_{l=1}^n \left(\sum \bar{I}_{2l-1} + \sum \underline{O}_{2l} \right), \sum_{l=1}^n \left(\sum \underline{I}_{2l-1} + \sum \bar{O}_{2l} \right) \right]$$

This, combined with (11) of Proposition 2 yields the result.

G Difference between \forall, \exists and \exists, \forall in the non-linear case

Example 10 (Difference between \forall, \exists and \exists, \forall). Consider as in Example 9, for a function f from \mathbb{R}^2 to \mathbb{R} the sets $R_{\forall\exists}(f) = \{z \mid \forall x_2, \exists x_1, z = f(x_1, x_2)\}$ and $R_{\exists\forall}(f) = \{z \mid \exists x_1, \forall x_2, z = f(x_1, x_2)\}$. In any case, $R_{\exists\forall}(f) \subseteq R_{\forall\exists}(f)$.

For non-linear f , the only difference with the linear case is that there may be isolated values for x_1 such that $f(x_1, x_2)$ does not depend on x_2 , which gives a finite set of isolated points for $R_{\exists\forall}(f)$, while $R_{\forall\exists}(f)$ can be a strict superset of $R_{\exists\forall}(f)$.

Take $f(x_1, x_2) = (x_1^2 - 1)x_2 + x_1$ for $x_1 \in [-1, 1]$ and $x_2 \in [-1, 1]$. For $x_1 = 1$ and $x_1 = -1$, $f(x_1, x_2) = x_1$, hence 1 and -1 belong to $R_{\exists\forall}(f)$, and $R_{\exists\forall}(f) = \{-1, 1\}$. A study of f reveals that $R_{\forall\exists}(f) = [-1, 1]$, which is a strict superset of $R_{\exists\forall}(f) = \{-1, 1\}$.

Indeed, a careful study of f reveals that:

- for $x_2 \in [-\frac{1}{2}, \frac{1}{2}]$, $f(x_1, x_2)$ is monotonic, and has exact range $[f(-1, x_2) = -1, f(1, x_2) = 1]$; when $x_2 \in [\frac{1}{2}, \infty[$,
- $f(x_1, x_2)$ has exact range $[-\frac{1+4x_2^2}{4x_2}, 1]$, with $-\frac{1+4x_2^2}{4x_2}$ having itself exact range $[-\frac{5}{4}, -1]$ ($f(\cdot, x_2)$ is decreasing from -1 to $-\frac{1}{2x_2}$ with value $-\frac{1+4x_2^2}{4x_2}$ and then increasing to 1),
- $f(x_1, x_2)$ has exact range $[-1, -\frac{1+4x_2^2}{4x_2}]$ when $x_2 \in]-\infty, -\frac{1}{2}]$ ($f(\cdot, x_2)$ is increasing from -1 to $-\frac{1}{2x_2}$ with value $-\frac{1+4x_2^2}{4x_2}$ which can range from 1 to $\frac{5}{4}$, and decreasing up to 1).

Thus the intersection of the ranges of $f(\cdot, x_2)$ for all x_2 is exactly $[-1, 1]$, and by Lemma 1, we conclude that $R_{\forall\exists}(f) = [-1, 1]$.

H Example 11

Example 11. Consider function $g : \mathbb{R}^3 \rightarrow \mathbb{R}$ given by

$$g(x_1, x_2, x_3) = \frac{x_1^2}{4} + (x_2 + 1)(x_3 + 2) + (x_3 + 3)^2.$$

On $[-1, 1]^3$, $\nabla_1 = |\frac{\partial g}{\partial x_1}| = |\frac{x_1}{2}| \in [0, \frac{1}{2}]$, $\nabla_2 = |\frac{\partial g}{\partial x_2}| = |x_3 + 2| \in [1, 3]$, $\nabla_3 = |\frac{\partial g}{\partial x_3}| = |x_2 + 1 + 2(x_3 + 3)| \in [4, 10]$, and $c = g(0, 0, 0) = 11$. Therefore, we can compute the outer and inner approximations O_i and I_i , $i = 1, 2, 3$, of Remark 6: $O_1 = [-\frac{1}{2}, \frac{1}{2}]$, $I_1 = 0$, $O_2 = [-3, 3]$, $I_2 = [-1, 1]$ and $O_3 = [-10, 10]$, $I_3 = [-4, 4]$. We can deduce an outer-approximation of the disturbance set:

$$\{z \mid \exists x_1 \in [-1, 1], \forall x_2 \in [-1, 1], \exists x_3 \in [-1, 1], z = g(x_1, x_2, x_3)\}$$

Let us first note that in order to apply Theorem 1 with $n = 2$, we must introduce a dummy universally quantified first variable, which means that all indices above should be added 1. Then, by a "direct" (adding 1 to variables indices) application of Theorem 1:

$$\begin{bmatrix} c & +\underline{O}_1 & +\overline{I}_2 & +\underline{O}_3, & c & +\overline{O}_1 & +\underline{I}_2 & +\overline{O}_3 \\ = [& 11 & -\frac{1}{2} & +1 & -10, & 11 & +\frac{1}{2} & -1 & +10 &] = [1.5, 20.5] \end{bmatrix}$$

In comparison, the sampling based estimation is [6.25, 16.25].

