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Motivated by the computation of some Feynman amplitudes, Broadhurst and Roberts recently conjectured and checked numerically to high precision a set of remarkable quadratic relations between the Bessel moments

$$\int_0^\infty I_0(t)^i K_0(t)^{k-i} t^{2j-1} dt \quad (i, j = 1, \dots, \lfloor (k-1)/2 \rfloor),$$

where $k \geq 1$ is a fixed integer and I_0 and K_0 denote the modified Bessel functions. We interpret these integrals and variants thereof as coefficients of the period pairing between middle de Rham cohomology and twisted homology of symmetric powers of the Kloosterman connection. Building on the general framework developed by Fresan, Sabbah and Yu (2020), this enables us to prove quadratic relations of the form suggested by Broadhurst and Roberts, which conjecturally comprise all algebraic relations between these numbers. We also make Deligne's conjecture explicit, thus explaining many evaluations of critical values of L -functions of symmetric power moments of Kloosterman sums in terms of determinants of Bessel moments.

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1. Introduction

Let $I_0(t)$ and $K_0(t)$ denote the modified Bessel functions of order zero, which are solutions to the ordinary differential equation $((t\partial_t)^2 - t^2)u = 0$. Since this equation has an irregular singularity at infinity, it does

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not come from geometry in the usual sense of encoding how periods vary in a family of algebraic varieties. However, certain integrals of monomials in $I_0(t)$ and $K_0(t)$ called *Bessel moments* are themselves periods, as shown for example by the identity (see [Vanhove 2014, (8.11)])

$$\int_0^\infty I_0(t)K_0(t)^{\ell+1}t \, dt = \frac{1}{2^\ell} \int_{x_i \geq 0} \frac{1}{(1 + \sum_{i=1}^\ell x_i)(1 + \sum_{i=1}^\ell 1/x_i) - 1} \prod_{i=1}^\ell \frac{dx_i}{x_i}.$$

In a series of papers and conference talks Broadhurst and Roberts [Broadhurst 2016; 2017a; 2017b; 2018; Broadhurst and Roberts 2019; Roberts 2017] put forward a program to understand the motivic origin of the Bessel moments

$$\int_0^\infty I_0(t)^a K_0(t)^b t^c \, dt. \tag{1.1}$$

An important insight of theirs was to look at counterparts of these integrals over finite fields, pursuing the analogy between the Bessel differential equation and the Kloosterman ℓ -adic sheaf. Roughly speaking, $I_0(t)$ and $K_0(t)$ correspond to the eigenvalues of Frobenius, and out of them one forms the k -th symmetric power moments of Kloosterman sums. The generating series of these moments over finite extensions of \mathbb{F}_p is a polynomial with integer coefficients. After removing some trivial factors and handling primes of bad reduction, a global L -function $L_k(s)$ is built with the reciprocals of these polynomials as local factors. Back at the beginning, Broadhurst, partly in joint work with Mellit [Broadhurst and Mellit 2016] and Roberts, Bloch, Kerr and Vanhove [Bloch et al. 2015], and Y. Zhou [2018a] proved or numerically checked in many cases that the critical values of these L -functions agree up to rational factors and powers of π with certain determinants of the Bessel moments (1.1).

For technical reasons, we shall make the change of variables $z = t^2/4$ and consider the associated rank-two vector bundle with connection on \mathbb{G}_m , which is called the *Kloosterman connection* and denoted by Kl_2 . Motives associated with symmetric powers of the Kloosterman connection were introduced in [Fresán et al. 2022]. Namely, for each integer $k \geq 1$, we constructed a motive M_k over the rational numbers, which is pure of weight $k + 1$, has rank $k' = \lfloor (k - 1)/2 \rfloor$ (resp. $k' - 1$) if k is not a multiple of 4 (resp. if k is a multiple of 4), and is endowed with a self-duality pairing

$$M_k \otimes M_k \rightarrow \mathbb{Q}(-k - 1) \tag{1.2}$$

that is symplectic if k is even and orthogonal if k is odd. By design, the L -function of this motive coincides with the above L -function $L_k(s)$. The main result of that paper was the computation of the Hodge numbers of M_k , which led to a proof that $L_k(s)$ extends meromorphically to the complex plane and satisfies the expected functional equation.

In this paper, we investigate the period realizations of the motives M_k . By design, the de Rham realization of M_k is isomorphic to the middle de Rham cohomology of the k -th symmetric power $\text{Sym}^k \text{Kl}_2$, which is defined as the image

$$H_{\text{dR,mid}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2) = \text{im}[H_{\text{dR,c}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2) \rightarrow H_{\text{dR}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)]$$

of compactly supported de Rham cohomology under the natural map to usual de Rham cohomology, and comes with a perfect intersection pairing S_k^{mid} realizing (1.2). Extending a computation from [Fresán et al. 2022, Proposition 4.14], we exhibit a basis of middle de Rham cohomology in Section 3, which is natural in that it is adapted to the Hodge filtration, and we present an explicit formula to compute the matrix of S_k^{mid} on this basis. If k is not a multiple of 4, the basis is simply given by the classes $\omega_i = [z^i v_0^k dz/z]$ for $1 \leq i \leq k'$, where v_0 is a specific section of Kl_2 .

Besides, we shall prove that the dual of the Betti realization of M_k is isomorphic to the middle twisted homology of $\text{Sym}^k \text{Kl}_2$, which is defined as the image

$$H_1^{\text{mid}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2) = \text{im}[H_1^{\text{rd}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2) \rightarrow H_1^{\text{mod}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)]$$

of rapid decay homology under the natural map to moderate growth homology. Elements of these homology groups are represented by linear combinations of twisted chains $c \otimes e$, where c is a path and e is a horizontal section of $\text{Sym}^k \text{Kl}_2$ that decays rapidly (resp. has moderate growth) on a neighborhood of the support of c . These conditions ensure that de Rham (resp. compactly supported de Rham) cohomology classes can be integrated along them, thus giving rise to a period pairing

$$P_k^{\text{mid}} : H_1^{\text{mid}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2) \otimes H_{\text{dR}, \text{mid}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2) \rightarrow \mathbb{C}.$$

This middle homology comes with a natural \mathbb{Q} -structure and, likewise to middle de Rham cohomology, a perfect intersection pairing B_k^{mid} realizing the transpose of (1.2). By analyzing the asymptotic behaviors of products of modified Bessel functions, we exhibit in Section 4 rapid decay homology classes α_i for $0 \leq i \leq k'$ whose images in middle homology are nonzero for $i \geq 1$.

Relying on the general results from the companion paper [Fresán et al. 2023], in particular the compatibility of the Betti and de Rham intersection pairings with the period pairing, we prove the following theorem. For simplicity, we only state it here when k is not a multiple of 4, postponing the full statements to Proposition 4.6, Theorems 3.17, 4.7, and 5.3, and Corollary 5.7.

Theorem 1.3. *Assume k is not a multiple of 4:*

- (1) *With respect to the basis $\{\omega_i\}_{1 \leq i \leq k'}$, the matrix of the de Rham intersection pairing S_k^{mid} is a lower-right triangular matrix with coefficients in \mathbb{Q} and (i, j) antidiagonal entries*

$$\begin{cases} (-2)^k \frac{k'!}{k!!} & \text{if } k \text{ is odd,} \\ \frac{(-1)^{k'+1}}{2^{k'}(j-i)} \cdot \frac{(k-1)!!}{(k'+1)!} & \text{if } k \text{ is even.} \end{cases}$$

- (2) *The middle homology classes $\{\alpha_i\}_{1 \leq i \leq k'}$ form a basis and the matrix of the Betti intersection pairing B_k^{mid} on this basis is given by*

$$B_k^{\text{mid}} = \left((-1)^{k-i} \frac{(k-i)!(k-j)!}{k!} \frac{\mathcal{B}_{k-i-j+1}}{(k-i-j+1)!} \right)_{1 \leq i, j \leq k'},$$

where \mathcal{B}_n denotes the n -th Bernoulli number.

(3) With respect to the bases $\{\alpha_i\}_{1 \leq i \leq k'}$ and $\{\omega_j\}_{1 \leq j \leq k'}$, the matrix of the period pairing P_k^{mid} consists of the Bessel moments

$$P_k^{\text{mid}} = \left((-1)^{k-i} 2^{k+1-2j} (\pi i)^i \int_0^\infty I_0(t)^i K_0(t)^{k-i} t^{2j-1} dt \right)_{1 \leq i, j \leq k'}$$

(4) The following quadratic relations hold:

$$P_k^{\text{mid}} \cdot (S_k^{\text{mid}})^{-1} \cdot {}^t P_k^{\text{mid}} = (-2\pi i)^{k+1} B_k^{\text{mid}}.$$

Quadratic relations of the shape $P_k^{\text{BR}} \cdot D_k^{\text{BR}} \cdot {}^t P_{k'}^{\text{BR}} = B_k^{\text{BR}}$ were conjectured by Broadhurst and Roberts [2018]. As we explain in Section 5.3, their matrices P_k^{BR} and B_k^{BR} coincide with ours up to different normalizations, but we were unfortunately unable to prove that, again up to normalization, the inverse of S_k^{mid} satisfies the recursive formulas defining their matrix D_k^{BR} . Nevertheless, we checked numerically that both matrices agree for $k \leq 22$, which is the limit for reasonable computation time with Maple.

Grothendieck’s period conjecture predicts that the transcendence degree of the field of periods of M_k agrees with the dimension of its motivic Galois group. Since the Betti intersection pairing is motivic, this is a subgroup of the general orthogonal group $GO_{k'}$ if k is odd and of the general symplectic group $GSp_{k'}$ (resp. $GSp_{k'-1}$) if k is even and not a multiple of 4 (resp. if k is a multiple of 4). Broadhurst and Roberts conjecture that this inclusion is an equality, which would mean that for fixed k the quadratic relations from Theorem 1.3 conjecturally exhaust all algebraic relations between the Bessel moments.

Finally, in Section 8 we make Deligne’s conjecture explicit for the critical values of $L_k(s)$ by identifying the periods that are expected to agree with them up to a rational factor with some determinants of Bessel moments already considered by Broadhurst and Roberts. Prior to that, we identify in Section 7 the period structure of the motive M_k with the period structure attached to the middle cohomology of $\text{Sym}^k \text{Kl}_2$ by means of Theorem 1.3. For that purpose, the appendix develops the necessary tools in a general setting of exponential mixed Hodge structures, complementing thereby the appendix of [Fresán et al. 2022].

Notation 1.4. We refer the reader to [Fresán et al. 2023] for the general setting of de Rham cohomology and twisted homology of vector bundles with connection, as well as the intersection forms and period pairings on these spaces. Throughout this article, we use the following notation and conventions:

- Given an integer $k \geq 1$, we set

$$k' = \lfloor (k - 1/2) \rfloor \quad (\text{i.e., } k = 2k' + 1 \text{ for odd } k \text{ and } k = 2(k' + 1) \text{ for even } k).$$

- Since the case where k is a multiple of 4 plays a special role throughout, we use the simplified common notation

$$\llbracket 1, k' \rrbracket = \begin{cases} \{1, \dots, k'\} & \text{if } 4 \nmid k, \\ \{1, \dots, k'\} \setminus \{k/4\} & \text{if } 4 \mid k, \end{cases}$$

so that

$$\#\llbracket 1, k' \rrbracket = \begin{cases} k' & \text{if } 4 \nmid k, \\ k' - 1 & \text{if } 4 \mid k. \end{cases}$$

We will consider square matrices indexed by $i, j \in \llbracket 1, k' \rrbracket$ that, when k is a multiple of 4, are obtained from a $k' \times k'$ matrix by deleting the row and column of index $k/4$.

- For integers $m \leq 0$, the factorial $m!$ and double factorial $m!!$ are given the value 1.
- For each integer $n \geq 0$, we denote by \mathcal{B}_n the n -th Bernoulli number, i.e., the n -th coefficient of the power series expansion

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n \frac{x^n}{n!}.$$

- The base torus is denoted by $\mathbb{G}_{m,z}$, and is regarded as included in the affine line with coordinate z . The coordinate $1/z$ is denoted by w . We also consider the degree two morphism $\rho_2 : \mathbb{G}_{m,t} \rightarrow \mathbb{G}_{m,z}$ which, at the ring level, is defined by $z \mapsto t^2/4$.

2. Pairings on the Kl_2 connection and its moments

In this section, we explain the algebraic duality pairing on $\text{Sym}^k \text{Kl}_2$ that gives rise to the de Rham intersection pairing. On the other hand, we endow the associated local system of flat sections $\text{Sym}^k \text{Kl}_2^\nabla$ with a \mathbb{Q} -structure and a topological duality pairing that will give rise to the Betti intersection pairing.

2.1. The Kl_2 connection. We first recall the definition of the Kl_2 connection, referring the reader to [Fresán et al. 2022, Section 4.1] for more details. We denote by $\mathbb{G}_{m,x}$ (resp. $\mathbb{G}_{m,z}$) the torus \mathbb{G}_m over the complex numbers with coordinate x (resp. z), and we define $f : \mathbb{G}_{m,x} \times \mathbb{G}_{m,z} \rightarrow \mathbb{A}^1$ as $f(x, z) = x + z/x$.

Let $\pi : \mathbb{G}_{m,x} \times \mathbb{G}_{m,z} \rightarrow \mathbb{G}_{m,z}$ denote the projection to the second factor and E^f the rank-one vector bundle with connection $(\mathcal{O}_{\mathbb{G}_{m,x} \times \mathbb{G}_{m,z}}, d + df)$ on $\mathbb{G}_{m,x} \times \mathbb{G}_{m,z}$. We define Kl_2 as the pushforward (in the sense of \mathcal{D} -modules) $\mathcal{H}^1 \pi_+ E^f$: this is a free $\mathcal{O}_{\mathbb{G}_m}$ -module of finite rank endowed with a connection having a regular singularity at the origin and an irregular one at infinity. Since the varieties we work with are all affine, it will be convenient to identify coherent sheaves with their global sections. To the sheaf $\mathcal{H}^1 \pi_+ E^f$ is then associated the module $H^1 \pi_+ E^f$ of global sections. Fixing the generator dx/x of relative differentials and denoting by ∂_x the partial derivative with respect to the variable x , we then have

$$\text{Kl}_2 = H^1 \pi_+ E^f = \text{coker}[\mathbb{C}[x, x^{-1}, z, z^{-1}] \xrightarrow{x\partial_x + (x-z/x)} \mathbb{C}[x, x^{-1}, z, z^{-1}]].$$

It follows that Kl_2 is the free $\mathbb{C}[z, z^{-1}]$ -module generated by the class v_0 of dx/x and the class v_1 of dx . The connection ∇ on Kl_2 satisfies

$$z\nabla_{\partial_z}(v_0, v_1) = (v_0, v_1) \cdot \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix},$$

so that v_0 is a solution to the differential equation $((z\partial_z)^2 - z)v = 0$.

Let $j : \mathbb{G}_{m,x} \hookrightarrow \mathbb{P}^1$ denote the inclusion. We write j_+^\dagger for the adjoint by duality of the pushforward j_+ , and similarly for π . The same argument as in [Malgrange 1991, Appendix 2, Proposition (1.7) page 217]

shows that the natural map $(j \times \text{Id})_{\dagger} E^f \rightarrow (j \times \text{Id})_{+} E^f$ is an isomorphism. Projecting to $\mathbb{G}_{m,z}$, we deduce that

$$H^1 \pi_{\dagger} E^f \rightarrow H^1 \pi_{+} E^f \tag{2.1}$$

is an isomorphism as well. Let us make this explicit. We set $x' = 1/x$. By an argument similar to that of [Fresán et al. 2023, Corollary 3.5], we can represent an element of $H^1 \pi_{\dagger} E^f$ as a pair $(\widehat{\psi}, \eta dx/x) = (\widehat{\psi}, -\eta dx'/x')$, where

- $\eta \in \mathbb{C}[x, x^{-1}, z, z^{-1}]$,
- $\widehat{\psi} = (\widehat{\psi}_0, \widehat{\psi}_{\infty})$, with

$$\widehat{\psi}_0 \in \mathbb{C}[z, z^{-1}][[x]][x^{-1}] \quad \text{and} \quad \widehat{\psi}_{\infty} \in \mathbb{C}[z, z^{-1}][[x']][x'^{-1}],$$

are such that the following holds:

$$(x \partial_x + (x - z/x)) \widehat{\psi}_0 = \iota_{\widehat{0}} \eta, \quad (x' \partial_{x'} + (zx' - 1/x')) \widehat{\psi}_{\infty} = -\iota_{\infty} \eta. \tag{2.2}$$

Here, $\iota_{\widehat{0}} : \mathbb{C}[x, x^{-1}, z, z^{-1}] \hookrightarrow \mathbb{C}[z, z^{-1}][[x]][x^{-1}]$ denotes the natural inclusion, and similarly for ι_{∞} . On these representatives, the natural morphism (2.1) is given by $(\widehat{\psi}, \eta dx/x) \mapsto \eta dx/x$. Checking that (2.1) is an isomorphism amounts to checking that, for any η as above, there exists a unique $\widehat{\psi}$ such that the (2.2) hold. Setting $\widehat{\psi}_0 = \sum_{n \geq n_0} \psi_{0,n}(z)x^n$ and $\widehat{\psi}_{\infty} = \sum_{n \geq n_{\infty}} \psi_{\infty,n}(z)x'^n$, $\iota_{\widehat{0}} \eta = \sum_n \eta_{0,n} x^n$, $\iota_{\infty} \eta = \sum_n \eta_{\infty,n} x'^n$, we determine $\psi_{0,n}(z)$ and $\psi_{\infty,n}(z)$ inductively by

$$\psi_{0,n+1} = z^{-1}(k\psi_{0,n} + \psi_{0,n-1} - \eta_{0,n}), \quad \psi_{\infty,n+1} = n\psi_{\infty,n} + z\psi_{\infty,n-1} + \eta_{\infty,n}, \tag{2.3}$$

thus showing explicitly that (2.1) is an isomorphism.

Example 2.4. The element of $H^1 \pi_{\dagger} E^f$ corresponding to v_0 (resp. v_1) is $(\widehat{\varphi}, dx/x)$ (resp. $(\widehat{\psi}, dx)$), where the elements $\widehat{\varphi}$ and $\widehat{\psi}$ determined by (2.3) satisfy (note that $\eta = 1$ resp. $\eta = x = 1/x'$)

$$\begin{cases} \varphi_{0, \leq 0} = 0, \\ \varphi_{0,1} = -z^{-1}, \\ \varphi_{\infty, \leq 0} = 0, \\ \varphi_{\infty,1} = 1, \end{cases} \quad \begin{cases} \psi_{0, \leq 1} = 0, \\ \psi_{\infty, < 0} = 0, \\ \psi_{\infty,0} = 1, \\ \psi_{\infty,1} = 0. \end{cases}$$

2.2. Algebraic duality on Kl_2 and its moments. Set

$$D = \{0, \infty\} = \mathbb{P}^1 \setminus \mathbb{G}_m.$$

Starting from the tautological pairing $E^f \otimes E^{-f} \rightarrow (\mathcal{O}_{\mathbb{G}_{m,x} \times \mathbb{G}_{m,z}}, d)$, we deduce a natural pairing

$$\langle \cdot, \cdot \rangle : H^1 \pi_{\dagger} E^f \otimes H^1 \pi_{+} E^{-f} \rightarrow H^2 \pi_{\dagger} \mathcal{O}_{\mathbb{G}_{m,x} \times \mathbb{G}_{m,z}} \xrightarrow{\text{res}_D} \mathbb{C}[z^{\pm 1}],$$

where the isomorphism res_D stands for the residue along D as in [Fresán et al. 2023, Section 3.c] (see also the proof of Lemma 2.5 below). Let $\iota : \mathbb{G}_{m,x} \times \mathbb{G}_{m,z} \rightarrow \mathbb{G}_{m,x} \times \mathbb{G}_{m,z}$ denote the involution $(x, z) \mapsto (-x, z)$.

Then $\iota^+ E^{-f} = E^f$, and this defines a canonical isomorphism $\mu : H^1 \pi_+ E^f \xrightarrow{\sim} H^1 \pi_+ E^{-f}$ since $\pi \circ \iota = \pi$. Let us set

$$(v_0^-, v_1^-) = \iota^*(v_0, v_1) = (dx/x, -dx),$$

that we consider as a basis of $H^1 \pi_+ E^{-f}$. Then the matrix of $z\nabla_{\partial_z}$ on $H^1 \pi_+ E^{-f}$ is equal to $\begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}$, and the above isomorphism reads $\mu(v_0, v_1) = (v_0^-, v_1^-)$.

Lemma 2.5. *The induced pairing*

$$\langle \cdot, \cdot \rangle_{\text{alg}} : H^1 \pi_+ E^f \otimes H^1 \pi_+ E^f \rightarrow \mathbb{C}[z^{\pm 1}]$$

defined by $\langle \cdot, \cdot \rangle_{\text{alg}} = \langle (2.1)^{-1} \cdot, \mu \cdot \rangle$ satisfies

$$\langle v_0, v_0 \rangle_{\text{alg}} = \langle v_1, v_1 \rangle_{\text{alg}} = 0, \quad \langle v_0, v_1 \rangle_{\text{alg}} = -\langle v_1, v_0 \rangle_{\text{alg}} = 1.$$

In other words, we get a skew-symmetric perfect pairing on Kl_2 :

$$\langle \cdot, \cdot \rangle_{\text{alg}} : (\text{Kl}_2, \nabla) \otimes (\text{Kl}_2, \nabla) \rightarrow (\mathcal{O}_{\mathbb{G}_m}, \mathfrak{d}), \tag{2.6}$$

which amounts to a canonical isomorphism $\lambda_{\text{alg}} : \text{Kl}_2 \rightarrow \text{Kl}_2^\vee$ with the dual connection endowed with the dual basis (v_0^\vee, v_1^\vee) , by setting

$$\begin{aligned} & \text{Kl}_2 \xrightarrow{\lambda_{\text{alg}}} \text{Kl}_2^\vee \\ (v_0, v_1) & \longmapsto (-v_1^\vee, v_0^\vee). \end{aligned}$$

Proof. We compute with the notation of [Example 2.4](#). We find, on the one hand,

$$\begin{aligned} \langle v_0, v_0 \rangle_{\text{alg}} &= \langle (\widehat{\varphi}, v_0), v_0^- \rangle = \text{res}_D \widehat{\varphi} \frac{dx}{x} = \varphi_{0,0} - \varphi_{\infty,0} = 0, \\ \langle v_1, v_1 \rangle_{\text{alg}} &= \langle (\widehat{\psi}, v_1), v_1^- \rangle = -\text{res}_D \widehat{\psi} dx = -\psi_{0,-1} + \psi_{\infty,1} = 0, \end{aligned}$$

and, on the other hand,

$$\begin{aligned} \langle v_0, v_1 \rangle_{\text{alg}} &= \langle (\widehat{\varphi}, v_0), v_1^- \rangle = -\text{res}_D \widehat{\varphi} dx = -\varphi_{0,-1} + \varphi_{\infty,1} = 1, \\ \langle v_1, v_0 \rangle_{\text{alg}} &= \langle (\widehat{\psi}, v_1), v_0^- \rangle = \text{res}_D \widehat{\psi} \frac{dx}{x} = \psi_{0,0} - \psi_{\infty,0} = -1. \end{aligned} \quad \square$$

For each $k \geq 1$, let \mathfrak{S}_k be the symmetric group acting on the tensor power $\text{Kl}_2^{\otimes k}$ by the natural permutation action. Let $\text{Sym}^k \text{Kl}_2$ be the symmetric power regarded as the \mathfrak{S}_k -invariant part of $\text{Kl}_2^{\otimes k}$. We consider the basis $\mathbf{u} = (u_a)_{0 \leq a \leq k}$ of $\text{Sym}^k \text{Kl}_2$ given by

$$u_a = v_0^{k-a} v_1^a = \frac{1}{|\mathfrak{S}_k|} \sum_{\sigma \in \mathfrak{S}_k} \sigma(v_0^{\otimes k-a} \otimes v_1^{\otimes a}),$$

in which the connection reads

$$z\partial_z u_a = (k-a)u_{a+1} + azu_{a-1} \quad (0 \leq a \leq k)$$

with the convention $u_{k+1} = 0$. The pairing (2.6) extends to $\text{Sym}^k \text{Kl}_2$, which is thus endowed with the following $(-1)^k$ -symmetric pairing (compatible with the connection):

$$\langle u_a, u_b \rangle_{\text{alg}} = \begin{cases} (-1)^a \frac{a!b!}{k!} & \text{if } a + b = k, \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

2.3. The \mathbb{Q} -structure of Kl_2^\vee and its moments.

The \mathbb{Q} -structure for a fixed nonzero z . We start by considering the \mathbb{Q} -structure on the fiber of the sheaf of analytic flat sections Kl_2^\vee at some $z \in \mathbb{G}_{m,z}$. We consider the function $f_z : \mathbb{G}_{m,x} \rightarrow \mathbb{A}^1$, defined as $f_z(x) = x + z/x$, where z is a fixed nonzero complex number, and $E^{f_z} = (\mathcal{O}_{\mathbb{G}_{m,x}}, d + df_z)$. Let \mathbb{P}_x^1 be the projective closure of $\mathbb{G}_{m,x}$ and let $\tilde{\mathbb{P}}^1$ be the real oriented blow-up along $D = \{0, \infty\}$, which is topologically a closed annulus. We denote by $\tilde{j} : \mathbb{G}_{m,x}^{\text{an}} \hookrightarrow \tilde{\mathbb{P}}^1$ the inclusion of the open annulus into the closed one. On $\tilde{\mathbb{P}}^1$, the de Rham complexes with rapid decay $\text{DR}^{\text{rd}}(E^{f_z})$ and with moderate growth $\text{DR}^{\text{mod}}(E^{f_z})$ have cohomology in degree zero only (see [Fresán et al. 2023, Theorem 2.30]), and the natural morphism $\text{DR}^{\text{rd}}(E^{f_z}) \rightarrow \text{DR}^{\text{mod}}(E^{f_z})$ is a quasiisomorphism. Indeed, the function e^{-f_z} on $\mathbb{G}_{m,x}^{\text{an}}$ has moderate growth near a point of the boundary $\partial\tilde{\mathbb{P}}^1$ if and only if it has rapid decay there. Above $x = 0$, this amounts to $\arg x \in \arg z + (-\pi/2, \pi/2) \bmod 2\pi$. Above $x = \infty$, this amounts to $\arg x \in (-\pi/2, \pi/2) \bmod 2\pi$. We denote by $\tilde{\mathbb{P}}_{\text{rd}}^1$ the open set which is the union of $\mathbb{G}_{m,x}^{\text{an}}$ and these two boundary open intervals, so that we have natural open inclusions

$$\mathbb{G}_{m,x}^{\text{an}} \xrightarrow{a_z} \tilde{\mathbb{P}}_{\text{rd}}^1 \xrightarrow{b_z} \tilde{\mathbb{P}}^1.$$

Then multiplication by e^{-f_z} yields an isomorphism of sheaves of \mathbb{C} -vector spaces

$$b_{z,!} a_{z,*} \mathbb{C}_{\mathbb{G}_{m,x}^{\text{an}}} \xrightarrow{\sim} \mathcal{H}^0 \text{DR}^{\text{rd}}(E^{f_z}) = \mathcal{H}^0 \text{DR}^{\text{mod}}(E^{f_z}). \tag{2.8}$$

Definition 2.9. The \mathbb{Q} -subsheaf $\mathcal{H}^0 \text{DR}^{\text{rd}}(E^{f_z})_{\mathbb{Q}} \subset \mathcal{H}^0 \text{DR}^{\text{rd}}(E^{f_z})$ is the image of $b_{z,!} a_{z,*} \mathbb{Q}_{\mathbb{G}_{m,x}^{\text{an}}}$ under the above isomorphism.

The Betti \mathbb{Q} -structure on $H_{\text{dR}}^1(\mathbb{G}_{m,x}, E^{f_z})$ (see [Fresán et al. 2023, Section 2.d]) is defined by means of (2.8) as

$$H_{\text{dR}}^1(\mathbb{G}_{m,x}, E^{f_z})_{\mathbb{Q}} = H^1(\tilde{\mathbb{P}}^1, b_{z,!} a_{z,*} \mathbb{Q}_{\mathbb{G}_{m,x}^{\text{an}}}) = H_c^1(\tilde{\mathbb{P}}_{\text{rd}}^1, a_{z,*} \mathbb{Q}_{\mathbb{G}_{m,x}^{\text{an}}}).$$

We denote by (recall that $z \neq 0$ is fixed):

- c_0^x the unit circle in $\mathbb{G}_{m,x}^{\text{an}}$ starting at 1 and oriented counterclockwise.
- c_z^x a smooth oriented path in $\mathbb{G}_{m,x}^{\text{an}}$ starting in a direction $\arg x$ contained in $\arg z + (-\pi/2, \pi/2) \bmod 2\pi$ at $x = 0$, intersecting c_0^x transversally only once, so that the local intersection number (Kronecker index) (c_z^x, c_0^x) is equal to one, and abutting to $x = \infty$ in a direction of $\arg x$ contained in $(-\pi/2, \pi/2) \bmod 2\pi$. The precise choice of c_z^x will be made later. We also consider the path c_z^{-x} obtained from c_z^x by applying the involution $\iota : x \mapsto -x$.

These paths define twisted cycles $\alpha_z^x = -c_z^x \otimes e^{-fz}$ and $\beta_z^x = c_0^x \otimes e^{-fz}$ in rapid decay homology $H_1^{\text{rd}}(\mathbb{G}_{m,x}, E^{fz})$ (see e.g., [Fresán et al. 2023, Section 2.d]). Similarly, we set

$$(\alpha_z^x)^\vee = -c_z^{-x} \otimes e^{fz} \quad \text{and} \quad (\beta_z^x)^\vee = -c_0^x \otimes e^{fz},$$

which define twisted cycles in $H_1^{\text{rd}}(\mathbb{G}_{m,x}, E^{-fz})$. As for the de Rham cohomology, the involution ι induces an isomorphism $H_1^{\text{rd}}(\mathbb{G}_{m,x}, E^{fz}) \rightarrow H_1^{\text{rd}}(\mathbb{G}_{m,x}, E^{-fz})$ sending a rapid decay chain $s \mapsto a(s) \otimes e^{-fz}$ to the rapid decay chain $s \mapsto -a(s) \otimes e^{fz}$, and thus inducing the corresponding Betti intersection pairing

$$H_1^{\text{rd}}(\mathbb{G}_{m,x}, E^{fz}) \otimes H_1^{\text{rd}}(\mathbb{G}_{m,x}, E^{fz}) \xrightarrow{\sim} H_1^{\text{rd}}(\mathbb{G}_{m,x}, E^{fz}) \otimes H_1^{\text{rd}}(\mathbb{G}_{m,x}, E^{-fz}) \rightarrow H_0(\mathbb{G}_{m,x}, \mathbb{C}) = \mathbb{C}.$$

This pairing is easily computed and has matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, thus showing that (α_z^x, β_z^x) is a \mathbb{Q} -basis of $H_1^{\text{rd}}(\mathbb{G}_{m,x}, E^{fz})_{\mathbb{Q}}$.

For applying [Fresán et al. 2023, Proposition 2.23], we use the topological duality pairing $\langle \cdot, \cdot \rangle_{\text{top}}$ on $H_1^{\text{rd}}(\mathbb{G}_{m,x}, E^{fz})$, which preserves the \mathbb{Q} -structure since it is induced by Poincaré–Verdier duality. The following relation holds; see [loc. cit., (3.10)]:

$$\langle \cdot, \cdot \rangle_{\text{top}} = 2\pi i \langle \cdot, \cdot \rangle_{\text{alg}}.$$

We let $(v_i^{\vee, \text{top}} = \frac{1}{2\pi i} v_i^\vee)$ denote the dual basis of (v_i) with respect to $\langle \cdot, \cdot \rangle_{\text{top}}$.

Proposition 2.10. *The \mathbb{Q} -vector space $H_{\text{dR}}^1(\mathbb{G}_{m,x}, E^{fz})_{\mathbb{Q}}$ is the \mathbb{Q} -span of*

$$\begin{aligned} e_0 &= \left(\frac{1}{2\pi i} \int_{c_0^x} e^{-fz} dx \right) \cdot v_0 - \left(\frac{1}{2\pi i} \int_{c_0^x} e^{-fz} \frac{dx}{x} \right) \cdot v_1 \quad \text{and} \\ e_1 &= - \left(\frac{1}{2\pi i} \int_{c_z^x} e^{-fz} dx \right) \cdot v_0 + \left(\frac{1}{2\pi i} \int_{c_z^x} e^{-fz} \frac{dx}{x} \right) \cdot v_1. \end{aligned}$$

Proof. From [loc. cit., Proposition 2.23], we deduce that $H_{\text{dR}}^1(\mathbb{G}_{m,x}, E^{fz})_{\mathbb{Q}}$ is the \mathbb{Q} -span of

$$P_1^{\text{rd, mod}}(\beta_z^x, v_0^{\vee, \text{top}})v_0 + P_1^{\text{rd, mod}}(\beta_z^x, v_1^{\vee, \text{top}})v_1 \quad \text{and} \quad P_1^{\text{rd, mod}}(\alpha_z^x, v_0^{\vee, \text{top}})v_0 + P_1^{\text{rd, mod}}(\alpha_z^x, v_1^{\vee, \text{top}})v_1, \quad (2.11)$$

where $P_1^{\text{rd, mod}} : H_1^{\text{rd}}(\mathbb{G}_{m,x}, E^{fz}) \otimes H_{\text{dR}}^1(\mathbb{G}_{m,x}, E^{fz}) \rightarrow \mathbb{C}$ denotes the period pairing from [loc. cit., Section 2.d]. We conclude with the identification

$$v_0^{\vee, \text{top}} = \frac{1}{2\pi i} v_1 = \frac{1}{2\pi i} [dx] \quad \text{and} \quad v_1^{\vee, \text{top}} = -\frac{1}{2\pi i} v_0 = -\frac{1}{2\pi i} [dx/x].$$

For example, the integral $\frac{1}{2\pi i} \int_{c_0^x} e^{-fz} dx$ is identified with the period

$$P_1^{\text{rd, mod}}(\beta_z^x, v_0^{\vee, \text{top}}).$$

□

The \mathbb{Q} -structure on Kl_2^∇ . We first recall some basic properties of the modified Bessel functions of order zero

$$\begin{aligned}
 I_0(t) &= \frac{1}{2\pi i} \oint \exp\left(-\frac{t}{2}\left(y + \frac{1}{y}\right)\right) \frac{dy}{y}, \\
 K_0(t) &= \frac{1}{2} \int_0^\infty \exp\left(-\frac{t}{2}\left(y + \frac{1}{y}\right)\right) \frac{dy}{y} \quad (|\arg t| < \pi/2),
 \end{aligned}
 \tag{2.12}$$

which are annihilated by the modified Bessel operator $(t\partial_t)^2 - t^2$. The function $I_0(t)$ is entire and satisfies $I_0(t) = I_0(-t)$. The function $K_0(t)$ extends analytically to a multivalued function on \mathbb{C}^\times satisfying the rule $K_0(e^{\pi i}t) = K_0(t) - \pi i I_0(t)$.

We have the following estimates as $t \rightarrow 0$ in any bounded ramified sector, by taking the real determination of $\log(t/2)$ when $t \in \mathbb{R}_{>0}$

$$I_0(t) = 1 + O(t^2), \quad K_0(t) = -(\gamma + \log(t/2)) + O(t^2 \log t),$$

where $\gamma = 0.5772\dots$ is the Euler constant. As a consequence, in such sectors,

$$I_0(t)^i K_0(t)^{k-i} = (-1)^{k-i} (\gamma + \log(t/2))^{k-i} + O(t^2 \log^{k-i} t). \tag{2.13}$$

On the other hand, we have the asymptotic expansions at infinity (see [Watson 1944, Section 7.23])

$$\begin{aligned}
 I_0(t) &\sim e^t \frac{1}{\sqrt{2\pi t}} \sum_{n=0}^\infty \frac{((2n-1)!!)^2}{2^{3n} n!} \frac{1}{t^n}, \quad |\arg t| < \pi/2 \\
 K_0(t) &\sim e^{-t} \sqrt{\frac{\pi}{2t}} \sum_{n=0}^\infty (-1)^n \frac{((2n-1)!!)^2}{2^{3n} n!} \frac{1}{t^n}, \quad |\arg t| < 3\pi/2 \\
 I_0(t)K_0(t) &\sim \frac{1}{2t} \sum_{n=0}^\infty \frac{((2n-1)!!)^3}{2^{3n} n!} \frac{1}{t^{2n}}.
 \end{aligned}
 \tag{2.14}$$

The latter is the unique formal solution in $1/t$ of the second symmetric power $(t\partial_t)^3 - 4t^2(t\partial_t) - 4t^2$ of the modified Bessel operator up to a scalar. Let us also note that these asymptotic expansions can be differentiated termwise. The Wronskian is given by $I_0(t)K_0'(t) - I_0'(t)K_0(t) = -1/t$.

We now assume that z varies in $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ and we choose a square root $t/2$ of z satisfying $\text{Re}(t) > 0$, that is, $\arg t \in (-\pi/2, \pi/2) \bmod 2\pi$. Due to formulas (2.11), since the integration paths can be made to vary in a locally constant way, we conclude that e_0, e_1 are sections of Kl_2^∇ on this domain, and their coefficients on the basis v_0, v_1 are holomorphic there. We will express them in terms of the modified Bessel functions. We set $x = (t/2)y$. On the one hand, we have

$$\frac{1}{2\pi i} \int_{c_0^x} e^{-f_z} \frac{dx}{x} = \frac{1}{2\pi i} \int_{(t/2)c_0^y} \exp\left(-\frac{t}{2}\left(y + \frac{1}{y}\right)\right) \frac{dy}{y} = \frac{1}{2\pi i} \int_{c_0^y} \exp\left(-\frac{t}{2}\left(y + \frac{1}{y}\right)\right) \frac{dy}{y} = I_0(t).$$

On the other hand, we now regard y as varying in $\mathbb{R}_{>0}$ and set $c_z^x = (t/2)\mathbb{R}_{>0}$, so that, for $x \in c_z^x$, both $\arg(z/x)$ and $\arg(x)$ belong to $(-\pi/2, \pi/2) \bmod 2\pi$. Then, similarly,

$$\frac{1}{2\pi i} \int_{c_z^x} e^{-f_z} \frac{dx}{x} = \frac{1}{2\pi i} \int_{\mathbb{R}_{>0}} \exp\left(-\frac{t}{2}\left(y + \frac{1}{y}\right)\right) \frac{dy}{y} = \frac{1}{\pi i} K_0(t).$$

By flatness of e_0, e_1 , we obtain therefore

$$e_0 = (t/2)I_0'(t)v_0 - I_0(t)v_1 \quad \text{and} \quad e_1 = \frac{1}{\pi i}(-t/2)K_0'(t)v_0 + K_0(t)v_1. \tag{2.15}$$

The pairing $\langle \cdot, \cdot \rangle_{\text{top}} = 2\pi i \langle \cdot, \cdot \rangle_{\text{alg}}$ being flat, it induces a nondegenerate pairing on the constant sheaf $\text{Kl}_2^\vee|_{\mathbb{C} \setminus \mathbb{R}_{\leq 0}}$ and we have there

$$\langle e_0, e_1 \rangle_{\text{top}} = 2\pi i \left((t/2)I_0'(t) \cdot \frac{1}{\pi i} K_0(t) - I_0(t) \cdot (t/2) \frac{1}{\pi i} K_0'(t) \right) = t(I_0'(t)K_0(t) - I_0(t)K_0'(t)) = 1,$$

according to the Wronskian relation. The other pairings are deduced from this one by skew-symmetry. We also obtain

$$v_0 = (2K_0(t)e_0 + 2\pi i I_0(t)e_1), \quad v_1 = t(K_0'(t)e_0 + \pi i I_0'(t)e_1). \tag{2.16}$$

In order to cross the cut $z \in \mathbb{R}_{<0}$, we note that the coefficients of e_0 , regarded as functions of $z \in \mathbb{C}$, are entire, while those of e_1 are multivalued holomorphic, and the monodromy operator T defined by analytic continuation along the path $\theta \mapsto e^\theta z$ ($\theta \in [0, 2\pi i]$) acts on e_1 as $T(e_1) = e_1 + e_0$. This shows that the \mathbb{Q} -structure of $\text{Kl}_2^\vee|_{\mathbb{C} \setminus \mathbb{R}_{\leq 0}}$ extends to a \mathbb{Q} -structure of Kl_2^\vee , which will be denoted by $(\text{Kl}_2^\vee)_{\mathbb{Q}}$. Moreover,

$$\langle \cdot, \cdot \rangle_{\text{top}} : (\text{Kl}_2^\vee)_{\mathbb{Q}} \otimes (\text{Kl}_2^\vee)_{\mathbb{Q}} \rightarrow \mathbb{Q}$$

is a nondegenerate skew-symmetric pairing, and the multivalued flat sections e_0 and e_1 satisfy $\langle e_0, e_1 \rangle_{\text{top}} = 1$.