Now, as $\bar{I}_3 + \underline{Q}_2 = 1 \geq \underline{I}_3 + \bar{O}_2 = -1$, we get by Theorem 1 an inner approximation of the disturbance set for g :

$$\begin{aligned} & [\quad c \quad +\underline{I}_1 \quad +\bar{O}_2 \quad +\underline{I}_3, \quad c \quad +\bar{I}_1 \quad +\underline{Q}_2 \quad +\bar{I}_3 \quad] \\ & = [\quad 11 \quad 0 \quad +3 \quad -4, \quad 11 \quad +0 \quad -3 \quad +4 \quad] = [10, 12] \end{aligned}$$

I Calculation of the Jacobian for Example 5

The Jacobian of φ with respect to x_0, y_0, θ_0, b_1 and a , $J_{i,x_0} = \frac{\partial \varphi_i}{\partial x_0}$, $J_{i,y_0} = \frac{\partial \varphi_i}{\partial y_0}$, $J_{i,\theta_0} = \frac{\partial \varphi_i}{\partial \theta_0}$, $J_{i,b_1} = \frac{\partial \varphi_i}{\partial b_1}$ and $J_{i,a} = \frac{\partial \varphi_i}{\partial a}$, for $i = x, y, \theta$ respectively, satisfies the following variational equation [20]:

$$\begin{aligned} \dot{J}_{x,x_0} &= -\sin(\theta)J_{\theta,x_0} & \dot{J}_{y,b_1} &= \cos(\theta)J_{\theta,b_1} \\ \dot{J}_{x,y_0} &= -\sin(\theta)J_{\theta,y_0} & \dot{J}_{y,a} &= \cos(\theta)J_{\theta,a} \\ \dot{J}_{x,\theta_0} &= -\sin(\theta)J_{\theta,\theta_0} & \dot{J}_{\theta,x_0} &= 0 \\ \dot{J}_{x,b_1} &= -\sin(\theta)J_{\theta,b_1} + 1 & \dot{J}_{\theta,y_0} &= 0 \\ \dot{J}_{x,a} &= -\sin(\theta)J_{\theta,a} & \dot{J}_{\theta,\theta_0} &= 0 \\ \dot{J}_{y,x_0} &= \cos(\theta)J_{\theta,x_0} & \dot{J}_{\theta,b_1} &= 0 \\ \dot{J}_{y,y_0} &= \cos(\theta)J_{\theta,y_0} & \dot{J}_{\theta,a} &= 1 \\ \dot{J}_{y,\theta_0} &= \cos(\theta)J_{\theta,\theta_0} \end{aligned}$$

with initial conditions $J_{i,x_0} = \delta_{i,x}$, $J_{i,y_0} = \delta_{i,y}$, $J_{i,\theta_0} = \delta_{i,\theta}$ where δ is the Kronecker symbol. Therefore, the only non-null entries of the Jacobian of φ are: $J_{x,x_0} = 1$, $J_{x,b_1} = t$, $J_{y,y_0} = 1$, $J_{\theta,\theta_0} = 1$, $J_{\theta,a} = t$, and also J_{x,θ_0} , $J_{x,a}$, J_{y,θ_0} and $J_{y,a}$ are given by the ODEs:

$$\begin{aligned} \dot{J}_{x,\theta_0} &= -\sin(\theta) \quad \text{with } \dot{J}_{x,\theta_0} = 0 \text{ at time } 0 \\ \dot{J}_{x,a} &= -\sin(\theta)t \quad \text{with } \dot{J}_{x,a} = 0 \text{ at time } 0 \\ \dot{J}_{y,\theta_0} &= \cos(\theta) \quad \text{with } \dot{J}_{y,\theta_0} = 0 \text{ at time } 0 \\ \dot{J}_{y,a} &= \cos(\theta)t \quad \text{with } \dot{J}_{y,a} = 0 \text{ at time } 0 \end{aligned}$$