The \mathbb{Q} -structure on $\text{Sym}^k \text{Kl}_2$. We naturally endow $\text{Sym}^k \text{Kl}_2$ with the pairing

$$\langle \cdot, \cdot \rangle_{\text{top}} = (2\pi i)^k \langle \cdot, \cdot \rangle_{\text{alg}}$$

and the \mathbb{Q} -structure $(\text{Sym}^k \text{Kl}_2)_{\mathbb{Q}}^\vee = \text{Sym}^k((\text{Kl}_2^\vee)_{\mathbb{Q}})$. The monomial sections

$$e_0^{k-a} e_1^a = \frac{1}{|\mathfrak{S}_k|} \sum_{\sigma \in \mathfrak{S}_k} \sigma(e_0^{\otimes k-a} \otimes e_1^{\otimes a}) \quad (0 \leq a \leq k) \tag{2.17}$$

form a basis of multivalued flat sections of the subsheaf $(\text{Sym}^k \text{Kl}_2)_{\mathbb{Q}}^\vee$ of $(\text{Kl}_2^{\otimes k})_{\mathbb{Q}}^\vee$ and satisfy

$$\langle e_0^{k-a} e_1^a, e_0^{k-b} e_1^b \rangle_{\text{top}} = \begin{cases} (-1)^a \frac{a!b!}{k!} & \text{if } a + b = k, \\ 0 & \text{otherwise.} \end{cases} \tag{2.18}$$

Lemma 2.19. *The coefficients of the flat sections $e_0^{k-a} e_1^a$ on the meromorphic basis $(u_b)_b$ of $\text{Sym}^k \text{Kl}_2$ have moderate growth at the origin. Moreover, they have moderate growth (resp. rapid decay) in a small sector centered at infinity and containing $\mathbb{R}_{>0}$ if and only if $a \leq k/2$ (resp. $a < k/2$).*

Proof. Since $I'_0(t)$ (resp. $K'_0(t)$) has an asymptotic expansion similar to that of $I_0(t)$ (resp. $K_0(t)$) at the origin and at infinity in the specified domains, the first statement follows from (2.15) and the definition (2.12) of I_0 and K_0 . Besides, the asymptotic expansion of I_0 and K_0 at infinity (2.14) implies the second statement by calculating the power of e^t in the products of I_0 and K_0 . □

3. de Rham pairing for $\text{Sym}^k \text{Kl}_2$

The main result of this section is the computation of the matrices of the de Rham pairings

$$\begin{aligned} & H_{\text{dR},c}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2) \otimes H_{\text{dR}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2) \xrightarrow{S} \mathbb{C} \\ & H_{\text{dR},\text{mid}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2) \otimes H_{\text{dR},\text{mid}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2) \xrightarrow{S_{\text{mid}}} \mathbb{C} \end{aligned} \tag{3.1}$$

with respect to suitable bases, taking into account the self-duality pairing induced by (2.7). Since the latter is $(-1)^k$ -symmetric, (3.1) is $(-1)^{k+1}$ -symmetric. We first make clear the bases in which we compute the matrices.

3.1. Bases of the de Rham cohomology. Let

$$\iota_{\widehat{0}} : \text{Sym}^k \text{Kl}_2 \rightarrow (\text{Sym}^k \text{Kl}_2)_{\widehat{0}} \quad \text{and} \quad \iota_{\infty} : \text{Sym}^k \text{Kl}_2 \rightarrow (\text{Sym}^k \text{Kl}_2)_{\infty}$$

denote the formalization of $\text{Sym}^k \text{Kl}_2$ at zero and infinity respectively, and $\widehat{\nabla}$ the induced connection. We can represent elements of $H_{\text{dR},c}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$ as pairs (\widehat{m}, η) as (see [Fresán et al. 2023, Corollary 3.5])

- $\widehat{m} = (\widehat{m}_0, \widehat{m}_{\infty})$ is a pair of formal germs in $(\text{Sym}^k \text{Kl}_2)_{\widehat{0}} \oplus (\text{Sym}^k \text{Kl}_2)_{\infty}$,
- η belongs to $\Gamma(\mathbb{G}_m, \Omega_{\mathbb{G}_m}^1 \otimes \text{Sym}^k \text{Kl}_2)$,

such that, denoting by $\widehat{\eta} = (\iota_{\widehat{0}}\eta, \iota_{\infty}\eta)$ the formal germ of η in

$$[\Omega_{\mathbb{P}^1, \widehat{0}}^1 \otimes (\text{Sym}^k \text{Kl}_2)_{\widehat{0}}] \oplus [\Omega_{\mathbb{P}^1, \infty}^1 \otimes (\text{Sym}^k \text{Kl}_2)_{\infty}],$$

\widehat{m} and η are related by $\widehat{\nabla}\widehat{m} = \widehat{\eta}$.

We can regard $H_{\text{dR},\text{mid}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$ as the image of the natural morphism

$$H_{\text{dR},c}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2) \rightarrow H_{\text{dR}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$$

sending a pair (\widehat{m}, η) to η . Recall that $H_{\text{dR},\text{mid}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$ has dimension k' if k is not a multiple of 4, and $k' - 1$ otherwise; see [Fresán et al. 2022, Proposition 4.12]. According to [Fresán et al. 2023, Remark 3.6], there exists a basis of $H_{\text{dR},c}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$ consisting of

- pairs $(\widehat{m}_i, 0)_i$ where $(\widehat{m}_i)_i$ is a basis of $\ker \widehat{\nabla}$ in $(\text{Sym}^k \text{Kl}_2)_{\widehat{0}} \oplus (\text{Sym}^k \text{Kl}_2)_{\infty}$, and
- a set of pairs $(\widehat{m}_j, \eta_j)_j$, of cardinality $\dim H_{\text{dR},\text{mid}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$, related as above such that $(\eta_j)_j$ are linearly independent in $H_{\text{dR}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$.

Furthermore, such a family $(\eta_j)_j$ is a basis of $H_{\text{dR,mid}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$.

We set (recall that $u_0 = v_0^k$)

$$\omega_i = [z^i u_0 dz/z] \in H_{\text{dR}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2), \quad 0 \leq i \leq k'. \tag{3.2}$$

It was proved in [Fresán et al. 2022, Proposition 4.14] that

$$\mathcal{B}_k = \{\omega_i \mid 0 \leq i \leq k'\}$$

is a basis of $H_{\text{dR}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$. This property also follows from the combination of Lemma 3.3, Proposition 3.8, and Theorem 3.17 below (see Remark 3.22).

We first determine which linear combinations of elements of \mathcal{B}_k belong to $H_{\text{dR,mid}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$. For each $i = 0, \dots, k'$, we look for the existence of

$$\widehat{m}_i = (\widehat{m}_{i,0}, \widehat{m}_{i,\infty}) \in (\text{Sym}^k \text{Kl}_2)_{\widehat{0}} \oplus (\text{Sym}^k \text{Kl}_2)_{\infty}$$

such that $\widehat{\nabla} \widehat{m}_i = (\iota_{\widehat{0}}(\omega_i), \iota_{\infty}(\omega_i))$.

Lemma 3.3 (solutions at $z = 0$). (1) *The subspace $\ker \widehat{\nabla} \subset (\text{Sym}^k \text{Kl}_2)_{\widehat{0}}$ has dimension one and a basis is given by $\widehat{m}_{0,0} = e_0^k$.*

(2) *There exists $\widehat{m}_{i,0}$ if and only if $i \geq 1$ and, in such case, there exists a unique $\widehat{m}_{i,0}$ belonging to $z\mathbb{C}[[z]] \cdot \mathbf{u}$.*

In fact, for any $j \geq 1$, there exists a unique $\widehat{m}_{j,0} \in z\mathbb{C}[[z]] \cdot \mathbf{u}$ with $\widehat{\nabla} \widehat{m}_{j,0} = \iota_{\widehat{0}}(z^j u_0 dz/z)$.

Proof. Set $(V, \nabla) = (\text{Sym}^k \text{Kl}_2, \nabla)$. We first claim that $\mathbb{C}[[z]] \cdot \mathbf{u}$ is equal to $V_{\widehat{0},0}$ (the 0-th step of the formal Kashiwara–Malgrange filtration at the origin, similar to that considered in the proof of [Fresán et al. 2023, Proposition 3.2]). Indeed, it is standard to show that there exists a formal (in fact convergent) base change $P(z) = \text{Id} + zP_1 + \dots$ such that the matrix of $\widehat{\nabla}$ in the basis $\mathbf{u}' = \mathbf{u} \cdot P(z)$ is constant and equal to a lower standard Jordan block with eigenvalue 0. It follows that \mathbf{u}' is a $\mathbb{C}[[z]]$ -basis of $V_{\widehat{0},0}$, and the claim follows, as well as the first point of the lemma.

Let us consider the second point. Setting $V_{\widehat{0},-1} = zV_{\widehat{0},0}$, we have recalled in [loc. cit.] that $z\partial_z : V_{\widehat{0},-1} \rightarrow V_{\widehat{0},-1}$ is bijective. The “if” part and its supplement follow. It remains to check that $\iota_{\widehat{0}}(u_0 dz/z)$ does not belong to the image of $\widehat{\nabla}$. It amounts to the same to replace u_0 with u'_0 defined above, and since u'_0 is the primitive vector of the matrix of $\widehat{\nabla}$, the assertion follows. \square

We now look for the solutions of $\widehat{\nabla} \widehat{m}_{i,\infty} = \iota_{\infty}(\omega_i)$ for $i = 1, \dots, k'$. For this purpose, we introduce the constants $\gamma_{k,i}$ as follows. Let us assume that $4 \mid k$. Recall that we have set $w = 1/z$ on \mathbb{G}_m . Write the asymptotic expansion as

$$2^k (I_0(t)K_0(t))^{k/2} \sim w^{k/4} \sum_{j=0}^{\infty} \gamma_{k,k/4+j} w^j, \tag{3.4}$$

so that we can define $\gamma_{k,i}$ by the residue

$$\gamma_{k,i} = \text{res}_{w=0} \frac{2^k (I_0(t)K_0(t))^{k/2}}{w^{i+1}}. \tag{3.5}$$

We have $\gamma_{k,i} = 0$ if $i < k/4$, $\gamma_{k,k/4} = 1$, and $\gamma_{k,i} > 0$ for all $i \in k/4 + \mathbb{N}^*$, e.g.,

$$\gamma_{k,1+k/4} = \frac{k}{2^6}, \quad \gamma_{k,2+k/4} = \frac{k(k+52)}{2^{13}}, \quad \gamma_{k,3+k/4} = \frac{k(k^2 + 156k + 13184)}{2^{19} \cdot 3}, \quad \dots \tag{3.6}$$

For what follows, it will be convenient to set

$$\gamma_{k,i} = 0 \quad (i \in k/4 + \mathbb{Z}) \text{ if } 4 \nmid k. \tag{3.7}$$

Proposition 3.8 (solutions at $z = \infty$). *Let us fix $i \in \{1, \dots, k'\}$:*

- (1) *For $4 \nmid k$, the equation $\widehat{\nabla} \widehat{m}_{i,\infty} = \iota_\infty(\omega_i)$ has a unique solution.*
- (2) *For $4 \mid k$ and $i \neq k/4$, the equation*

$$\widehat{\nabla} \widehat{m}_{i,\infty} = \iota_\infty(\omega_i - \gamma_{k,i} \omega_{k/4})$$

has a solution (in fact a one-dimensional affine space of solutions) where $\gamma_{k,i}$ is given by (3.5). Moreover, the subspace $\ker \widehat{\nabla} \subset (\text{Sym}^k \text{Kl}_2)_\infty$ is generated by the formal expansion $\widehat{m}_{k/4,\infty}$ of $2^k(\pi i)^{k/2}(e_0 e_1)^{k/2}$.

In fact, for any $j \geq 1$, there exists $\widehat{m}_{j,\infty}$ satisfying $\widehat{\nabla} \widehat{m}_{j,\infty} = \iota_\infty(z^j - \gamma_{k,j} z^{k/4}) u_0 dz/z$.

The second assertion of 3.8(2) is easy to check from the formal structure at infinity of $\text{Sym}^k \text{Kl}_2$; see [Fresán et al. 2022, Proposition 4.6(3)]. We set

$$\begin{aligned} \widehat{m}_0 &= (\widehat{m}_{0,0}, 0) \text{ (i.e., } \widehat{m}_{0,\infty} = 0) && \text{for all } k, \\ \widehat{m}_{k/4} &= (0, \widehat{m}_{k/4,\infty}) \text{ (i.e., } \widehat{m}_{k/4,0} = 0) && \text{if } 4 \mid k, \end{aligned}$$

and (see Notation 1.4)

$$\omega'_i = \begin{cases} 0 & \text{if } i = 0 \text{ and } i = k/4, \\ \omega_i - \gamma_{k,i} \omega_{k/4} & \text{if } i \in \llbracket 1, k' \rrbracket, \end{cases} \tag{3.9}$$

so that $\omega'_i = \omega_i$ if $4 \nmid k$ and $1 \leq i \leq k'$, or if $4 \mid k$ and $1 \leq i < k/4$. Once we know that \mathcal{B}_k is a basis of $H_{\text{dR}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$, using the convention (3.7) we derive the following from [Fresán et al. 2023, Remark 3.6(4)]:

Corollary 3.10. *The following set of $k' + 1$ elements*

$$\mathcal{B}_{k,c} = (\widehat{m}_i, \omega'_i)_{0 \leq i \leq k'} = \begin{cases} \{(\widehat{m}_0, 0), (\widehat{m}_1, \omega_1), \dots, (\widehat{m}_{k'}, \omega_{k'})\} & \text{if } 4 \nmid k, \\ \{(\widehat{m}_0, 0), (\widehat{m}_1, \omega'_1), \dots, (\widehat{m}_{k/4}, 0), \dots, (\widehat{m}_{k'}, \omega'_{k'})\} & \text{if } 4 \mid k, \end{cases}$$

is a basis of $H_{\text{dR},c}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$.

Let us consider the subset $\mathcal{B}_{k,\text{mid}}$ of the k' (resp. $(k' - 1)$) dimensional subspace $H_{\text{dR},\text{mid}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$ of $H_{\text{dR}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$ if $4 \nmid k$ (resp. if $4 \mid k$):

$$\mathcal{B}_{k,\text{mid}} = \begin{cases} \{\omega_i \mid i \in \llbracket 1, k' \rrbracket\} & \text{if } 4 \nmid k, \\ \{\omega'_i \mid i \in \llbracket 1, k' \rrbracket\} & \text{if } 4 \mid k. \end{cases}$$

Corollary 3.11. *The set $\mathcal{B}_{k,\text{mid}}$ is a basis of $H_{\text{dR},\text{mid}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$.*

Remark 3.12. Part of this result can also be proved as a consequence of [Fresán et al. 2022, Proposition 4.21(2) and Theorem 1.8]. However, the present proof does not rely on Hodge theory.

Proof of Proposition 3.8. Recall that e_0, e_1 are defined by (2.15) and ω_i by (3.2). Let us set $\bar{e}_1 = \pi i e_1$. Then $v_0 = 2(K_0 e_0 + I_0 \bar{e}_1)$ and

$$\omega_i = - \sum_{a=0}^k \binom{k}{a} \frac{2^k I_0^a K_0^{k-a}}{w^i} e_0^{k-a} \bar{e}_1^a \frac{dw}{w}.$$

We have to examine if there exist $\xi_{i,a}$ in some extension of $\mathbb{C}((w))$ such that

$$w \partial_w \xi_{i,a} = -w^{-i} (2^k I_0^a K_0^{k-a}) \quad \text{and} \quad \sum_{a=0}^k \binom{k}{a} \xi_{i,a} e_0^{k-a} \bar{e}_1^a \in (\text{Sym}^k \text{Kl}_2)_\infty \quad \text{if } 4 \nmid k,$$

and a similar property in the case $4 \mid k$. Then one takes

$$\widehat{m}_{i,\infty} = \sum_{a=0}^k \binom{k}{a} \xi_{i,a} e_0^{k-a} \bar{e}_1^a.$$

We write

$$\frac{2^k I_0^a K_0^{k-a}}{w^i} \sim \begin{cases} \sqrt{\pi}^{k-2a} e^{-2(k-2a)/\sqrt{w}} w^{k/4-i} \cdot F_a & a \neq k/2, \\ w^{k/4-i} \left(\sum_{n=0}^\infty \frac{((2n-1)!!)^3}{2^{5n} n!} w^n \right)^{k/2} & a = k/2, \end{cases}$$

with $F_a \in 1 + \sqrt{w}\mathbb{Q}[[\sqrt{w}]]$. When $a \neq k/2$ there exists a unique $\xi_{i,a}$ with the expansion

$$\xi_{i,a} \sim \frac{\sqrt{\pi}^{k-2a}}{(k-2a)} e^{-2(k-2a)/\sqrt{w}} w^{k/4-i+1/2} \cdot G_{i,a} \tag{3.13}$$

for some $G_{i,a} \in -1 + \sqrt{w}\mathbb{Q}[[\sqrt{w}]]$. Moreover, when expressed as a combination of monomials $v_0^{k-b} v_1^b$, such $\xi_{i,a} e_0^{k-a} \bar{e}_1^a$ has no exponential factor and, if σ denotes the action $w^{1/4} \mapsto iw^{1/4}$ so that $\mathbb{C}((w)) = \mathbb{C}((w^{1/4}))^\sigma$, one has $\sigma(\xi_{i,a} e_0^{k-a} \bar{e}_1^a) = \xi_{i,k-a} e_0^a \bar{e}_1^{k-a}$. When $4 \mid (k+2)$ and $a = k/2$, the exponents of w in the expansion of $2^k w^{-i-1} I_0^a K_0^{k-a}$ are in $\frac{1}{2} + \mathbb{Z}$ and one takes $\xi_{i,k/2}$ satisfying

$$\xi_{i,k/2} \sim \frac{w^{k/4-i}}{(k/4-i)} \cdot G_i$$

with $G_i \in -1 + w\mathbb{Q}[[w]]$. Then the factor $\xi_{i,k/2} (e_0 \bar{e}_1)^{k/2}$ has no exponential part in its expression in terms of v_0, v_1 and is invariant under σ . Finally, when $4 \mid k, k \geq 8$ and $a = k/2$, the residue

$$\gamma_{k,i} = \text{res}_{w=0} \frac{2^k I_0^{k/2} K_0^{k/2}}{w^{i+1}},$$

vanishes if and only if $i < k/4$. Therefore, for $i \geq k/4$ there exists $\xi_{i,k/2} \in \mathbb{Q}((w))$ such that

$$w \partial_w \xi_{i,k/2} = -(w^{-i} - \gamma_{k,i} w^{-k/4}) (I_0 K_0)^{k/2} dw.$$

In this case, $\xi_{i,k/2} (e_0 \bar{e}_1)^{k/2}$ has no exponential factor in combinations of v_0, v_1 and is invariant under σ . The proof is complete. □

Remark 3.14. In the case where $4 \nmid k$, the $\widehat{m}_{i,\infty}$ are uniquely determined since there is no horizontal formal section at $z = \infty$. When $4 \mid k$, the $\widehat{m}_{i,\infty}$ are unique up to adding a multiple of the formal horizontal section $\widehat{m}_{k/4,\infty}$ defined in Proposition 3.8(2). To normalize the choice, we take the series $\xi_{i,k/2}$ above to have no constant term. This normalization then fixes the computations of periods below.

3.2. Computation of the de Rham pairing. We aim at computing the matrix S_k^{mid} of the pairing

$$H_{\text{dR,mid}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2) \otimes H_{\text{dR,mid}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2) \rightarrow \mathbb{C}$$

induced by the self-duality pairing (2.7), with respect to the basis $\mathcal{B}_{k,\text{mid}}$. By [Fresán et al. 2023, Proposition 3.12], the matrix S_k^{mid} is equal to the sum of the matrices having (i, j) entries respectively:

- If $4 \nmid k$ and $i, j = 1, \dots, k'$,

$$\text{res}_{z=0} \langle \widehat{m}_{i,0}, z^j u_0 dz/z \rangle_{\text{alg}} \quad \text{and} \quad -\text{res}_{w=0} \langle \widehat{m}_{i,\infty}, w^{-j} u_0 dw/w \rangle_{\text{alg}}.$$

- A similar formula following Proposition 3.8(2) if $4 \mid k$.

For $i \in \{1, \dots, k'\}$ we set

$$\widehat{m}_{i,\infty} = \sum_{a=0}^k \mu_{a,i}(w) u_a, \quad \mu_{a,i}(w) = \sum_{\ell \gg -\infty} \mu_{a,i,\ell} w^\ell.$$

We can already note that, according to Lemma 3.3, $\langle \widehat{m}_{i,0}, z^j u_0 dz/z \rangle_{\text{alg}}$ has no residue (and a similar assertion if $4 \mid k$) so S_k^{mid} is determined by the residues at infinity. It follows from (2.7) that, if $4 \nmid k$, we have

$$S_{k;i,j}^{\text{mid}} = (-1)^{k+1} \mu_{k,i,j}, \quad i, j = 1, \dots, k'. \tag{3.15}$$

If $4 \mid k$, S_k^{mid} is the $(k' - 1) \times (k' - 1)$ -matrix given by the formula

$$S_{k;i,j}^{\text{mid}} = (-1)^{k+1} \begin{cases} \mu_{k,i,j} & \text{if } i \text{ or } j < k/4, \\ (\mu_{k,i,j} - \gamma_{k,j} \mu_{k,i,k/4}) & \text{if } i \text{ and } j > k/4, \end{cases} \quad i, j \in \llbracket 1, k' \rrbracket. \tag{3.16}$$

In other words, in the matrix given by (3.15) we delete the row $i = k/4$ and the column $j = k/4$ and we add to it the matrix having entry (i, j) equal to $(-1)^k \gamma_{k,j} \mu_{k,i,k/4}$ for $i, j > k/4$. According to [Fresán et al. 2023, Corollary 3.14] and (2.7), the matrix S_k^{mid} is $(-1)^{k+1}$ -symmetric.