In order to find an over-approximation of these entries of the Jacobian, we use here a simple mean-value theorem, given that $\theta(t) \in [-0.015, 0.015]$ for $t \in [0, 0.15]$:

$$\begin{aligned} J_{x,\theta_0} &= -\sin([-0.015, 0.015])t \\ &\in [-1.309 \cdot 10^{-4}, 1.309 \cdot 10^{-4}] \\ J_{x,a} &= -[0, 0.5]\sin([-0.015, 0.015])t \\ &\in [-6.545 \cdot 10^{-5}, 6.545 \cdot 10^{-5}] \\ J_{y,\theta_0} &= \cos([-0.015, 0.015])t \\ &\in [0, 0.5] \\ J_{y,a} &= [0, 0.5]\cos([-0.015, 0.015])t \\ &\in [0, 0.25] \end{aligned}$$

By Remark 6, this gives the following inner and outer approximations for all parameters x_0, y_0, θ_0, a and b_1 , and all components x, y and θ of φ :

$$\begin{aligned}
& - I_{x,a} = 0, O_{x,a} = [-6.545 \cdot 10^{-7}, 6.545 \cdot 10^{-7}], I_{x,x_0} = O_{x,x_0} = [-0.1, 0.1], \\
& I_{x,\theta_0} = 0, O_{x,\theta_0} = [-1.309 \cdot 10^{-6}, 1.309 \cdot 10^{-6}], I_{x,b_1} = 0, O_{x,b_1} = [-0.005, 0.005], \\
& - I_{y,a} = 0, O_{y,a} = [-0, 0.0025], I_{y,y_0} = O_{y,y_0} = [-0.1, 0.1], I_{y,\theta_0} = 0, \\
& O_{y,\theta_0} = [-0, 0.005], \\
& - I_{\theta,\theta_0} = O_{\theta,\theta_0} = [-0.01, 0.01], I_{\theta,a} = 0, O_{\theta,a} = [0, 0.005],
\end{aligned}$$

We note also that $\frac{\partial x}{\partial t} = \cos(\theta) + b_1 \in [0.989999965, 1.01]$ thus $I_{x,t} = [0, 0.494999982]$, $O_{x,t} = [0, 0.505]$, similarly, $I_{y,t} = 0$, $O_{y,t} = [-\sin(0.015)/2, \sin(0.015)/2] = [-1.309 \cdot 10^{-4}, 1.309 \cdot 10^{-4}]$ and $I_{\theta,t} = 0$, $O_{\theta,t} = [-0.005, 0.005]$.

J Computation of components y and θ for Example 5

Inner-approximation for the y component of φ is computed similarly as for x , its lower bound is:

$$\begin{array}{ccccccc}
y_c & +I_{y,a} & +I_{y,x_0} & +I_{y,y_0} & +I_{y,\theta_0} & +\bar{O}_{y,b_1} & +I_{y,t} \\
= 0 & +0 & +0 & -0.1 & +0 & +0 & +0
\end{array}$$

which is equal to -0.1, and its upper bound:

$$\begin{array}{ccccccc}
y_c & +\bar{I}_{y,a} & +\bar{I}_{y,x_0} & +\bar{I}_{y,y_0} & +\bar{I}_{y,\theta_0} & +\underline{O}_{y,b_1} & +\bar{I}_{y,t} \\
= 0 & +0 & +0 & +0.1 & +0 & +0 & +0
\end{array}$$

which is equal to 0.1. Therefore the inner-approximation for y is equal to $[-0.1, 0.1]$.

Outer-approximation for the y component of φ has as lower bound:

$$\begin{array}{ccccccc}
y_c & +\underline{O}_{y,a} & +\underline{O}_{y,x_0} & +\underline{O}_{y,y_0} & +\underline{O}_{y,\theta_0} & +\bar{I}_{y,b_1} & +\underline{O}_{y,t} \\
= 0 & -0,0025 & -0 & -0.1 & -0,005 & +0 & -1.309 \cdot 10^{-4}
\end{array}$$

which is equal to 0.1076309, and its upper bound:

$$\begin{array}{ccccccc}
y_c & +\bar{O}_{y,a} & +\bar{O}_{y,x_0} & +\bar{O}_{y,y_0} & +\bar{O}_{y,\theta_0} & +\underline{I}_{y,b_1} & +\bar{O}_{y,t} \\
= 0 & +0.0025 & +0 & +0.1 & +0,005 & -0 & +1.309 \cdot 10^{-4}
\end{array}$$

which is equal to 0.1076309. Therefore the outer-approximation for y is equal to $[0.1076309, 0.1076309]$.