Theorem 3.17. *The matrix S_k^{mid} is lower-right triangular (i.e., the entries (i, j) are zero if $i + j \leq k'$) and the antidiagonal entry on the i -th row is equal to*

$$\begin{cases} (-2)^k \frac{k'!}{k'!!} & \text{if } k \text{ is odd,} \\ \frac{(-1)^{k'+1}}{2^{k'}(k'+1-2i)} \cdot \frac{(k-1)!!}{(k'+1)!} & \text{if } k \text{ is even.} \end{cases}$$

Proof. We keep notation from the proof of [Proposition 3.8](#). For odd k , the entries of S_k^{mid} are

$$\begin{aligned} \text{res}_{w=0} \langle \widehat{m}_{i,\infty}, \omega_j \rangle_{\text{alg}} &= - \sum_{a,b=0}^k \binom{k}{a} \binom{k}{b} \text{res}_{w=0} \left\langle \xi_{i,a} e_0^{k-a} \bar{e}_1^a, \frac{2^k I_0^b K_0^{k-b}}{w^{j+1}} e_0^{k-b} \bar{e}_1^b \, dw \right\rangle_{\text{alg}} \\ &= - \frac{1}{2^k} \sum_{a,b=0}^k \binom{k}{a} \binom{k}{b} \text{res}_{w=0} \left\langle \xi_{i,a} e_0^{k-a} e_1^a, \frac{2^k I_0^b K_0^{k-b}}{w^{j+1}} e_0^{k-b} e_1^b \, dw \right\rangle_{\text{top}} \\ &= - \frac{1}{2^k} \sum_{a=0}^k (-1)^a \binom{k}{a} \text{res}_{w=0} \left(\xi_{i,a} \frac{2^k I_0^{k-a} K_0^a}{w^{j+1}} \, dw \right) \\ &= - \frac{1}{2^k} \sum_{a=0}^k \frac{(-1)^a \binom{k}{a}}{k-2a} \text{res}_{w=0} (w^{k'-i-j} F_{k-a} G_{i,a} \, dw) \end{aligned}$$

where $F_{k-a}, -G_{i,a} \in 1 + \sqrt{w}\mathbb{Q}[[\sqrt{w}]]$. Clearly, the last residue vanishes if $i + j \leq k'$. If $i + j = k' + 1$, we find

$$\text{res}_{w=0} \langle \widehat{m}_{i,\infty}, \omega_j \rangle = 2^{-k} \sum_{a=0}^k \frac{(-1)^a \binom{k}{a}}{k-2a}.$$

We conclude the case where k is odd with the next lemma.

Lemma 3.18. *For any $k \geq 1$, we have*

$$\sum_{\substack{0 \leq a \leq k \\ a \neq k/2}} \frac{(-1)^a \binom{k}{a}}{k-2a} = \begin{cases} 2^k (-2)^{k'} \frac{k'!}{k!!} & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

Proof. If k is even, replacing a with $k - a$ in the sum shows that the sum is equal to its opposite, and hence vanishes. We thus assume that k is odd and set

$$f_k(x) = \sum_{a=0}^k \binom{k}{a} \frac{x^{k-2a}}{k-2a}.$$

Then $f_k(x) = \int_1^x (x + 1/x)^k \, dx/x$. Besides, one has

$$\begin{aligned} f_k(x) &= \int_0^{\log x} (e^t + e^{-t})^k \, dt = 2^k \int_0^{\log x} \cosh^k t \, dt \quad (x = e^t) \\ &= 2^k \left[\frac{\sinh t \cosh^{k-1} t}{k} \Big|_{t=\log x} + \frac{k-1}{k} \int_0^{\log x} \cosh^{k-2} t \, dt \right] \\ &= \frac{1}{k} \left(x - \frac{1}{x} \right) \left(x + \frac{1}{x} \right)^{k-1} + 4 \frac{k-1}{k} f_{k-2}(x). \end{aligned}$$

By evaluating $f_k(i)$ inductively, one obtains the desired equality. □

Assume $4 \mid (k + 2)$, so that $k/2 = k' + 1$ is odd. Then the entries of S_k^{mid} are

$$\begin{aligned} & \text{res}_{w=0} \langle \widehat{m}_{i,\infty}, \omega_j \rangle \\ &= -\frac{1}{2^k} \left[\sum_{a \neq k/2} (-1)^a \binom{k}{a} \text{res}_{w=0} \left(\xi_{i,a} \frac{2^k I_0^{k-a} K_0^a}{w^{j+1}} dw \right) - \binom{k}{k/2} \text{res}_{w=0} \left(\xi_{i,k/2} \frac{2^k (I_0 K_0)^{k/2}}{w^{j+1}} dw \right) \right] \\ &= -\frac{1}{2^k} \left[\sum_{a \neq k/2} \frac{(-1)^a \binom{k}{a}}{(k-2a)} \text{res}_{w=0} (w^{k'-i-j+1/2} F_{k-a} G_{i,a} dw) - \frac{2 \binom{k}{k/2}}{(k'+1-2i)} \text{res}_{w=0} (w^{k'-i-j} F G_i dw) \right], \end{aligned}$$

where

$$F(w) = \left(\sum_{n=0}^{\infty} \frac{((2n-1)!!)^3}{2^{5n} n!} w^n \right)^{k/2}.$$

Again, the residue is zero if $i + j \leq k'$. On the other hand, if $i + j = k' + 1$, the first term in the above expression is zero, according to the lemma above, and since $F(0) = 1$ and $G_i(0) = -1$, we have

$$\text{res}_{w=0} \langle \widehat{m}_{i,\infty}, \omega_j \rangle = -\binom{k}{k/2} [2^{k-1} (k' + 1 - 2i)]^{-1}.$$

The computation in the case where $4 \nmid k$ is similar. □

Example 3.19 ($k = 5$). We have $k' = 2$ and

$$S_5^{\text{mid}} = \begin{pmatrix} 0 & 8/15 \\ 8/15 & \mu_{5,2,2} \end{pmatrix}.$$

From the proof of the theorem above, one has

$$\mu_{5,2,2} = \frac{-1}{2^5} \sum_{a=0}^5 \frac{(-1)^a \binom{5}{a}}{5-2a} \text{res}_{w=0} \left(F_{5-a} G_{2,a} \frac{dw}{w^2} \right).$$

A direct computation yields

$$-\sum_{a=0}^5 \frac{(-1)^a \binom{5}{a}}{5-2a} F_{5-a} G_{2,a} \sim \frac{256}{15} + \frac{2^9 \cdot 13}{3^3 \cdot 5^5} w \pmod{w^2 \mathbb{Q}[[w]]}.$$

Therefore,

$$S_5^{\text{mid}} = \begin{pmatrix} 0 & 8/15 \\ 8/15 & 2^4 \cdot 13/3^3 \cdot 5^3 \end{pmatrix}.$$

Example 3.20 ($k = 6$). We have $k' = 2$ and $i = 1, 2$. Due to skew-symmetry, the antidiagonal entries are enough to determine S_6^{mid} . [Theorem 3.17](#) gives

$$S_6^{\text{mid}} = \begin{pmatrix} 0 & -5/8 \\ 5/8 & 0 \end{pmatrix}.$$

Corollary 3.21. *The ordered basis $\mathcal{B}_{k,\text{mid}}$ of $H_{\text{dr,mid}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$ is adapted to the Hodge filtration.*

Proof. For odd k , [Fresán et al. 2022, Proposition 4.21(2) and Theorem 1.8(1)] imply the claim. For even k , [loc. cit.] only gives compatibility for half of the basis $\mathcal{B}_{k,\text{mid}}$. However, since the Poincaré pairing respects the Hodge filtration, compatibility holds for the whole $\mathcal{B}_{k,\text{mid}}$ by the above theorem. \square

Remark 3.22 (the matrix S_k). The matrix of the pairing

$$S_k : H_{\text{dR},c}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2) \otimes H_{\text{dR}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2) \rightarrow \mathbb{C}$$

in the basis $\mathcal{B}_{k,c} \otimes \mathcal{B}_k$ is obtained with similar residue formulas as for S_k^{mid} , due to [Fresán et al. 2023, Proposition 3.12]:

(1) If $4 \nmid k$,

$$S_{k;i,j} = \begin{cases} S_{k;i,j}^{\text{mid}} & \text{if } 1 \leq i, j \leq k', \\ 0 & \text{if } i = 0 \text{ and } j = 1, \dots, k'. \end{cases}$$

On the other hand, we have $S_{k;0,0} = \text{res}_{z=0} \langle e_0^k, u_0 dz/z \rangle_{\text{alg}} = 1$. Last, for $i \geq 1$, we use the expressions (2.15) and obtain

$$\begin{aligned} S_{k;i,0} &= \text{res}_{w=0} \left\langle \widehat{m}_{i,\infty}, u_0 \frac{dz}{z} \right\rangle_{\text{alg}} \\ &= \sum_{a=0}^k \binom{k}{a} \text{res}_{w=0} \left\langle \xi_{i,a} e_0^{k-a} \bar{e}_1^a, u_0 \frac{dz}{z} \right\rangle_{\text{alg}} \\ &= \sum_{a=0}^k \binom{k}{a} \text{res}_{w=0} \left\langle \xi_{i,a} \left[\frac{t}{2} I_0' v_0 - I_0 v_1 \right]^{k-a} \left[-\frac{t}{2} K_0' v_0 + K_0 v_1 \right]^a, u_0 \frac{dz}{z} \right\rangle_{\text{alg}} \\ &= \sum_{a=0}^k (-1)^a \binom{k}{a} \text{res}_{w=0} \xi_{i,a} I_0^{k-a} K_0^a \frac{dz}{z}. \end{aligned}$$

Since

$$\xi_{i,a} I_0^{k-a} K_0^a \sim \begin{cases} \frac{1}{2^{k(k-2a)}} w^{(k+1)/2-i} G_{i,a} & a \neq k/2, \\ \frac{1}{2^{k(k/4-i)}} w^{k/2-i} G_i & a = k/2 \end{cases}$$

with $G_{i,a}, G_i \in \mathbb{Q}[[\sqrt{w}]]$, we conclude that $S_{k;i,0} = 0$. In other words, S_k takes the form

$$\begin{pmatrix} 1 & 0 \\ 0 & S_k^{\text{mid}} \end{pmatrix}.$$

In particular, we get

$$\det S_k = \det S_k^{\text{mid}} = \begin{cases} (-1)^{k'(k'+1)/2} (2^{k'} k'! / k!!)^{k'} & \text{if } k \text{ is odd,} \\ ((k-1)!!)^{k'} / [(2^{k'} (k'+1)!)^{k'} ((k'-1)!!)^2] & \text{if } 4 \mid (k+2). \end{cases}$$

(2) If $4 \mid k$, the matrix S_k is obtained from the matrix $\begin{pmatrix} 1 & 0 \\ 0 & S_k^{\text{mid}} \end{pmatrix}$ by adding a row $i = k/4$ and a column $j = k/4$ that we compute now. For the row $i = k/4$, we note that

$$\langle \widehat{m}_{k/4,\infty}, v_0^k \rangle_{\text{alg}} = 2^k (I_0 K_0)^{k/2} = w^{k/4} \sum_{j \geq 0} \gamma_{k,k/4+j} w^j,$$

so that

$$S_{k;k/4,k/4+j} = \begin{cases} 0 & \text{if } j < 0, \\ -\gamma_{k,k/4+j} & \text{if } j \geq 0. \end{cases}$$

For the column $j = k/4$, we compute as above that $S_{k;0,k/4} = 0$ and, for $i \geq 1$ and $\neq k/4$,

$$S_{k;i,k/4} = \text{res}_{z=\infty} \langle \widehat{m}_{i,\infty}, \omega_{k/4} \rangle_{\text{alg}}.$$

In particular, by setting $k'' = \lfloor (k - 1/4) \rfloor$ and recalling $\gamma_{k,k/4} = 1$, we get

$$\det S_k = -\det S_k^{\text{mid}} = -[2^{-k'-1}(k - 1)!((k' + 1)!)^{-1}]^{k'-1} (k'!)^{-2}. \tag{3.23}$$

4. Betti intersection pairing for $\text{Sym}^k \text{Kl}_2$

In this section, we exhibit natural bases of the rapid decay, the moderate growth, and the middle homology spaces denoted respectively by

$$H_1^{\text{rd}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2), \quad H_1^{\text{mod}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2), \quad \text{and} \quad H_1^{\text{mid}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2).$$

We then compute the Betti pairing B_k as introduced in [Fresán et al. 2023, Section 2.d]. Bear in mind that the notation there keeps track of the degree of the homology spaces; as this degree is always equal to 1 here, we omit it, but we remember the exponent k of the symmetric power. We keep the setting of Section 2.3. While we used $\langle \cdot, \cdot \rangle_{\text{alg}}$ on $\text{Sym}^k \text{Kl}_2$ to compute the de Rham intersection matrix S_k , we will use the topological pairing $\langle \cdot, \cdot \rangle_{\text{top}}$ on $\text{Sym}^k \text{Kl}_2^\vee$ to compute the Betti pairing B_k , which is thus defined over \mathbb{Q} .

We consider the following C^∞ chains on \mathbb{P}_z^1 diffeomorphic to their images:

- $\mathbb{R}_+ = [0, \infty]$, oriented from 0 to $+\infty$,
- $c_0 =$ unit circle, starting at 1 and oriented counterclockwise,
- $c_+ = [1, \infty]$, oriented from +1 to $+\infty$.

According to Lemma 2.19, the $\lfloor k/2 \rfloor + 1$ twisted chains

$$\beta_j = \mathbb{R}_+ \otimes e_0^j e_1^{k-j}, \quad 0 \leq j \leq \lfloor k/2 \rfloor,$$

have moderate growth and define $\lfloor k/2 \rfloor + 1$ elements of $H_1^{\text{mod}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$, still denoted by β_j .

Besides, the twisted chain $\alpha_0 = c_0 \otimes e_0^k$ is a twisted cycle with compact support, and hence has rapid decay, since e_0 is invariant by monodromy. We obtain other twisted chains with compact support as follows. For each integer $n \geq 1$, let us set

$$C_n(a) = \frac{(-1)^{a-1}}{na} \binom{n}{a}, \quad 1 \leq a \leq n.$$

Lemma 4.1. *Let $n \geq 1$ be an integer:*

(1) *The sequence $(C_n(a))_a$ is the unique solution to the linear relations*

$$\sum_{a=1}^n C_n(a)a = \frac{1}{n} \quad \text{and} \quad \sum_{a=1}^n C_n(a)a^r = 0 \quad \text{if } 2 \leq r \leq n.$$

(2) *The sequence $(C_n(a))_a$ is the unique solution to the linear relations*

$$\sum_{a=1}^n C_n(a)a^{n+1} = (-1)^{n-1}(n-1)! \quad \text{and} \quad \sum_{a=1}^n C_n(a)a^r = 0 \quad \text{if } 2 \leq r \leq n.$$

Proof. Direct simplification of Cramer’s rule in solving the systems of linear equations. □

Lemma 4.2. *For integers $n \geq 1$ and $r \geq 0$, one has*

$$\sum_{a=1}^{n+r} C_{n+r}(a) \sum_{b=1}^a b^n = \begin{cases} \frac{(-1)^n}{n+r} \mathcal{B}_n & \text{if } r \geq 1, \\ \frac{(-1)^n}{n} \mathcal{B}_n + (-1)^{n-1} \frac{(n-1)!}{n+1} & \text{if } r = 0. \end{cases}$$

Proof. Replacing $\sum_{b=1}^a b^n$ with Bernoulli’s formula

$$\sum_{b=1}^a b^n = \frac{1}{n+1} \sum_{\ell=0}^n (-1)^\ell \binom{n+1}{\ell} \mathcal{B}_\ell a^{n+1-\ell}, \tag{4.3}$$

we can rewrite the left-hand side as

$$\sum_{a=1}^{n+r} C_{n+r}(a) \sum_{b=1}^a b^n = \frac{1}{n+1} \sum_{\ell=0}^n (-1)^\ell \binom{n+1}{\ell} \mathcal{B}_\ell \sum_{a=1}^{n+r} C_{n+r}(a) a^{n+1-\ell}.$$

Then [Lemma 4.1\(1\)](#) gives the assertion for $r \geq 1$ and, for $r = 0$, we use [Lemma 4.1\(2\)](#) instead. □

Since $T^a e_1 = e_1 + a e_0$ for all $a \geq 1$, it follows from [Lemma 4.1\(1\)](#) that, for each $1 \leq i \leq k$, the twisted chain with compact support

$$\sum_{a=1}^{k-i+1} C_{k-i+1}(a) c_0^a \otimes e_0^{i-1} e_1^{k-i+1}$$

has boundary $\{1\} \otimes e_0^i e_1^{k-i}$. As a consequence, the following $(k' + 1)$ twisted chains are twisted cycles, whose classes are elements in $H_1^{\text{rd}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$:

$$\begin{cases} \alpha_0 = c_0 \otimes e_0^k \\ \alpha_i = -\frac{(-1)^{k-i}(k-i)!}{k-i+2} \alpha_0 + c_+ \otimes e_0^i e_1^{k-i} + \sum_{a=1}^{k-i+1} C_{k-i+1}(a) c_0^a \otimes e_0^{i-1} e_1^{k-i+1} \quad (1 \leq i \leq k'). \end{cases} \tag{4.4}$$

The natural map

$$H_1^{\text{rd}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2) \rightarrow H_1^{\text{mod}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$$

is given on these twisted cycles by

$$\alpha_i \mapsto \begin{cases} 0 & \text{if } i = 0, \\ \beta_i & \text{if } 1 \leq i \leq k' \end{cases} \tag{4.5}$$

by shrinking the circle c_0 and extending the half-line c_+ .

Proposition 4.6. *If $0 \leq j \leq \lfloor k/2 \rfloor$, we have $B_k(\alpha_0, \beta_j) = -\delta_{0,j}$ and, if $1 \leq i \leq k'$,*

$$B_k(\alpha_i, \beta_j) = (-1)^{k-i} \frac{(k-i)!(k-j)!}{k!} \frac{B_{k-i-j+1}}{(k-i-j+1)!}.$$

With a small abuse of notation, we also denote by B_k the matrix $(B_{k;i,j})_{1 \leq i,j \leq k'}$ with

$$B_{k;i,j} = B_k(\alpha_i, \beta_j).$$

Proof. For the first assertion, since the intersection index (c_0, \mathbb{R}_+) is equal to -1 , we have by (2.18)

$$B_k(\alpha_0, \beta_j) = \begin{cases} -\langle e_0^k, e_0^j e_1^{k-j} \rangle_{\text{top}} = 0 & \text{if } j \neq 0, \\ -\langle e_0^k, e_1^k \rangle_{\text{top}} = -1 & \text{if } j = 0. \end{cases}$$

Let us compute $B_k(\alpha_i, \beta_j)$ if $1 \leq i \leq k'$ and $0 \leq j \leq \lfloor k/2 \rfloor$. Fix some $\theta_o \in (0, \pi)$ and let $x_o = \exp(i\theta_o)$. To achieve the computation, we move the ray c_+ by adding the scalar $(x_o - 1)$ and let the circle c_0 start at x_o . Then the component $c_0^a \otimes e_0^{i-1} e_1^{k-i+1}$ in the deformed α_i meets β_j physically a times at the same point $1 \in \mathbb{C}^\times$ with intersection index -1 . At the b -th intersection ($1 \leq b \leq a$), the factor $e_0^{i-1} e_1^{k-i+1}$ becomes $e_0^{i-1} (e_1 + be_0)^{k-i+1}$ and (2.18) gives

$$\langle e_0^{i-1} (e_1 + be_0)^{k-i+1}, e_0^j e_1^{k-j} \rangle_{\text{top}} = (-1)^j \binom{k-j+1}{j} \binom{k}{j}^{-1} b^{k-i-j+1}.$$

For $j \geq 1$, since $B_k(\alpha_0, \beta_j) = 0$, we obtain, by adding these contributions and taking into account the intersection indices,

$$B_k(\alpha_i, \beta_j) = (-1)^{j+1} \binom{k-i+1}{j} \binom{k}{j}^{-1} \sum_{a=1}^{k-i+1} C_{k-i+1}(a) \sum_{b=1}^a b^{k-i-j+1}.$$

The asserted equality follows by applying Lemma 4.2 with $r \geq 1$.

If $j = 0$, then $B_k(\alpha_i, \beta_j)$ writes

$$\frac{(-1)^{k-i} (k-i)!}{k-i+2} - \sum_{a=1}^{k-i+1} C_{k-i+1}(a) \sum_{b=1}^a b^{k-i+1} = (-1)^{k-i} \frac{B_{k-i+1}}{k-i+1},$$

after Lemma 4.2 with $r = 0$. □

Theorem 4.7. (1) *The family $(\alpha_i)_{0 \leq i \leq k'}$ is a basis of $H_1^{\text{rd}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$.*

(2) *The family (β_j) with $0 \leq j \leq k'$ (resp. with $0 \leq j \leq k'+1$ and $j \neq 1$) is a basis of $H_1^{\text{mod}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$ if $4 \nmid k$ (resp. if $4 \mid k$).*

(3) *The family (β_j) with $1 \leq j \leq k'$ (resp. with $2 \leq j \leq k'$) is a basis of $H_1^{\text{mid}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$ if $4 \nmid k$ (resp. if $4 \mid k$).*

Notation 4.8. If $4 \mid k$, we shift the indices of the bases α and β as follows:

$$\text{for } i \in \llbracket 1, k' \rrbracket, \quad \alpha'_i = \begin{cases} \alpha_{i+1} & \text{if } i < k/4, \\ \alpha_i & \text{if } i > k/4, \end{cases}$$

and similarly for β' . We set $B_k^{\text{mid}} = (B_{k;i,j}^{\text{mid}})_{i,j \in \llbracket 1, k' \rrbracket}$ with

$$B_{k;i,j}^{\text{mid}} = \begin{cases} B_k(\alpha_i, \beta_j) & \text{if } 4 \nmid k, \\ B_k(\alpha'_i, \beta'_j) & \text{if } 4 \mid k. \end{cases}$$

In particular, $B_k^{\text{mid}} = B_k$ if $4 \nmid k$. **Theorem 4.7(3)** implies that B_k^{mid} is an invertible matrix.

Theorem 4.7 is a straightforward consequence of **Propositions 4.6** and **4.9** below.

Proposition 4.9. (1) Let B_k denote the matrix of size k' having entries $B_k(\alpha_i, \beta_j)$ with

- $1 \leq i, j \leq k'$ if $4 \nmid k$,
- $1 \leq i \leq k'$ and $2 \leq j \leq k/2$ if $4 \mid k$.

Then

$$\det B_k = \begin{cases} [k! \prod_{a=1}^{k'} \binom{k}{a}]^{-1} & \text{if } k \text{ is odd,} \\ [k!(k'+1) \prod_{a=1}^{k'} \binom{k}{a}]^{-1} & \text{if } 4 \mid (k+2), \\ [k! \prod_{a=2}^{k'} \binom{k}{a}]^{-1} & \text{if } 4 \mid k. \end{cases}$$

(2) If $4 \mid k$, let B'_k denote the matrix of size $k' - 1$ having entries $B_k(\alpha_i, \beta_j)$ with $2 \leq i, j \leq k'$. Then

$$\det B'_k = \left[\frac{k}{4} (k')^2 \prod_{a=2}^{k'} \binom{k}{a} \right]^{-1}.$$

(Note that $B_k^{\text{mid}} = B_k$ if $4 \nmid k$ and $B_k^{\text{mid}} = B'_k$ if $4 \mid k$).

Some determinants of Bernoulli numbers. We will make use of the following lemma:

Lemma 4.10. *The following identities hold:*

$$\det \left(\frac{B_{i+j}}{(i+j)!} \right)_{1 \leq i, j \leq n} = \frac{(-1)^{n(n-1)/2} (2n+1)!!}{2^{n(n+1)} [3!!5!! \cdots (2n+1)!!]^2}. \tag{4.11}$$

$$\det \left(\frac{B_{i+j+1}}{(i+j+1)!} \right)_{1 \leq i, j \leq n} = \begin{cases} \frac{(-1)^{n/2}}{2^{n(n+2)} [3!!5!! \cdots (2n+1)!!]^2} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{4.12}$$

Proof. We follow the principle in [Krattenthaler 2005, Section 5.4]. Recall that, for each $n \geq 0$, the Lommel polynomials $h_{n,\nu}(x)$ satisfy

$$J_{\nu+n}(z) = h_{n,\nu}(z^{-1})J_\nu(z) - h_{n-1,\nu+1}(z^{-1})J_{\nu-1}(z),$$

where $J_\nu(z)$ is the Bessel function of the first kind of order ν . Let

$$f_n(x) = \frac{1}{(2n+1)!!} h_{n,1/2}(x) \in \mathbb{Q}[x] \quad (n \geq 0).$$

The following hold:

- $f_n(x)$ is monic of degree n with $f_0(x) = 1$, $f_1(x) = x$, and $\{f_n(x)\}$ satisfies the recursive relation

$$f_{n+1}(x) = xf_n(x) - b_n f_{n-1}(x), \quad b_n = \frac{1}{(2n+1)(2n+3)}, \quad n \geq 1.$$

- $\{f_n(x)\}$ forms an orthogonal family with respect to the linear functional

$$L(f(x)) = \sum_{m=1}^{\infty} \frac{1}{(m\pi)^2} (f(1/m\pi) + f(-1/m\pi)).$$

For the above; see [Koelink and Van Assche 1995, Section 1]. We have the moments $\mu_r = L(x^r)$ of the linear functional

$$\mu_r = \frac{1}{\pi^{r+2}} \sum_{m=1}^{\infty} \frac{1}{m^{r+2}} + \frac{1}{(-m)^{r+2}} = \frac{(-1)^{r/2} 2^{r+2} \mathcal{B}_{r+2}}{(r+2)!} \geq 0, \quad r \geq 0$$

(in particular, $\mu_0 = \frac{1}{3}$). With these data and by applying [Krattenthaler 2005, Theorem 29], one readily obtains

$$\det(\mu_{i+j})_{0 \leq i, j \leq n-1} = \frac{(2n+1)!!}{[3!!5!! \dots (2n+1)!!]^2}, \quad \det(\mu_{i+j+1})_{0 \leq i, j \leq n-1} = \begin{cases} \frac{(-1)^{n/2}}{[3!!5!! \dots (2n+1)!!]^2} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

The asserted formulas follow immediately by a simple matrix manipulation. □

Remark 4.13. Let

$$\Theta_n = \left(\frac{\mathcal{B}_{2a+2b-2}}{(2a+2b-2)!} \right)_{1 \leq a, b \leq n}, \quad \Theta'_n = \left(\frac{\mathcal{B}_{2a+2b}}{(2a+2b)!} \right)_{1 \leq a, b \leq n}.$$

By rearranging columns and rows, one has

$$\begin{aligned} \left(\frac{\mathcal{B}_{i+j}}{(i+j)!} \right)_{1 \leq i, j \leq n} &\sim \begin{cases} \Theta_{n/2} \oplus \Theta'_{n/2} & \text{if } n \text{ is even,} \\ \Theta_{n+1/2} \oplus \Theta'_{(n-1/2)} & \text{if } n \text{ is odd,} \end{cases} \\ \left(\frac{\mathcal{B}_{i+j+1}}{(i+j+1)!} \right)_{1 \leq i, j \leq n} &\sim \begin{cases} \begin{pmatrix} 0 & \Theta'_{n/2} \\ \Theta'_{n/2} & 0 \end{pmatrix} & \text{if } n \text{ is even,} \\ \text{singular} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Therefore, (4.11) inductively implies the equalities

$$\det \Theta_n = \frac{1}{2^{2n^2} 3!!5!! \dots (4n-1)!!}, \quad \det \Theta'_n = \frac{(-1)^n}{2^{2n(n+1)} 3!!5!! \dots (4n+1)!!}$$

and (4.12) is a consequence of (4.11). The evaluation of a variant of $\det \Theta_n$ is also considered in [Zhang and Chen 2014, Corollary 2], although their formula does not seem to be correct. Both this approach and that of [loc. cit.] use the orthogonal family of Lommel polynomials.

Proof of Proposition 4.9. Let $\Delta_n = (\delta_{n+1}(i+j))_{1 \leq i, j \leq n}$. For integers m, n with $m \geq n \geq 0$, let

$$D_{m,n}^\pm = \text{diag}((\pm 1)^m m!, (\pm 1)^{m-1} (m-1)!, \dots, (\pm 1)^n n!).$$

(1) For k odd, we have

$$B_k = \frac{1}{k!} D_{k-1, k'+1}^- \Delta_{k'} \left(\frac{\mathcal{B}_{i+j}}{(i+j)!} \right)_{1 \leq i, j \leq k'} \Delta_{k'} D_{k-1, k'+1}^+$$

and by (4.11), one obtains

$$\det B_k = \frac{k!}{2^{k'(k'+1)} (k!)^{k'}} \prod_{a=1}^{k'} \left[\frac{(k'+a)!}{(2a+1)!!} \right]^2 = \left[k! \prod_{a=1}^{k'} \binom{k}{a} \right]^{-1}.$$

For $4 \mid (k+2)$, we have

$$B_k = \frac{1}{k!} D_{k-1, k'+2}^- \Delta_{k'} \left(\frac{\mathcal{B}_{i+j+1}}{(i+j+1)!} \right)_{1 \leq i, j \leq k'} \Delta_{k'} D_{k-1, k'+2}^+$$

and

$$\det B_k = \frac{1}{2^{k'(k'+2)} (k!)^{k'}} \prod_{a=1}^{k'} \left[\frac{(k'+1+a)!}{(2a+1)!!} \right]^2 = \left[k!(k'+1) \prod_{a=2}^{k'} \binom{k}{a} \right]^{-1}$$

by (4.12).

For $4 \mid k$, we have

$$B_k = \frac{1}{k!} D_{k-2, k'+1}^- \Delta_{k'} \left(\frac{\mathcal{B}_{i+j}}{(i+j)!} \right)_{1 \leq i, j \leq k'} \Delta_{k'} D_{k-1, k'+2}^+$$

and

$$\det B_k = \frac{(k-1)!(k-1)!!}{2^{k'(k'+1)} (k!)^{k'} (k'+1)!} \prod_{a=1}^{k'} \left[\frac{(k'+a)!}{(2a+1)!!} \right]^2 = \left[k! \prod_{a=2}^{k'} \binom{k}{a} \right]^{-1}.$$

(2) We have

$$B'_k = \frac{1}{k!} D_{k-2, k'+2}^- \Delta_{k'-1} \left(\frac{\mathcal{B}_{i+j+1}}{(i+j+1)!} \right)_{1 \leq i, j \leq k'-1} \Delta_{k'-1} D_{k-2, k'+2}^+$$

and

$$\det B'_k = \frac{1}{2^{k'^2-1} (k!)^{k'-1}} \prod_{a=1}^{k'-1} \left[\frac{(k'+1+a)!}{(2a+1)!!} \right]^2 = \left[\frac{k}{4} (k')^2 \prod_{a=2}^{k'} \binom{k}{a} \right]^{-1}. \quad \square$$

5. Quadratic relations between periods and Bessel moments

In this section, we express the period pairing between rapid decay homology and de Rham cohomology of $\text{Sym}^k \text{Kl}_2$ in terms of Bessel moments and we obtain quadratic relations between them by specializing to our setting the general results from [Fresán et al. 2023].

5.1. Quadratic relations between periods. We use the topological pairing $\langle \cdot, \cdot \rangle_{\text{top}}$ on Kl_2 , which is compatible with the \mathbb{Q} -structure of Kl_2^∇ from Section 2.3. Recall that the induced pairing $\langle \cdot, \cdot \rangle_{\text{top}}$ on $\text{Sym}^k \text{Kl}_2$ is $(-1)^k$ -symmetric. The period pairing $P_1^{\text{rd,mod}}$ was defined in [Fresán et al. 2023], where the index 1 referred to the degree of rapid decay homology and moderate growth de Rham cohomology. Here we denote it by $P_k^{\text{rd,mod}}$, in order to emphasize that we are dealing with the k -th symmetric power and since there are no other nontrivial (co)homological degrees at play. Recall the de Rham cohomology classes ω_i from (3.2) and the rapid decay cycles α_i from (4.4). Since $(\alpha_i)_{0 \leq i \leq k'}$ is a basis of $H_1^{\text{rd}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$ by Theorem 4.7(1) and $(\omega_i)_{0 \leq i \leq k'}$ is a basis of $H_{\text{dR}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$ by [Fresán et al. 2022, Proposition 4.14], we deduce from the perfectness of the pairing $P_k^{\text{rd,mod}}$ (see [Fresán et al. 2023, Corollary 2.11]) that the $(k' + 1) \times (k' + 1)$ period matrix $(P_{k;i,j}^{\text{rd,mod}})_{0 \leq i, j \leq k'}$ defined by

$$P_{k;i,j}^{\text{rd,mod}} = P_k^{\text{rd,mod}}(\alpha_i, \omega_j)$$

is invertible. Thanks to the identity (2.16) relating v_0 to e_0 and the change of variables $t = 2\sqrt{z}$, the first row of this matrix reads

$$P_k^{\text{rd,mod}}(\alpha_0, \omega_j) = \int_{c_0} \langle e_0^k, v_0^k \rangle_{\text{top}} z^j \frac{dz}{z} = \int_{c_0} (2\pi i)^k I_0(2\sqrt{z})^k z^j \frac{dz}{z} = (2\pi i)^{k+1} \delta_{0,j}, \tag{5.1}$$

from which we immediately derive:

Proposition 5.2. *The $k' \times k'$ period matrix $P_k = (P_{k;i,j}^{\text{rd,mod}})_{1 \leq i, j \leq k'}$ is invertible.* □

The pure part of the pairing $P_k^{\text{rd,mod}}$ arises from the pairing between middle homology and middle de Rham cohomology. According to (4.5) and Theorem 4.7(3), the elements $(\alpha_i)_{i \in \llbracket 1, k' \rrbracket}$ map to a basis of $H_1^{\text{mid}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$ whenever $4 \nmid k$. If $4 \mid k$, we instead consider the images of the shifted elements $(\alpha'_i)_{i \in \llbracket 1, k' \rrbracket}$, as introduced in Notation 4.8. Regarding $H_{\text{dR, mid}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$, Corollary 3.11 gives the basis $(\omega_i)_{i \in \llbracket 1, k' \rrbracket}$ (resp. $(\omega'_i)_{i \in \llbracket 1, k' \rrbracket}$) if $4 \nmid k$ (resp. if $4 \mid k$), where ω'_i is modified as in (3.9). With this notation, the *middle period matrix* is defined as follows:

$$P_k^{\text{mid}} = (P_{k;i,j}^{\text{mid}})_{i, j \in \llbracket 1, k' \rrbracket} = \begin{cases} (P_k^{\text{rd,mod}}(\alpha_i, \omega_j))_{i, j \in \llbracket 1, k' \rrbracket} & \text{if } 4 \nmid k, \\ (P_k^{\text{rd,mod}}(\alpha'_i, \omega'_j))_{i, j \in \llbracket 1, k' \rrbracket} & \text{if } 4 \mid k. \end{cases}$$

In particular, $P_k^{\text{mid}} = P_k$ if $4 \nmid k$.

Recall the matrices S_k^{mid} from Section 3.2 and B_k^{mid} from Notation 4.8. In the current setting, the general method to express middle quadratic relations explained in [Fresán et al. 2023, Section 3.f], namely (3.21) therein yields the following result:

Theorem 5.3 (middle quadratic relations for $\text{Sym}^k \text{Kl}_2$). *The middle periods of $\text{Sym}^k \text{Kl}_2$ satisfy the following quadratic relations:*

$$(-2\pi i)^{k+1} B_k^{\text{mid}} = P_k^{\text{mid}} \cdot (S_k^{\text{mid}})^{-1} \cdot {}^t P_k^{\text{mid}}.$$

In particular, the matrix P_k^{mid} is invertible. □

5.2. Bessel moments as periods. We consider the power moments of the modified Bessel functions I_0 and K_0 defined by

$$\text{BM}_k(i, j) = (-1)^{k-i} 2^{k-j} (\pi i)^i \int_0^\infty I_0(t)^i K_0(t)^{k-i} t^j dt,$$

where the indices i and j are subject to the constraints

$$0 \leq i \leq k' \text{ and } j \geq 0 \quad \text{or, if } k \text{ is even, } i = \frac{k}{2} \text{ and } 0 \leq j \leq k' - 1.$$

For this range of indices, it results from the asymptotic expansion at infinity (2.14) that the improper integral $\text{BM}_k(i, j)$ converges. In what follows, such moments will occur only for odd j .

Proposition 5.4. *For all $1 \leq i, j \leq k'$, the following equality holds:*

$$P_{k;i,j}^{\text{rd,mod}} = \text{BM}_k(i, 2j - 1).$$

Proof. In view of the definition (4.4) of the twisted cycles α_i , we need to compute the integrals along c_0^a of $\langle e_0^{a-1} e_1^{k-a+1}, v_0^k \rangle_{\text{top}} z^j dz/z$ for $a = 1, \dots, k - i + 1$ and the integral along c_+ of $\langle e_0^i e_1^{k-i}, v_0^k \rangle_{\text{top}} z^j dz/z$.

Let us first remark that, for $\varepsilon \in (0, 1]$, we can replace c_0 and c_+ with the scalings $c_{0,\varepsilon}$ and $c_{+,\varepsilon}$ by ε defined with the base point ε instead of 1, leading to a twisted cycle $\alpha_{i,\varepsilon}$ equivalent to α_i ($1 \leq i \leq k'$). We will show

$$\lim_{\varepsilon \rightarrow 0} \int_{c_{0,\varepsilon}^a} \langle e_0^{a-1} e_1^{k-a+1}, v_0^k \rangle_{\text{top}} z^j \frac{dz}{z} = 0, \quad a = 1, \dots, k - i + 1, \tag{5.5}$$

$$\lim_{\varepsilon \rightarrow 0} \int_{c_{+,\varepsilon}} \langle e_0^i e_1^{k-i}, v_0^k \rangle_{\text{top}} z^j \frac{dz}{z} = \text{BM}_k(i, 2j - 1) \quad \text{for } 1 \leq i, j \leq k'. \tag{5.6}$$

We first show that the limits (5.5) are zero. According to (2.18), we only need to compute the coefficient of v_0^k in $e_0^{k-a+1} e_1^{a-1}$, which is equal to $I_0^{a-1} K_0^{k-a+1}$ up to a constant (see (2.16)). It remains to check that

$$\int_{\arg t=0}^{a\pi i} I_0(t)^{a-1} K_0(t)^{k-a+1} t^{2j-1} dt \rightarrow 0 \quad \text{when } |t| = 2\sqrt{\varepsilon} \text{ and } \varepsilon \rightarrow 0.$$

We can use the estimate (2.13) to compute the integral, and the assumption $j \geq 1$ implies that the absolute value of the integral tends to zero with ε .

For (5.6), recall that the coefficient of v_0^k in $e_0^{k-i} e_1^i$ is equal to $2^k (\pi i)^i \binom{k}{i} I_0^i K_0^{k-i}$, and thus (see (2.18))

$$\langle e_0^i e_1^{k-i}, v_0^k \rangle_{\text{top}} = (-1)^{k-i} \binom{k}{i}^{-1} 2^k (\pi i)^i \binom{k}{i} I_0^i K_0^{k-i} = (-1)^{k-i} 2^k (\pi i)^i I_0^i K_0^{k-i}.$$

The assertion (5.6) then follows from the relation $z^j dz/z = 2^{-2j+1} t^{2j-1} dt$. □

Corollary 5.7. *The matrix P_k^{mid} satisfies, for $i, j \in \llbracket 1, k' \rrbracket$,*

$$P_{k;i,j}^{\text{mid}} = \begin{cases} \text{BM}_k(i, 2j - 1) & \text{if } 4 \nmid k, \\ \text{BM}_k(i + 1, 2j - 1) - \gamma_{k,j} \text{BM}_k(i + 1, k') & \text{if } 4 \mid k \text{ and } i < k/4, \\ \text{BM}_k(i, 2j - 1) - \gamma_{k,j} \text{BM}_k(i, k') & \text{if } 4 \mid k \text{ and } i > k/4. \end{cases}$$

Example 5.8. Consider the case $k = 8$. Taking the determinant of the quadratic relations from [Theorem 5.3](#) and using the computation [\(3.23\)](#) and [Proposition 4.9\(2\)](#) yield

$$(\det P_8^{\text{mid}})^2 = (2\pi i)^{18} \det B_8^{\text{mid}} \det S_8^{\text{mid}} = \frac{-5^2 \pi^{18}}{2^4 3^2}.$$

On the other hand, by [Corollary 5.7](#) and the equalities $\gamma_{8,1} = 0$ and $\gamma_{8,3} = \frac{1}{8}$ from [\(3.6\)](#), we explicitly have

$$P_8^{\text{mid}} = \begin{pmatrix} (\pi i)^2 & \\ & -(\pi i)^3 \end{pmatrix} A \begin{pmatrix} 2^7 & \\ & 2^2 \end{pmatrix},$$

with

$$A = \int_0^\infty \begin{pmatrix} I_0(t)^2 K_0(t)^6 t & 2I_0(t)^2 K_0(t)^6 t^5 - I_0(t)^2 K_0(t)^6 t^3 \\ I_0(t)^3 K_0(t)^5 t & 2I_0(t)^3 K_0(t)^5 t^5 - I_0(t)^3 K_0(t)^5 t^3 \end{pmatrix} dt.$$

One can check numerically that $\det A$ is positive, and hence $\det A = 5\pi^4 / (2^{11} \cdot 3)$. Using the linear relations $BM_8^c(1, r) + BM_8^c(3, r) = 0$ from [Corollary 6.12](#) below for $r = 1, 5$, this verifies the numerical evaluation made in [\[Broadhurst and Roberts 2018, \(2.5\)\]](#).