Inner-approximation for the θ component of φ is computed similarly, its lower bound is:

$$\begin{array}{ccccccc}
\theta_c & +\underline{I}_{\theta,a} & +\underline{I}_{\theta,x_0} & +\underline{I}_{\theta,y_0} & +\underline{I}_{\theta,\theta_0} & +\bar{O}_{\theta,b_1} & +\underline{I}_{\theta,t} \\
= 0 & +0 & +0 & +0 & -0.01 & +0 & +0
\end{array}$$

which is equal to -0.01, and its upper bound:

$$\begin{array}{ccccccc}
\theta_c & +\bar{I}_{\theta,a} & +\bar{I}_{\theta,x_0} & +\bar{I}_{\theta,y_0} & +\bar{I}_{\theta,\theta_0} & +\underline{O}_{\theta,b_1} & +\bar{I}_{\theta,t} \\
= 0 & +0 & +0 & +0 & +0.01 & +0 & +0
\end{array}$$

which is equal to 0.01. Therefore the inner-approximation for θ is equal to $[-0.01, 0.01]$.

Outer-approximation for the θ component of φ has as lower bound:

$$\begin{array}{rcccccccc} \theta_c & +\underline{Q}_{\theta,a} & +\underline{Q}_{\theta,x_0} & +\underline{Q}_{\theta,y_0} & +\underline{Q}_{\theta,\theta_0} & +\bar{I}_{\theta,b_1} & +\underline{Q}_{\theta,t} \\ = 0 & +0 & +0 & +0 & -0.01 & +0 & -0.005 \end{array}$$

which is equal to -0.02, and its upper bound:

$$\begin{array}{rcccccccc} \theta_c & +\bar{O}_{\theta,a} & +\bar{O}_{\theta,x_0} & +\bar{O}_{\theta,y_0} & +\bar{O}_{\theta,\theta_0} & +\underline{L}_{\theta,b_1} & +\bar{O}_{\theta,t} \\ = 0 & +0.005 & +0 & +0 & +0.01 & +0 & +0.005 \end{array}$$

which is equal to 0.02. Therefore the outer-approximation for θ is equal to $[-0.02, 0.02]$.

K Proof of Theorem 2

The principle is the same as in the case of one alternation of, for all and there exists quantifiers, treated in [21].

For each $i \in \{1, \dots, n\}$, function π^i associates to each x_j for $j \in \{k_{2i} + 1, \dots, k_{2i+1}\}$ the index $l \in \{1, \dots, m\}$ of the unique output component of the function in which it will be existentially quantified.

First suppose π^i are jointly surjective (i.e. the union of their image is $\{1, \dots, m\}$). For each of the m quantified problems for z_j o Theorem 2, for all $i = 1, \dots, n$, and for all $l_j^i \in J_{E,z_j}^i$ we can associate the continuous selection

$$g_{l_j^i} \left(z_j, (x_p)_{p \in \{1, \dots, k_{2i-1}\} \cup J_{A,z_j}^i} \right)$$

by [19], since f is elementary. For a given $(z_1, \dots, z_m) \in \mathbf{z}$, let us define the continuous map g that associates to each $(x_1, \dots, x_u) \in \mathbf{x}$,

$$\left((g_{l_j^1})_{j \in \{1, \dots, m\}}, \dots, (g_{l_j^n})_{j \in \{1, \dots, m\}} \right)$$

which can be completed adding identities on the components which are not defined, to be in value in $\mathbf{x} \subseteq \mathbb{R}^m$. By Brouwer fixed point theorem, $\forall z \in \mathbf{z}$, there is a fixed point $x^z \in \mathbf{x}$ of g , which thus satisfies $z = f(x^z)$ as the π^i 's are jointly surjective. We observe that the fixed point obtained does depend on the quantified variables, in the same order than in the original quantified problem $R\mathcal{P}(f)$.

Finally, if the π^i 's are not jointly surjective, there exist z_i in which no input variable is existentially quantified. The corresponding under-approximation will be empty or reduced to a point and the previous proof still holds on the other components.

L Example 7 detailed

There are several possible quantified formulas giving a 2D inner-approximation for Example 7.

One of them is, for all z_1 and z_2 :

$$\boxed{\exists x_1}, \forall x_2, \forall x_3, \boxed{\exists x_4}, z_1 = f_1(x_1, x_2, x_3, x_4) \quad (24)$$

$$\forall x_1, \forall x_2, \forall x_4, \boxed{\exists x_3}, z_2 = f_2(x_1, x_2, x_3, x_4) \quad (25)$$

where we indicated above the existential quantifiers for each group $\forall\exists$ within a framed box. By Theorem 1, we get an empty set for z_1 as defined by Equation (24), since the constraints of Proposition 1 are not satisfied: the contribution of the existentially quantified x_4 is $[-1, 1]$ whereas the universally quantified x_2 and x_3 account for $[-4, 4]$, which thus cannot be fully compensated.

Similarly Equation (25) yields an empty inner-approximation since it does not satisfy the constraints of Proposition 1, the contribution of x_3 being too small to counteract the contribution of x_1, x_2 and x_4 .

Another possibility is to interpret the following quantified formulas, for all z_1 and z_2 :

$$\boxed{\exists x_1}, \forall x_2, \forall x_4, \boxed{\exists x_3}, z_1 = f_1(x_1, x_2, x_3, x_4) \quad (26)$$

$$\forall x_1, \forall x_2, \forall x_3, \boxed{\exists x_4}, z_2 = f_2(x_1, x_2, x_3, x_4) \quad (27)$$

This time, the conditions of Proposition 1 for obtaining a non-empty inner-approximation are met and we get for Equation (26):

$$\begin{aligned} & [z_1^c - \|\Delta_{x_1}\| + \|\Delta_{x_2, x_4}\| - \|\Delta_{x_3}\|, z_1^c + \|\Delta_{x_1}\| - \|\Delta_{x_2, x_4}\| + \|\Delta_{x_3}\|] \\ & = [2 \quad -2 \quad +1 +1 \quad -3, \quad 2 \quad +2 \quad -1 -1 \quad +3] \end{aligned}$$

which is equal to $[-1, 5]$, and for Equation (27):

$$\begin{aligned} & [z_2^c + \|\Delta_{x_1, x_2, x_4}\| - \|\Delta_{x_3}\|, z_2^c - \|\Delta_{x_1, x_2, x_4}\| + \|\Delta_{x_3}\|] \\ & = [-1 \quad +1 +1 +1 \quad -5, \quad -1 \quad -1 -1 -1 \quad +5] \end{aligned}$$

which is equal to $[-3, 1]$. Hence $[-1, 5] \times [-3, 1]$ is in the set $R_{\exists\forall\exists}(f)$.

Finally, there are two other possibilities for finding joint inner-approximations of $R_{\exists\forall\exists}(f)$ of Equation (14):

$$\forall x_1, \forall x_2, \forall x_3, \boxed{\exists x_4}, z_1 = f_1(x_1, x_2, x_3, x_4) \quad (28)$$

$$\boxed{\exists x_1}, \forall x_2, \forall x_4, \boxed{\exists x_3}, z_2 = f_2(x_1, x_2, x_3, x_4) \quad (29)$$

For the same reason as for Equations (24) and (25), Equations (28) and (29) yield empty inner-approximations since the contribution to it of the last existentially quantified variables is the same in both cases (e.g. $[-1, 1]$ for x_4 in Equations (24) and (28)), and is not big enough to compensate for the next universally quantified variables (e.g. $[3, -3]$ for variable x_3 in Equations (24) and (28)). Similarly for the last choice:

$$\forall x_1, \forall x_2, \forall x_4, \boxed{\exists x_3}, z_1 = f_1(x_1, x_2, x_3, x_4) \quad (30)$$

$$\boxed{\exists x_1}, \forall x_2, \forall x_3, \boxed{\exists x_4}, z_2 = f_2(x_1, x_2, x_3, x_4) \quad (31)$$

for which the Equation (30) gives an empty set, and Equation (31) gives $[-1 - 1 + 1 + 1 - 5, -1 + 1 - 1 - 1 + 5]$ which is equal to $[-3, 3]$. Overall, this choice of alternation of quantifiers allowed by Theorem 2 gives an empty joint inner-approximation for the quantified problem $R_{\exists\forall\exists}(f)$ of Equation (14).

M Example Motion- n

We consider the Dubbins vehicle of Example 4, and compute the x component at time nT , x_n , with $T = 0.5s$, given a piecewise constant control a_i on each of the control periods $[(i-1)T, iT]$, which is given by the following function below, after integration of the dynamics:

$$x_n = x_0 + \sum_{k=1}^n \frac{1}{a_k} \left(\sin \left(\theta_0 + T \sum_{l=1}^n a_l \right) - \sin \left(\theta_0 + T \sum_{l=1}^{n-1} a_l \right) \right) + T \sum_{k=1}^n b_k$$

and we consider the following motion planning problem (up to a small relaxation δ):

$$\exists x_0, \exists \theta_0, \exists a_1, \forall b_1, \dots, \exists a_n, \forall b_n, \exists \delta, x = x_n(x_0, \theta_0, a_1, b_1, \dots, a_n, b_n) + \delta$$