5.3. Relation to the conjecture of Broadhurst and Roberts. Broadhurst and Roberts [\[2018, Section 5\]](#) use different normalizations to state their conjectural quadratic relations. Instead of the above matrices B_k and P_k , they consider matrices that we shall denote by B_k^{BR} and P_k^{BR} (the notation B_N and F_N is used in [\[loc. cit.\]](#), the index k being occupied by what is k' here). To compare their matrices with ours, we introduce auxiliary square matrices

$$U_{k'} = \text{antidiag}(1, \dots, 1), \quad R_{k'} = \text{antidiag}(i, i^2, \dots, i^{k'}), \quad T_{k'} = \text{diag}(-4, (-4)^2, \dots, (-4)^{k'})$$

of size k' , where the antidiagonal entries are listed down from the top corner. By [Propositions 4.6](#) and [5.4](#), the matrices B_k^{BR} and P_k^{BR} relate to ours as

$$U_{k'} B_k^{\text{BR}} U_{k'} = \frac{k!}{2^{k+1}} (i^{(k+i+j-1)} \cdot B_{k;i,j})_{1 \leq i, j \leq k'} \quad \text{and} \quad U_{k'} P_k^{\text{BR}} = \frac{1}{(-2\sqrt{\pi})^{k+1}} \cdot ((-4)^j i^j P_{k;i,j}^{\text{rd,mod}})_{1 \leq i, j \leq k'},$$

whence the identities

$$B_k^{\text{BR}} = -\frac{i^{(k+1)} k!}{2^{k+1}} {}^t R_{k'} \cdot B_k \cdot R_{k'} \quad \text{and} \quad P_k^{\text{BR}} = \frac{1}{(-2\sqrt{\pi})^{k+1}} {}^t R_{k'} \cdot P_k \cdot T_{k'}. \tag{5.9}$$

Besides, Broadhurst and Roberts [\[2018, page 7\]](#) define matrices $D_k^{\text{BR}} = (D_{k;i,j}^{\text{BR}})_{1 \leq i, j \leq k'}$ with rational coefficients and conjecture that, for all integers $k \geq 1$, the quadratic relation

$$P_k^{\text{BR}} \cdot D_k^{\text{BR}} \cdot {}^t P_k^{\text{BR}} = B_k^{\text{BR}}.$$

holds. In the direction of this conjecture, we obtain:

Corollary 5.10. *If $4 \nmid k$, then the matrix D_k defined as*

$$D_k = (-1)^k k! (T_{k'} \cdot S_k^{\text{mid}} \cdot T_{k'})^{-1}$$

satisfies

$$P_k^{\text{BR}} \cdot D_k \cdot {}^t P_k^{\text{BR}} = B_k^{\text{BR}}.$$

Proof. Under the assumption on k , we have $P_k = P_k^{\text{mid}}$ and $B_k = B_k^{\text{mid}}$. The statement then follows from the quadratic relations of [Theorem 5.3](#) and the equalities (5.9). \square

If $4 \mid k$, we set

$$B_k^{\text{BR}} = -\frac{i^{-(k+1)}k!}{2^{k+1}} R'_{k'} \cdot B_k^{\text{mid}} \cdot R'_{k'} \quad \text{and} \quad P_k^{\text{BR}} = \frac{1}{(-2\sqrt{\pi})^{k+1}} R'_{k'} \cdot P_k^{\text{mid}} \cdot T'_{k'},$$

where we denote by $R'_{k'}$ (resp. $T'_{k'}$) the matrix obtained from $R_{k'}$ (resp. $T_{k'}$) by deleting the row and the column of index $k/4$, that we consider as indexed by $[[1, k']]^2$. We define the matrix D'_k indexed by $[[1, k']]^2$ so that it satisfies the relation (see (3.16))

$$D'_k = (-1)^k k! (T'_{k'} \cdot S_k^{\text{mid}} \cdot T'_{k'})^{-1}.$$

Corollary 5.11. *If $4 \mid k$, then the matrix D'_k satisfies*

$$P_k^{\text{BR}} \cdot D'_k \cdot {}^t P_k^{\text{BR}} = B_k^{\text{BR}}.$$

Remark 5.12. Numerical computations show that, for all integers $k \leq 22$ that are not multiples of 4, the equality $D_k = D_k^{\text{BR}}$ holds. On the other hand, for $k = 8, 12, 16, 20$, the matrix D'_k coincides with the matrix D_k^{BR} obtained from D_k^{BR} by deleting the row and the column of index $k/4$. These computations also seem to suggest that $k!(S_k^{\text{mid}})^{-1}$ has integral coefficients. Is it true for all k ?

6. The full period matrices

In [Theorem 5.3](#), we emphasized quadratic relations for the middle periods to make the link with the conjecture of Broadhurst and Roberts. However, quadratic relations hold between the rapid-decay versus moderate periods

$$P_k^{\text{rd,mod}} : H_1^{\text{rd}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2) \otimes H_{\text{dR}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2) \rightarrow \mathbb{C}$$

and the moderate versus rapid-decay periods

$$P_k^{\text{mod,rd}} : H_1^{\text{mod}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2) \otimes H_{\text{dR,c}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2) \rightarrow \mathbb{C}.$$

Namely, it follows from [\[Fresán et al. 2023, Remark 2.15\]](#) that

$$(-2\pi i)^{k+1} B_k^{\text{rd,mod}} = P_k^{\text{rd,mod}} \cdot (S_k)^{-1} \cdot {}^t P_k^{\text{mod,rd}},$$

where $B_k^{\text{rd,mod}}$ and S_k stand for the complete Betti and de Rham intersection pairings. In this section, we consider the bases $\mathcal{B}_{k,c}$, \mathcal{B}_k and $(\alpha_i)_i, (\beta_i)_i$ as defined in [Section 3.1](#) and [Theorem 4.7](#), and we compute the entries of the matrices of $P_k^{\text{rd,mod}}$ and $P_k^{\text{mod,rd}}$ which do not appear in P_k^{mid} .

6.1. The complete period matrix $\mathbf{P}^{\text{rd,mod}}$. In order to finish the computation of the complete period matrix $(\mathbf{P}_{k;i,j}^{\text{rd,mod}})_{0 \leq i, j \leq k'}$, we are left, according to (5.1) and Proposition 5.4, with computing the terms $\mathbf{P}_{k;i,0}^{\text{rd,mod}}$ for $i = 1, \dots, k'$. In such a range, we consider regularized Bessel moments defined as follows, according to the expansions (2.13) and (2.14).

Definition 6.1 (regularized Bessel moments). For all i such that $0 \leq i \leq k'$, the functions

$$G_{k,i}(\varepsilon) = \int_{\varepsilon}^{\infty} I_0(t)^i K_0(t)^{k-i} \frac{dt}{t} + \frac{(-1)^{k-i}}{k-i+1} (\gamma + \log(\varepsilon/2))^{k-i+1}$$

are holomorphic on small sectors containing $\varepsilon \in \mathbb{R}_{>0}$ and have finite limit as $\varepsilon \rightarrow 0^+$. The regularized Bessel moments are defined as

$$\text{BM}_k^{\text{reg}}(i, -1) = (-1)^{k-i} 2^{k+1} (\pi i)^i \lim_{\varepsilon \rightarrow 0^+} G_{k,i}(\varepsilon).$$

Proposition 6.2. *We have*

$$\mathbf{P}_k^{\text{rd,mod}}(\alpha_i, \omega_0) = \text{BM}_k^{\text{reg}}(i, -1), \quad \text{for } 1 \leq i \leq k'.$$

Proof. We argue as for (5.5) and (5.6). If $1 \leq i \leq k'$, let

$$\mathbf{P}'_k(\alpha_i, \omega_0) = \mathbf{P}_k^{\text{rd,mod}}\left(\alpha_i + \frac{(-1)^{k-i} (k-i)!}{k-i+2} \alpha_0, \omega_0\right),$$

so that

$$\mathbf{P}_k^{\text{rd,mod}}(\alpha_i, \omega_0) = \mathbf{P}'_k(\alpha_i, \omega_0) - \frac{(-1)^{k-i} (k-i)!}{k-i+2} \mathbf{P}_k^{\text{rd,mod}}(\alpha_0, \omega_0). \tag{6.3}$$

On the one hand, by scaling the chains c_+ and c_0 by $\varepsilon \in \mathbb{R}_{>0}$ and letting $\varepsilon' = 2\sqrt{\varepsilon}$, we find

$$\begin{aligned} \mathbf{P}'_k(\alpha_i, \omega_0) &= \int_{c_{+,\varepsilon}} \langle e_0^i e_1^{k-i}, v_0^k \rangle_{\text{top}} \frac{dz}{z} + \sum_{a=1}^{k-i+1} C_{k-i+1}(a) \int_{c_{0,\varepsilon}^a} \langle e_0^{i-1} e_1^{k-i+1}, v_0^k \rangle_{\text{top}} \frac{dz}{z} \\ &= (-1)^{k-i} 2^k (\pi i)^i \int_{\varepsilon}^{\infty} I_0(2\sqrt{z})^i K_0(2\sqrt{z})^{k-i} \frac{dz}{z} \\ &\quad + (-1)^{k-i+1} 2^k (\pi i)^{i-1} \sum_{a=1}^{k-i+1} C_{k-i+1}(a) \int_{c_{0,\varepsilon}^a} I_0(2\sqrt{z})^{i-1} K_0(2\sqrt{z})^{k-i+1} \frac{dz}{z} \\ &= (-1)^{k-i} 2^{k+1} (\pi i)^i \int_{\varepsilon'}^{\infty} I_0(t)^i K_0(t)^{k-i} \frac{dt}{t} \\ &\quad + (-1)^{k-i+1} 2^{k+1} (\pi i)^{i-1} \sum_{a=1}^{k-i+1} C_{k-i+1}(a) \int_{c_{0,\varepsilon'}^a} I_0(t)^{i-1} K_0(t)^{k-i+1} \frac{dt}{t} \\ &= (-1)^{k-i} 2^{k+1} (\pi i)^i \int_{\varepsilon'}^{\infty} I_0(t)^i K_0(t)^{k-i} \frac{dt}{t} \\ &\quad + 2^{k+1} (\pi i)^{i-1} \sum_{a=1}^{k-i+1} C_{k-i+1}(a) \int_{c_{0,\varepsilon'}^a} [(\gamma + \log(t/2))^{k-i+1} + O(t^2 \log^{k-i+1} t)] \frac{dt}{t}. \end{aligned}$$

Lemma 4.1 yields

$$\begin{aligned} \sum_{a=1}^{k-i+1} C_{k-i+1}(a) \int_{c_{0,\varepsilon'}^{a/2}} (\gamma + \log(t/2))^{k-i+1} \frac{dt}{t} \\ = \frac{1}{k-i+2} \sum_{a=1}^{k-i+1} C_{k-i+1}(a) [(\gamma + \log(\varepsilon'/2) + \pi i a)^{k-i+2} - (\gamma + \log(\varepsilon'/2))^{k-i+2}] \\ = \frac{\pi i}{k-i+1} (\gamma + \log(\varepsilon'/2))^{k-i+1} + \frac{(-1)^{k-i} (k-i)!}{k-i+2} (\pi i)^{k-i+2}. \end{aligned}$$

Letting $\varepsilon \rightarrow 0^+$, one obtains

$$P'_k(\alpha_i, \omega_0) = \text{BM}_k^{\text{reg}}(i, -1) + \frac{(-1)^{k-i} (k-i)!}{k-i+2} (2\pi i)^{k+1}.$$

On the other hand, $P_k^{\text{rd,mod}}(\alpha_0, \omega_0) = (2\pi i)^{k+1}$ as computed in (5.1), and (6.3) gives the desired formula. □

6.2. The complete period matrix $P_k^{\text{mod,rd}}$. The period matrix $P_k^{\text{mod,rd}}$ is defined by

$$P_{k;i,j}^{\text{mod,rd}} = P_k^{\text{mod,rd}}(\beta_i, (\widehat{m}_j, \omega'_j)), \quad \begin{cases} 0 \leq i, j \leq k' & \text{if } 4 \nmid k, \\ \begin{cases} 0 \leq i \leq k/2, i \neq 1 \\ 0 \leq j \leq k' \end{cases} & \text{if } 4 \mid k, \end{cases}$$

(see the notation of (3.9) for ω'_j) and is nondegenerate (argument similar to that for $P_k^{\text{rd,mod}}$). According to [Fresán et al. 2023, Section 3.f], its middle part is equal to P_k^{mid} already computed:

- If $4 \nmid k$, we are left with computing $P_{k;i,j}^{\text{mod,rd}}$ for $i, j \in [0, k']$ and i or $j = 0$.
- If $4 \mid k$, we are left with computing $P_{k;i,j}^{\text{mod,rd}}$ with

$$\begin{cases} i = 0, 2, \dots, k/2, j = 0, k/4, \\ i = 0, k/2, j = 1, \dots, k', j \neq k/4. \end{cases}$$

Definition 6.4 (regularized Bessel moments, continued). If $4 \mid k$ and $k/4 < j \leq k'$, the function $\varepsilon \mapsto H_{k,j}(\varepsilon)$ defined by (see (3.5))

$$H_{k,j}(\varepsilon) = \int_0^{1/\varepsilon} (I_0(t)K_0(t))^{k/2} \left(t^{2j} - \frac{\gamma_{k,j}}{2^{k/2-2j}} t^{k/2} \right) \frac{dt}{t} - \frac{1}{2^{k-2j+1}} \sum_{n=1}^j \frac{\gamma_{k,j-n}}{n(4\varepsilon^2)^n}$$

is holomorphic near $\mathbb{R}_{>0}$ with finite limit when $\varepsilon \rightarrow 0^+$. We set

$$\text{BM}_k^{\text{reg}}(k/2, 2j-1) = 2^{k-2j+1} (\pi i)^{k/2} \lim_{\varepsilon \rightarrow 0^+} H_{k,j}(\varepsilon).$$

Proposition 6.5. (1) If $4 \nmid k$, we have:

- (a) $P_{k;i,0}^{\text{mod,rd}} = (-1)^k \delta_{i,0}, i = 0, \dots, k'$.
- (b) $P_{k;0,j}^{\text{mod,rd}} = \text{BM}_k(0, 2j-1), j = 1, \dots, k'$.

(2) If $4 \mid k$, we have:

- (a) $P_{k;i,0}^{\text{mod,rd}} = \delta_{i,0}, i = 0, \dots, k/2.$
- (b) $P_{k;i,k/4}^{\text{mod,rd}} = -(2\pi i)^{k/2} 2^{k/2} \binom{k}{k/2}^{-1} \delta_{i,k/2}, i = 0, \dots, k/2.$
- (c) $P_{k;0,j}^{\text{mod,rd}} = \text{BM}_k(0, 2j - 1) - \gamma_{k,j} \text{BM}_k(0, k'), j = 1, \dots, k', j \neq k/4.$
- (d) $P_{k;k/2,j}^{\text{mod,rd}} = \begin{cases} \text{BM}_k(k/2, 2j - 1) & 1 \leq j < k/4, \\ \text{BM}_k^{\text{reg}}(k/2, 2j - 1) & k/4 < j \leq k'. \end{cases}$

Proof. For (a) and (a), we note that

$$P_k^{\text{mod,rd}}(\beta_i, (\widehat{m}_0, 0)) = \langle e_0^i e_1^{k-i}, e_0^k \rangle_{\text{top}} = (-1)^k \delta_{i,0}.$$

Similarly, for (b) (see Proposition 3.8 and [Fresán et al. 2023, Proposition 3.18]),

$$P_k^{\text{mod,rd}}(\beta_i, (\widehat{m}_{k/4}, 0)) = -\langle e_0^i e_1^{k-i}, 2^k (\pi i)^{k/2} (e_0 e_1)^{k/2} \rangle_{\text{top}} = -2^k (\pi i)^{k/2} \binom{k}{k/2}^{-1} \delta_{i,k/2}.$$

For the rest, since the coefficients of the cycle β_i in terms of $\{u_a\}$, have logarithmic growth at 0, and $\widehat{m}_{j,0}$ is holomorphic and vanishes (Lemma 3.3), the contribution $\lim_{z \rightarrow 0} \langle e_0^i e_1^{k-i}, \widehat{m}_{j,0} \rangle_{\text{top}}(z)$ of $\langle \beta_i, \widehat{m}_{j,0} \rangle_{\text{top}}$ in [Fresán et al. 2023, Proposition 3.18] in the period pairing is zero. For $i \neq k/2$ or $i = k/2, 1 \leq j < k/4$, the coefficient $(-1)^{k-i} (\pi i)^i \xi_{j,i}$ of $\langle \beta_i, \widehat{m}_{j,\infty} \rangle_{\text{top}}$ (see (3.13)) is holomorphic near $\mathbb{R}_{>0}$ and vanishes at ∞ , so the contribution of $\widehat{m}_{j,\infty}$ is zero too. In this case, one therefore has

$$P_k^{\text{mod,rd}}(\beta_i, (\widehat{m}_j, \omega'_j)_{\text{top}}) = \int_{\mathbb{R}_+} \langle e_0^i e_1^{k-i}, \omega'_j \rangle_{\text{top}} = \text{BM}_k(i, 2j - 1) - \gamma_{k,j} \text{BM}_k(i, k')$$

as in the proof of Proposition 5.4. This completes the cases (b), (c) and the first part of (d).

It remains to check the second part of (d), with $k/4 < j \leq k'$. Let

$$\xi = (-4\pi i I_0 K_0)^{k/2} z^{j-1} dz \quad \text{and} \quad \eta = (-4\pi i I_0 K_0)^{k/2} z^{k/4-1} dz.$$

We have

$$\langle (e_0 e_1)^{k/2}, \omega'_j \rangle_{\text{top}} = \langle (e_0 e_1)^{k/2}, (\omega_j + \text{res}_{z=\infty}(\omega_j) \omega_{k/4}) \rangle_{\text{top}} = \xi + \text{res}_{z=\infty}(\xi) \eta.$$

Note that $\widehat{m}_{j,\infty}$ is defined so that $d\langle (e_0 e_1)^{k/2}, \widehat{m}_{j,\infty} \rangle_{\text{top}} = \xi + \text{res}_{z=\infty}(\xi) \eta$ and $\langle (e_0 e_1)^{k/2}, \widehat{m}_{j,\infty} \rangle_{\text{top}}$ has no constant term in the fractional Laurent series expansion in $1/z$. We then obtain

$$P_k^{\text{mod,rd}}(\alpha_{k/2}, (\widehat{m}_j, \omega'_j)) = \lim_{\tau \rightarrow \infty} \int_0^\tau \xi + \text{res}_{z=\infty}(\xi) \eta - \langle (e_0 e_1)^{k/2}, \widehat{m}_{j,\infty} \rangle_{\text{top}}(\tau) = \text{BM}_k^{\text{reg}}(k/2, 2j - 1). \quad \square$$

Example 6.6. Respectively for $k = 3, 4$, the complete quadratic relations lead to the equalities of the regularized Bessel moments

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \int_\varepsilon^\infty I_0(t) K_0(t)^2 \frac{dt}{t} + \frac{1}{3} (\gamma + \log \varepsilon/2)^3 &= \frac{3}{2} \int_0^\infty K_0(t)^3 t dt \cdot \int_0^\infty I_0(t) K_0(t)^2 t dt, \\ \lim_{\varepsilon \rightarrow 0^+} \int_\varepsilon^\infty I_0(t) K_0(t)^3 \frac{dt}{t} - \frac{1}{4} (\gamma + \log \varepsilon/2)^4 &= \frac{\pi^4}{120}. \end{aligned}$$

Remark 6.7 (determinant of P_k). A formula for the determinant of the matrix P_k in Proposition 5.2, which implies in particular its nonvanishing, was conjectured by Broadhurst and Mellit [2016] and Broadhurst [2016, Conjecture 4 and 7] and proved by Zhou [2018b]. Since $\det S_k$ is computed in Remark 3.22 and the determinant of the Betti intersection matrix is computed in Proposition 4.9 in order to prove its nonvanishing, the quadratic relations also lead to a computation of $\det P_k$ up to sign.

Remark 6.8 (dimension of the linear span of Bessel moments). Zhou [2019b, Theorem 1.2], shows that, for each i such that $1 \leq i \leq k'$ (resp. for $i = 0$), the \mathbb{Q} -vector space E_i spanned by the Bessel moments $BM_k(i, 2j - 1)$ for all $j \geq 1$ has dimension at most k' (resp. at most $k' + 1$). From the point of view of the present paper, the upper bound $\dim_{\mathbb{Q}} E_i \leq k'$ when $1 \leq i \leq k'$ is a direct consequence of the fact that the $BM_k(i, 2j - 1)$ result from pairing the fixed rapid decay cycle α_i with the de Rham cohomology classes $\omega_j = [z^j u_0 dz/z]$. Provided $j \geq 1$ and k is not a multiple of 4, all these ω_j lie in the middle de Rham cohomology $H_{\text{dR},\text{mid}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$, which has dimension k' . If k is a multiple of 4, the periods $P_k^{\text{mid}}(\alpha_i, \cdot)$ are generated by $BM_k(i, 2j - 1) - \gamma_{k,j} BM_k(i, k')$ for $j \in [1, k']$, with $\gamma_{k,k/4} = 1$. For each $j \geq 1$, the class $(z^j - \gamma_{k,j} z^{k/4})u_0 dz/z$ lies in $H_{\text{dR},\text{mid}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$ and has period $BM_k(i, 2j - 1) - \gamma_{k,j} BM_k(i, k')$ when paired with α_i . Therefore, $\dim_{\mathbb{Q}} E_i \leq k'$ holds. In the case $i = 0$, the periods $P_k^{\text{mod,rd}}(\alpha_0, \cdot)$ are generated by

$$1 \quad \text{and} \quad BM_k(0, 2j - 1) - \gamma_{k,j} BM_k(0, k') \quad (1 \leq j \leq k').$$

Again, for each $j \geq 1$, there exists \widehat{m}_j such that the class

$$(\widehat{m}_j, (z^j - \gamma_{k,j} z^{k/4})u_0 dz/z)$$

lies in $H_{\text{dR},\text{c}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$ and has period

$$BM_k(0, 2j - 1) - \gamma_{k,j} BM_k(0, k')$$

when paired with α_0 . Hence, $\dim_{\mathbb{Q}} E_0 \leq k' + 1$, as this is the dimension of de Rham cohomology with compact support.

6.3. Linear relations. When k is even, there is a unique nontrivial linear relation among the $k' + 2$ classes $(\beta_j)_{0 \leq j \leq k/2}$ in $H_1^{\text{mod}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$. We determine explicitly the moderate 2-chain that yields this relation.

Lemma 6.9. For two integers n, r with $0 \leq r \leq 2n$,

$$\sum_{i=0}^{\min\{n, 2n-r\}} (-1)^i \binom{n}{i} \binom{2n-i}{r} = \begin{cases} 0 & \text{if } 0 \leq r \leq n - 1, \\ \binom{n}{r-n} & \text{if } n \leq r \leq 2n. \end{cases}$$

Proof. We provide two approaches; the second one has been provided by Hao-Yun Yao of NTU.

Let a_r be the sum in the left hand side. Then the polynomial $f(x) = \sum_{r=0}^{2n} a_r x^r$ is the coefficient of y^n in the expansion of

$$g(x, y) = (y - 1)^n \sum_{r=0}^n (x + 1)^{2n-r} y^r = (x + 1)^n (y - 1)^n \sum_{j=0}^n (x + 1)^{n-j} y^j.$$

Thus one obtains

$$f(x) = (x + 1)^n ((x + 1) - 1)^n = x^n (x + 1)^n$$

and the assertion follows.

Alternatively, let S be the set of subsets of $\{1, 2, \dots, 2n\}$ consisting of r elements and define $T = \{A \in S \mid \{1, 2, \dots, n\} \subset A\}$. Members of T are obtained by choosing $(r - n)$ elements from $\{n + 1, \dots, 2n\}$ so that the cardinality $|T|$ equals 0 if $0 \leq r < n$ or $\binom{n}{r-n}$ if $n \leq r \leq 2n$. Let $S_c = \{A \in S \mid c \notin A\}$. One has $S = T \sqcup (S_1 \cup \dots \cup S_n)$ and $|S_{c_1} \cap \dots \cap S_{c_i}| = \binom{2n-i}{r}$ for $1 \leq c_1 < \dots < c_i \leq 2n$. The inclusion-exclusion principle then gives the formula. □

Proposition 6.10. *Let $\tilde{\mathbb{P}}^1$ be the real oriented blow-up of \mathbb{P}^1 along $\{0, \infty\}$ as in Section 2.3, and consider the simplicial 2-chain*

$$\rho : \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x, y, x + y \leq 1\} \rightarrow \tilde{\mathbb{P}}^1, \quad \rho(x, y) = \tan \frac{\pi(x + y)}{2} \exp(4i \tan^{-1}(y/x)),$$

which covers $\tilde{\mathbb{P}}^1$ once. If k is even, the singular chain

$$\Delta = \rho \otimes \sum_{i=0}^{k/2} (-1)^i \binom{k/2}{i} e_0^i e_1^{k-i}$$

is of moderate growth, and the relation

$$\sum_{i=0}^{\lfloor (k-2)/4 \rfloor} \binom{k/2}{2i+1} \beta_{2i+1} = 0$$

holds in $H_1^{\text{mod}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$.

Proof. It follows as direct consequences of the monodromy relation and Lemma 6.9, which also imply that

$$\frac{-1}{2} \partial \Delta = \sum_{i=0}^{\lfloor (k-2)/4 \rfloor} \binom{k/2}{2i+1} \beta_{2i+1}. \quad \square$$

If $4 \mid (k + 2)$, choosing the principal determination of $w^{1/2}$ near $\mathbb{R}_{>0}$, we write, in a way similar to (3.4), the asymptotic expansion

$$2^k (I_0(t) K_0(t))^{k/2} \sim w^{k/4} \sum_{n=0}^{\infty} \gamma'_{k, n+k/4} w^n.$$

As in Definition 6.4, one checks that, for $j \in [(k + 2)/4, k']$, the difference

$$H_{k,j}(\varepsilon) = \int_0^{1/\varepsilon} (I_0(t)K_0(t))^{k/2} t^{2j} \frac{dt}{t} - \frac{1}{2^{k-2j+1}\varepsilon} \sum_{n=0}^j \frac{\gamma'_{k,j-n-1/2}}{(2n+1)(4\varepsilon^2)^n}$$

is holomorphic near $\mathbb{R}_{>0}$ with finite limit as $\varepsilon \rightarrow 0$. Set

$$\text{BM}_k^{\text{reg}}(k/2, 2j - 1) = -2^{k-2j+1}(\pi i)^{k/2} \lim_{\sigma \rightarrow 0^+} H_{k,j}(\sigma).$$

In this case, one has

$$P_k^{\text{mod,rd}}(\beta_{k/2}, (\widehat{m}_j, \omega'_j)) = \begin{cases} 0 & \text{if } j = 0, \\ \text{BM}_k(k/2, 2j - 1) & \text{if } 1 \leq j \leq (k - 2)/4, \\ \text{BM}_k^{\text{reg}}(k/2, 2j - 1) & \text{if } (k + 2)/4 \leq j \leq k' \end{cases}$$

as in Proposition 6.5(2). To unify the situations, we introduce the compactly supported version of the Bessel moments.

Definition 6.11. Assume k is even. We set

$$\text{BM}_k^c(i, 2j - 1) = \begin{cases} \text{BM}_k^{\text{reg}}(k/2, 2j - 1) & \text{if } \begin{cases} i = k/2, \\ \lfloor k/4 \rfloor + 1 \leq j \leq k', \end{cases} \\ \text{BM}_k(i, 2j - 1) - \gamma_{k,j} \text{BM}_k(i, k') & \text{if } 4 \mid k, 0 \leq i \leq k', \\ \text{BM}_k(i, 2j - 1) & \text{otherwise.} \end{cases}$$

Corollary 6.12 (sum rule identities). Assume k is even and set $k'' = \lfloor (k - 1)/4 \rfloor$. The linear relations among Bessel moments

$$\sum_{i=0}^{k''} \binom{k/2}{2i+1} \text{BM}_k^c(2i + 1, 2j - 1) = 0 \tag{6.13}$$

hold for all j such that

$$\begin{cases} 1 \leq j \leq 2k'' & \text{if } 4 \mid (k + 2), \\ 1 \leq j \leq 2k'' + 1, j \neq k/4 & \text{if } 4 \mid k. \end{cases}$$

Remark 6.14. For $j \in [1, k']$, these *sum rule identities* were proved by Zhou by analytic means; see [Zhou 2019a, (1.3)] for $4 \mid (k + 2)$ and [loc. cit., (1.5)] for $4 \mid k$, where the second argument of the Bessel moments is also allowed to be even. Our proof, closer to the spirit of the Kontsevich–Zagier period conjecture, produces the relation (6.13) simply from the Stokes formula.

7. Comparison of period structures

In [Fresán et al. 2022], we have introduced the Nori motives M_k defined over \mathbb{Q} , whose definition we recall in Section 7.1, and we have explained how the Hodge filtration of their Hodge realization can be computed in terms of symmetric powers of the Kloosterman connection on \mathbb{G}_m and their irregular Hodge filtration. We aim at making Deligne’s conjecture on critical values explicit for them in Section 8. In this

section as a preparation, we focus on the comparisons of various \mathbb{Q} -structures. For that purpose, we only retain from each Nori motive M_k

- its *period realization* (over $\text{Spec } \mathbb{Q}$, see [Section A4](#)), that is, the \mathbb{Q} -vector spaces $(M_k)_{\text{dR}}$ (de Rham realization) and $(M_k)_{\text{B}}$ (Betti realization), together with the comparison isomorphism $\text{comp} : \mathbb{C} \otimes_{\mathbb{Q}} (M_k)_{\text{B}} \simeq \mathbb{C} \otimes_{\mathbb{Q}} (M_k)_{\text{dR}}$, and
- the action F_{∞} on $(M_k)_{\text{B}}$ induced by geometric complex conjugation.

On the other hand, the computations of the previous sections lead us to define a (transposed) period structure (over $\text{Spec } \mathbb{Q}$) as follows:

- The de Rham \mathbb{Q} -vector space $H_{\text{dR,mid}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)_{\mathbb{Q}}$ is the \mathbb{Q} -vector space generated by the family $\mathcal{B}_{k,\text{mid}}$ of [Corollary 3.11](#).
- The (dual) Betti \mathbb{Q} -vector space $H_1^{\text{mid}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)_{\mathbb{Q}}$ is the \mathbb{Q} -vector space generated by the family (β_j) of twisted cycles of [Theorem 4.7\(3\)](#).
- The period pairing is the pairing P_k^{mid} .

There is an obvious notion of morphism of period structures.

Theorem 7.1. *The period structure $((M_k)_{\text{dR}}, (M_k)_{\text{B}}, \text{comp})$ of the motive M_k is isomorphic to the transpose of $(H_{\text{dR,mid}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)_{\mathbb{Q}}, H_1^{\text{mid}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)_{\mathbb{Q}}, P_k^{\text{mid}})$.*

Furthermore, we will give an explicit description under this correspondence of the action F_{∞} on $H_1^{\text{mid}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)_{\mathbb{Q}}$.

We proceed in two steps. Firstly in [Section 7.1](#), in an analogous way to the method used in [\[Fresán et al. 2022\]](#), we realize the mixed Hodge structure corresponding to M_k together with its associated period structure as the exponential mixed Hodge structure and the associated period structure attached to a function on a smooth variety equipped with the action of a finite group. We also analyze the automorphism of the Betti fiber of this exponential mixed Hodge structure induced by the complex conjugation on the variety underlying M_k . In particular, we avoid computing periods directly on the variety underlying the motive M_k . The tools for this part are developed in the appendix in which the period realization of exponential mixed Hodge structures is investigated, extending the focus on the de Rham realization in the appendix of [\[loc. cit.\]](#). Then in [Section 7.2](#), we compare these objects to those of the previous sections by making explicit differential forms of higher degree and higher dimensional twisted cycles that correspond to the bases obtained for $\text{Sym}^k \text{Kl}_2$ there. The period matrix computed in [Proposition 5.4](#) is thus a period matrix for M_k , and the action of the conjugation is explicit since the Bessel moments we consider are either real or purely imaginary.

7.1. The motives M_k . Let $y = (y_1, \dots, y_k)$ be the Cartesian coordinates of the torus \mathbb{G}_m^k defined over \mathbb{Q} . Upon \mathbb{G}_m^k , consider the action of $\mathfrak{S}_k \times \mu_2$ where the symmetric group \mathfrak{S}_k acts by permuting the variables y and the group $\mu_2 = \{\pm 1\}$ acts as $y_i \mapsto \pm y_i$. Denote by $g_k : \mathbb{G}_m^k \rightarrow \mathbb{A}^1$ the regular function given by the

Laurent polynomial

$$g_k(y_1, \dots, y_k) = \sum_{i=1}^k \left(y_i + \frac{1}{y_i} \right)$$

and let $\mathcal{X}_0 = (g_k)$ be the associated closed subvariety of \mathbb{G}_m^k . Then \mathcal{X}_0 is invariant under the action of $\mathfrak{S}_k \times \mu_2$ and hence the latter acts on various cohomology spaces, or on the (Nori) motives $H^{k-1}(\mathcal{X}_0)$ and $H_c^{k-1}(\mathcal{X}_0)$ of degree $k-1$ of \mathcal{X}_0 . Let $\text{sgn}: \mathfrak{S}_k \times \mu_2 \rightarrow \mathbb{Q}^\times$ be the product of the sign character on \mathfrak{S}_k and the trivial one on μ_2 . We define the pure motive M_k of weight $k+1$ over \mathbb{Q} by taking the sgn -isotypic part

$$M_k = \text{gr}_{k+1}^W [H_c^{k-1}(\mathcal{X}_0)(-1)]^{\mathfrak{S}_k \times \mu_2, \text{sgn}},$$

where W indicates the weight filtration; see [Fresán et al. 2022, (3.1)]. The Betti realization $(M_k)_B$ and the de Rham realization $(M_k)_{\text{dR}}$ of M_k are the \mathbb{Q} -vector spaces of dimension $\lfloor (k-1/2) \rfloor - \delta_{4\mathbb{Z}}(k)$ obtained by replacing H_c by the corresponding cohomology functors (with compact support).

Let \mathcal{X} be the complex variety defined by the base change of \mathcal{X}_0 to \mathbb{C} . It is shown in [loc. cit.] that the base change

$$\mathbb{C} \otimes (M_k)_{\text{dR}} = \text{gr}_{k+1}^W [H_{\text{dR},c}^{k-1}(\mathcal{X})(-1)]^{\mathfrak{S}_k \times \mu_2, \text{sgn}}$$

of the de Rham realization $(M_k)_{\text{dR}}$ is identified with $H_{\text{dR},\text{mid}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)$ discussed in the previous sections.

The pair (U, \tilde{f}_k) . We consider the torus $U_0 = \mathbb{G}_{m,t} \times \mathbb{G}_m^k$ over \mathbb{Q} with its \mathbb{C} -extension U , endowed with

- the action of \mathfrak{S}_k permuting the coordinates on \mathbb{G}_m^k ,
- the action of μ_2 sending (t, y) to $\pm(t, y)$,
- the involution ι sending t to t and y_i to $-y_i$ for each i ,
- the antilinear involution conj on the analytic manifold $U(\mathbb{C})$ induced by the conjugation of coordinates, which commutes with all the above actions.

Let $\tilde{f}_k: U_0 = \mathbb{G}_{m,t} \times \mathbb{G}_m^k \rightarrow \mathbb{A}^1$ denote the Laurent polynomial $(t/2) \cdot g_k$, where g_k is defined above. Then \tilde{f}_k is left invariant by the action of \mathfrak{S}_k , μ_2 and conj , and satisfies $\iota^* \tilde{f}_k = -\tilde{f}_k$.

Conjugation acts on $\Omega^p(U)$ by changing a p -form $\omega(t, y)$ to $\overline{\omega(\bar{t}, \bar{y})}$. Therefore, $\text{conj}^*(d\tilde{f}_k) = d\tilde{f}_k$ and conj^* induces an involution on $H_{\text{dR},c}^{k+1}(U, \tilde{f}_k)$, $H_{\text{dR}}^{k+1}(U, \tilde{f}_k)$ and $H_{\text{dR},\text{mid}}^{k+1}(U, \tilde{f}_k)$ which commutes with the action induced by $\mathfrak{S}_k \times \mu_2$ and ι^* .

According to [Fresán et al. 2022, Theorem 3.8], the associated mixed Hodge structure $(M_k)_H$ is identified with the exponential mixed Hodge structure $H_{\text{mid}}^{k+1}(U, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \text{sgn}}$. In particular, they have isomorphic period realizations; see the Appendix.

The de Rham realization $(M_k)_{\text{dR}}$. The pair (U, \tilde{f}_k) , together with the action of $\mathfrak{S}_k \times \mu_2$, is the \mathbb{C} -extension of the pair (U_0, \tilde{f}_k) defined over \mathbb{Q} . As explained in Section A4, the de Rham cohomologies $H_{\text{dR},?}^{k+1}(U, \tilde{f}_k)$ ($? = \emptyset, c, \text{mid}$) are also endowed with the \mathbb{Q} -structure $H_{\text{dR},?}^{k+1}(U_0, \tilde{f}_k)$, and therefore so are the sgn -isotypic components $H_{\text{dR},?}^{k+1}(U, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \text{sgn}}$ with the \mathbb{Q} -structure $H_{\text{dR},?}^{k+1}(U_0, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \text{sgn}}$.

Lemma 7.2. *The isomorphism $\mathrm{gr}_{k+1}^W \mathbf{H}_{\mathrm{dR},c}^{k-1}(\mathcal{X})(-1)^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}} \simeq \mathbf{H}_{\mathrm{dR},\mathrm{mid}}^{k+1}(U, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}}$ of [Fresán et al. 2022, Theorem 3.8] identifies the \mathbb{Q} -vector spaces $(\mathbf{M}_k)_{\mathrm{dR}}$ and $\mathbf{H}_{\mathrm{dR},\mathrm{mid}}^{k+1}(U_0, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}}$.*

Proof. We start from Example A.28, on noting that $\tilde{f}_k = t g_k$. We thus obtain isomorphisms (setting $Z_0 = \mathbb{A}_t^1 \times \mathcal{X}_0$)

$$\mathbf{H}_{\mathrm{dR},c}^{k+1}(Z_0)^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}} \simeq \mathbf{H}_{\mathrm{dR},c}^{k+1}(\mathbb{A}_t^1 \times \mathbb{G}_m^k/\mathbb{Q}, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}} \simeq \mathbf{H}_{\mathrm{dR},c}^{k+1}(U_0, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}},$$

where the second isomorphism is obtained as in the proof of [Fresán et al. 2022, Theorem 3.8]. After extension of scalars from \mathbb{Q} to \mathbb{C} , these isomorphisms become the de Rham part of the isomorphisms considered in the proof of [loc. cit.]. Since the left-hand sides are defined from Nori motives, the weight filtration is defined on them. We note that $(\mathbf{M}_k)_{\mathrm{dR}} \simeq \mathbf{H}_{\mathrm{dR},c}^{k+1}(Z_0)^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}}/W_k \mathbf{H}_{\mathrm{dR},c}^{k+1}(Z_0)^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}}$ because this holds after extension of scalars. Furthermore, the composition

$$\mathbf{H}_{\mathrm{dR},c}^{k+1}(Z_0)^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}} \xrightarrow{\sim} \mathbf{H}_{\mathrm{dR},c}^{k+1}(U_0, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}} \rightarrow \mathbf{H}_{\mathrm{dR}}^{k+1}(U_0, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}}$$

whose image is $\mathbf{H}_{\mathrm{dR},\mathrm{mid}}^{k+1}(U_0, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}}$ by definition, has kernel equal to $W_k \mathbf{H}_{\mathrm{dR},c}^{k+1}(Z_0)^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}}$, because this holds after extension of scalars. This proves the lemma. \square

The Betti realization $(\mathbf{M}_k)_{\mathrm{B}}$. We use the notation and apply the results of Section A3, more specifically Formulas (A.19), (A.20) and (A.21), to the pair (U, \tilde{f}_k) . Since \tilde{f}_k is invariant under the action of $\mathfrak{S}_k \times \mu_2$ on U , as well as under the action of complex conjugation $\mathrm{conj} : U(\mathbb{C}) \rightarrow U(\mathbb{C})$, the families of supports Φ_{rd} and Φ_{mod} are also invariant under these actions. There is thus a natural action of $\mathfrak{S}_k \times \mu_2$ on $\mathbf{H}_{\mathbb{C}}^{k+1}(\tilde{U}_{\mathrm{rd}}(D), \mathbb{Q})$ and $\mathbf{H}_{\mathbb{C}}^{k+1}(\tilde{U}_{\mathrm{mod}}(D), \mathbb{Q})$. We thus have

$$\begin{aligned} (\mathbf{M}_k)_{\mathrm{B}} &\simeq \mathbf{H}_{\mathrm{B},\mathrm{mid}}^{k+1}(U, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}} \\ &= \mathrm{im}[\mathbf{H}_{\mathrm{B},c}^{k+1}(U, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}} \rightarrow \mathbf{H}_{\mathrm{B}}^{k+1}(U, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}}] \\ &\simeq \mathrm{im}[\mathbf{H}_{\mathbb{C}}^{k+1}(\tilde{U}_{\mathrm{rd}}(D), \mathbb{Q})^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}} \rightarrow \mathbf{H}_{\mathbb{C}}^{k+1}(\tilde{U}_{\mathrm{mod}}(D), \mathbb{Q})^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}}]. \end{aligned} \tag{7.3}$$

Lemma 7.4. *Under the identification (7.3), the action of the conjugation on $(\mathbf{M}_k)_{\mathrm{B}}$ coincides with that on $\mathbf{H}_{\mathrm{B},\mathrm{mid}}^{k+1}(U, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}}$.*

Proof. We start from the identification of Example A.31. Since the action of $\mathfrak{S}_k \times \mu_2$ on $\mathbb{A}_t^1 \times \mathbb{G}_m^k$ commutes with conj , we obtain an identification

$$(\mathbf{M}_k)_{\mathrm{B}} \simeq \mathbf{H}_{\mathrm{B},\mathrm{mid}}^{k+1}(\mathbb{A}_t^1 \times \mathbb{G}_m^k, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}}$$

compatible with conj^* . Since the isomorphism (see [Fresán et al. 2022, Proof of Theorem 3.8])

$$\mathbf{H}_{\mathrm{B},\mathrm{mid}}^{k+1}(\mathbb{A}_t^1 \times \mathbb{G}_m^k, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}} \simeq \mathbf{H}_{\mathrm{B},\mathrm{mid}}^{k+1}(U, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}}$$

is clearly compatible with conj^* the desired result follows. \square

The Betti realization is also given by (see (A.22))

$$(\mathbf{M}_k)_{\mathrm{B}} \simeq \mathrm{im}[\mathbf{H}_{k+1}^{\mathrm{rd}}(U, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}} \rightarrow \mathbf{H}_{k+1}^{\mathrm{mod}}(U, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}}]$$

and carries the involution $F_\infty = \text{conj}_k$. Indeed, the expression (A.24) makes explicit the interpretation of the elements of $H_{k+1}^{\text{rd}}(U, \tilde{f}_k)$ in terms of twisted cycles. It is moreover clear that, in this expression, the closed subspace $\partial_{\text{rd},R}U$ of $U(\mathbb{C})$, upon which $\exp(-\tilde{f}_k)$ is sufficiently small, is invariant under both the action of $\mathfrak{S}_k \times \mu_2$ and the complex conjugation on $U(\mathbb{C})$, which thus provides the action of $\mathfrak{S}_k \times \mu_2$ and F_∞ on $H_{k+1}^{\text{rd}}(U, \tilde{f}_k)$.

Self-duality and period pairing. The involution ι yields the identifications

$$\mu : H_?^{k+1}(U, \tilde{f}_k) \xrightarrow{\sim} H_?^{k+1}(U, -\tilde{f}_k), \quad ? = \emptyset, \mathfrak{c},$$

compatible with the action of $\mathfrak{S}_k \times \mu_2$, and leads to the self-duality up to a Tate twist by $\mathbb{Q}(k+1)$ of the mixed Hodge structure $(M_k)_\mathbb{H} \simeq H_{\text{mid}}^{k+1}(U, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \text{sgn}}$.

The nondegenerate period pairing

$$P^{\text{rd}, \text{mod}} : H_{k+1}^{\text{rd}}(U, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \text{sgn}} \otimes H_{\text{dR}}^{k+1}(U, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \text{sgn}} \rightarrow \mathbb{C},$$

is computed as follows. Given a twisted cycle

$$\alpha \in H_{k+1}^{\text{rd}}(U, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \text{sgn}} = H_{k+1}(\tilde{U}_{\text{rd}}, \partial \tilde{U}_{\text{rd}}, \mathbb{Q})^{\mathfrak{S}_k \times \mu_2, \text{sgn}},$$

we choose for each large enough $R > 0$ a representative $\alpha_R \in H_{k+1}(U, \partial_{\text{rd},R}U, \mathbb{Q})$ sgn-invariant under $\mathfrak{S}_k \times \mu_2$. We also represent a de Rham class in the space $H_{\text{dR}}^{k+1}(U, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \text{sgn}}$ by a top differential form ω on U which is sgn-invariant under $\mathfrak{S}_k \times \mu_2$. Then

$$P^{\text{rd}, \text{mod}}(\alpha, [\omega]) = \lim_{R \rightarrow \infty} \int_{\alpha_R} e^{-\tilde{f}_k} \iota^* \omega.$$

7.2. Proof of Theorem 7.1. The goal of this section is to identify the period structure

$$(H_{\text{dR}, \text{mid}}^{k+1}(U_0, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \text{sgn}}, H_{\text{B}, \text{mid}}^{k+1}(U, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \text{sgn}}, \text{comp})$$

as defined from Sections A2 and A3 with the transpose of

$$(H_{\text{dR}, \text{mid}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)_\mathbb{Q}, H_1^{\text{mid}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)_\mathbb{Q}, P_k^{\text{mid}}),$$

ending thereby the proof of Theorem 7.1, and to make explicit the action of F_∞ on $H_1^{\text{mid}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)_\mathbb{Q}$.

Summary of the proof.

Step 1 We consider the family $(w_j)_{0 \leq j \leq k'}$ in $H_{\text{dR}}^{k+1}(U_0, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \text{sgn}}$ as defined in [Fresán et al. 2022, proof of Proposition 4.21], which has been shown to form a \mathbb{Q} -basis of this space.

Step 2 We construct a family of rapid decay cycles (τ_i) in $H_{k+1}^{\text{rd}}(U, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \text{sgn}}$ such that the matrix $(P^{\text{rd}, \text{mod}}(\tau_i, w_j))_{0 \leq i, j \leq k'}$ is equal to $(P_{k; i, j}^{\text{rd}, \text{mod}})_{0 \leq i, j \leq k'}$ obtained from Propositions 5.4 and 6.2. Since this matrix is nondegenerate and (w_j) is a basis of $H_{\text{dR}}^{k+1}(U, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \text{sgn}}$, we conclude that (τ_i) is a basis of $H_{\text{rd}}^{k+1}(U, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \text{sgn}}$.

Step 3 According to [Remark A.25](#), the period realization of $H^{k+1}(U_0, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \text{sgn}}$ is isomorphic to

$$(\mathbb{Q} \cdot (w_j), \mathbb{Q} \cdot (\tau_i), (P_{i,j}^{\text{rd,mod}})_{0 \leq i, j \leq k'}),$$

hence to

$$(\mathbb{Q} \cdot (\omega_j), \mathbb{Q} \cdot (\alpha_i), (P_{k;i,j}^{\text{rd,mod}})_{0 \leq i, j \leq k'}),$$

that is,

$$(H_{\text{dR}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)_{\mathbb{Q}}, H_1^{\text{rd}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)_{\mathbb{Q}}, P_k^{\text{rd,mod}}).$$

Step 4 Owing to the duality argument explained in [\[Fresán et al. 2023, Lemma 2.27\]](#), the same property holds for the period realization of $H_c^{k+1}(U_0, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \text{sgn}}$ and

$$(H_{\text{dR},c}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)_{\mathbb{Q}}, H_1^{\text{mod}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)_{\mathbb{Q}}, P_k^{\text{mod,rd}}).$$

Step 5 That the same property holds true for the period realization of $H_{\text{mid}}^{k+1}(U_0, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \text{sgn}}$ and

$$(H_{\text{dR},\text{mid}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)_{\mathbb{Q}}, H_1^{\text{mid}}(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2)_{\mathbb{Q}}, P_k^{\text{mid}})$$

needs a supplementary argument when $k \equiv 0 \pmod 4$.

Step 6 Lastly, that conjugation is compatible with these identifications amounts to checking whether the entries of the period matrix are real or purely imaginary.

A basis of $H_{\text{dR}}^{k+1}(U_0, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \text{sgn}}$. Let us consider the class $[\tilde{\omega}_j]$ of

$$\tilde{\omega}_j = 2^{1-2j} t^{2j-1} dt \wedge \frac{dy_1}{y_1} \wedge \dots \wedge \frac{dy_k}{y_k}$$

in $H_{\text{dR}}^{k+1}(U_0, \tilde{f}_k)$ and set

$$w_j = \frac{1}{2k!} \sum_{\sigma \in \mathfrak{S}_k \times \mu_2} \text{sgn}(\sigma) \cdot \sigma([\tilde{\omega}_j]) \in H_{\text{dR}}^{k+1}(U_0, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \text{sgn}}. \tag{7.5}$$

Through the isomorphism (see [\[Fresán et al. 2022, Proposition 2.13\]](#))

$$H_{\text{dR}}^1(\mathbb{G}_m, \text{Sym}^k \text{Kl}_2) \simeq H_{\text{dR}}^{k+1}(U, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \text{sgn}},$$

each class $[z^j v_0^k dz/z]$ in the basis \mathcal{B}_k corresponds to the class w_j ; see [\[loc. cit., proof of Proposition 4.21\]](#). As a consequence, $(w_j)_{0 \leq j \leq k'}$ forms a basis of $H_{\text{dR}}^{k+1}(U_0, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \text{sgn}}$ satisfying $\iota^* w_j = w_j$.

A family of rapid decay cycles. For each i such that $0 \leq i \leq k'$, we define rapid decay cycles $\tilde{\alpha}_i$ as follows. First, let $\tilde{\alpha}_0$ be the bounded cycle

$$\tilde{\alpha}_0 = \prod_{p=0}^k c_p,$$

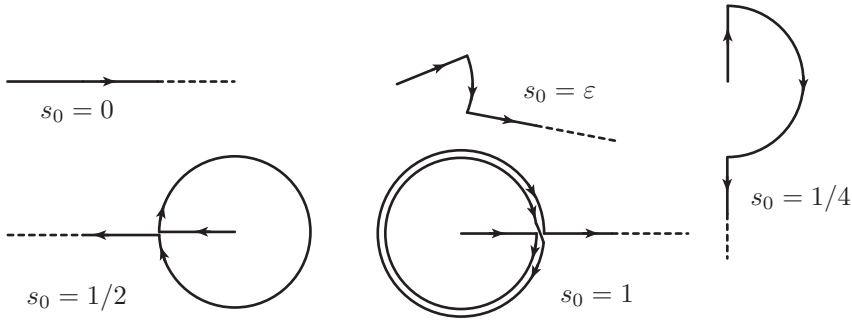


Figure 1. The path $s_p \mapsto y_p(s_0, s_p)$ for s_0 fixed and $r \leq p \leq k$.

where c_0 is the circle $|t| = 1$ and c_p is the circle $|y_p| = 1$ for $p = 1, \dots, k$, both oriented counterclockwise. For the remaining unbounded cycles, we start with the chain

$$\tilde{\alpha}'_i = [1, +\infty) \times \left(\prod_{p=1}^i c_p \right) \times (\mathbb{R}_{>0})^{k-i}$$

on $U(\mathbb{C})$, where $[1, +\infty)$ is relative to the variable t and the open octant $(\mathbb{R}_{>0})^{k-i}$ is relative to the variables y_p for $p = i + 1, \dots, k$. The orientation is the natural one on each half-line, and given by the order of the variables for $\tilde{\alpha}'_i$. Since $\text{Re}(y_p + 1/y_p) \geq -2$ on c_p and ≥ 2 on $\mathbb{R}_{>0}$, by noting that $k - i > i$ for $1 \leq i \leq k'$, one sees that $e^{-\tilde{f}_k}$ has rapid decay along $\tilde{\alpha}'_i$. The rapid decay chain $\tilde{\alpha}'_i$ has boundary

$$\partial \tilde{\alpha}'_i = -\{1\} \times \left(\prod_{p=1}^i c_p \right) \times (\mathbb{R}_{>0})^{k-i},$$

and hence is not a cycle. To kill the boundary, we mimic the construction of the cycles α_i from (4.4) by introducing, for each r such that $1 \leq r \leq k'$, the chain $\tilde{\gamma}_r$ defined as the image of the map $[0, 1]^r \times (0, 1)^{k-r+1} \rightarrow U(\mathbb{C})$ given by

$$(s_0, \dots, s_k) \mapsto \begin{cases} t = e^{2\pi i s_0}, \\ y_p = e^{2\pi i s_p} & 1 \leq p \leq r-1, \\ y_p = \begin{cases} e^{2\pi i s_0} \cdot 4s_p & 0 < s_p \leq 1/4, \\ e^{2\pi i s_0(2-4s_p)} & 1/4 \leq s_p \leq 3/4, \\ e^{-2\pi i s_0} (1 + \tan \frac{\pi}{2}(4s_p - 3)) & 3/4 \leq s_p < 1, \end{cases} & r \leq p \leq k. \end{cases}$$

(See Figure 1.) The chain $\tilde{\gamma}_r$ decays rapidly for $e^{-\tilde{f}_k}$ with boundary

$$\partial \tilde{\gamma}_r = \{1\} \times \left(\prod_{p=1}^{r-1} c_p \right) \times \prod_{p=r}^k (\mathbb{R}_{>0} - 2c_p) - \{1\} \times \left(\prod_{p=1}^{r-1} c_p \right) \times (\mathbb{R}_{>0})^{k-r}.$$

For each integer $m \geq 1$, there is a unique sequence $(\theta_m(r))_{0 \leq r \leq m-1}$ of rational numbers satisfying the identity

$$\sum_{r=0}^{m-1} \theta_m(r) a^r [(a+b)^{m-r} - b^{m-r}] = ab^{m-1} \tag{7.6}$$

in the polynomial ring $\mathbb{Q}[a, b]$. Explicitly, the first two values are $\theta_m(0) = 1/m$ and $\theta_m(1) = -1/2$ and the recursion

$$\sum_{r=0}^{\ell-1} \binom{m-r}{m-\ell} \theta_m(r) = 0 \tag{7.7}$$

holds if $1 < \ell < m$. Set

$$\tilde{\theta}_m = \sum_{r=0}^{m-1} \frac{\theta_m(r)}{m-r+1}.$$

For each i such that $1 \leq i \leq k'$, let

$$\tilde{\alpha}_i = -(-2)^{k-i} \tilde{\theta}_{k-i+1} \tilde{\alpha}_0 + \frac{1}{(k-i+1)!} \sum_{\sigma \in \mathfrak{S}_{k-i+1}} \sigma \left(\tilde{\alpha}'_i + \sum_{r=i}^k (-2)^{r-i-1} \theta_{k-i+1}(r-i) \tilde{\gamma}_r \right),$$

where the symmetric group \mathfrak{S}_j acts by permuting the last j components. We compute the boundary of the terms involving $\tilde{\gamma}_r$. Since the average over the group action makes the positions of circles and lines from $\partial \tilde{\gamma}_r$ equidistributed in the last $k-i+1$ components, for each fixed j such that $1 \leq j \leq k-i$ and any $\sigma \in \mathfrak{S}_{k-i+1}$, the coefficient in the boundary of the term $\sigma(\{1\} \times \prod_{p=1}^{i+j-1} c_p \times \prod_{p=i+j}^k \mathbb{R}_{>0})$, resulting from $\tilde{\gamma}_r$ for $i \leq r \leq i+j-1$, is equal to

$$\begin{aligned} \sum_{r=i}^{i+j-1} (-2)^{r-i-1} \theta_{k-i+1}(r-i) \binom{k+1-r}{i+j-r} (-2)^{i+j-r} &= (-2)^{j-1} \sum_{\ell=0}^{j-1} \theta_{k-i+1}(\ell) \binom{k-i+1-\ell}{j-\ell} \\ &= \begin{cases} 1 & \text{if } j = 1, \\ 0 & \text{if } 1 < j < k-i+1, \end{cases} \end{aligned}$$

where the last equality follows from (7.7). One concludes that the $\tilde{\alpha}_i$ define rapid decay cycles.

Remark 7.8. In fact, $\theta_m(r) = \binom{m}{r} \mathcal{B}_r / m$ if $0 \leq r < m$, and $\tilde{\theta}_m = -\mathcal{B}_m / m$ for all $m \geq 1$. To see this, plugging $a = 1$ into (7.6) and summing the resulting equations for $b = 1, \dots, n$, one obtains

$$\sum_{\ell=0}^{m-1} n^{m-\ell} \sum_{r=0}^{\ell} \binom{m-r}{m-\ell} \theta_m(r) = \sum_{b=1}^n b^{m-1},$$

and hence by Bernoulli's formula (4.3),

$$\sum_{r=0}^{\ell} \binom{m-r}{m-\ell} \theta_m(r) = \frac{(-1)^\ell}{m} \binom{m}{\ell} \mathcal{B}_\ell, \quad 0 \leq \ell < m.$$

Using (7.7) and the equalities $\theta_m(0) = \mathcal{B}_0/m$, $\theta_m(1) = \mathcal{B}_1$ and $\mathcal{B}_{2p+1} = 0$ for $p \geq 1$, we obtain the identity for $\theta_m(r)$. Therefore,

$$\tilde{\theta}_m = \frac{1}{m} \sum_{r=0}^{m-1} \frac{1}{m-r+1} \binom{m}{r} \mathcal{B}_r = \frac{1}{m(m+1)} \sum_{r=0}^{m-1} \binom{m+1}{r} \mathcal{B}_r,$$

and the identity for $\tilde{\theta}_m$ follows from the recursive relation $\sum_{r=0}^m \binom{m+1}{r} \mathcal{B}_r = 0$ for all $m \geq 1$.

Lemma 7.9. *For any positive integer m , the following identity holds in $\mathbb{Q}[a, b]$:*

$$\sum_{r=0}^{m-1} \frac{\theta_m(r)}{m-r+1} a^{r-1} [(a+b)^{m-r+1} - b^{m-r+1}] = \tilde{\theta}_m a^m + \frac{b^m}{m}. \tag{7.10}$$

Proof. Integrating equation (7.6) with respect to b from 0 to b yields the formula. □

Proposition 7.11. *The period pairing*

$$\text{Prd.mod} : \mathbf{H}_{k+1}^{\text{rd}}(U, \tilde{f}_k) \otimes \mathbf{H}_{\text{dR}}^{k+1}(U, \tilde{f}_k) \rightarrow \mathbb{C}$$

is given on the rapid decay cycles $\tilde{\alpha}_i$ and the differential forms $\tilde{\omega}_j$ by

$$\text{Prd.mod}(\tilde{\alpha}_i, \tilde{\omega}_j) = \begin{cases} 2(2\pi i)^{k+1} \delta_{0,j} & \text{if } i = 0, \\ (-1)^{k-i} \text{BM}_k^{\text{reg}}(i, -1) & \text{if } j = 0 \text{ and } 1 \leq i \leq k', \\ (-1)^{k-i} \text{BM}_k(i, 2j-1) & \text{if } 1 \leq i, j \leq k'. \end{cases}$$

Proof. It is clear that $\text{Prd.mod}(\tilde{\alpha}_0, \tilde{\omega}_j) = 2(2\pi i)^{k+1} \delta_{0,j}$.

For the rest, as in the proof of Proposition 6.2, for $\varepsilon \in \mathbb{R}_{>0}$, we let $\tilde{\alpha}'_{i,\varepsilon} = \varepsilon \tilde{\alpha}'_i$ and $\tilde{\gamma}_{r,\varepsilon} = \varepsilon \tilde{\gamma}_r$ be the scalings of $\tilde{\alpha}'_i$ and $\tilde{\gamma}_r$, which are homologous to $\tilde{\alpha}'_i$ and $\tilde{\gamma}_r$, respectively. Consider the case $j = 0$. By the limiting behavior (2.13) and Definition 6.1 of the regularized Bessel moments, we have the limiting behaviors

$$\begin{aligned} \int_{\tilde{\alpha}'_{i,\varepsilon}} e^{-\tilde{f}_k} \tilde{\omega}_0 &= 2^{k+1} (\pi i)^i \int_{\varepsilon}^{\infty} I_0(t)^i K_0(t)^{k-i} \frac{dt}{t} \\ &\sim_{\varepsilon \rightarrow 0^+} (-1)^{k-i} \text{BM}_k^{\text{reg}}(i, -1) + \frac{(-1)^{k-i+1} 2^{k+1} (\pi i)^i}{k-i+1} (\gamma + \log(\varepsilon/2))^{k-i+1}, \\ \int_{\tilde{\gamma}_{r,\varepsilon}} e^{-\tilde{f}_k} \tilde{\omega}_0 &= 2^{k+1} (\pi i)^{r-1} \int_{|t|=\varepsilon} I_0(t)^{r-1} K_0(t)^{k-r+1} \frac{dt}{t} \\ &\sim_{\varepsilon \rightarrow 0^+} \frac{(-1)^{k-r+1} 2^{k+1} (\pi i)^{r-1}}{k-r+2} \cdot [(2\pi i + \gamma + \log(\varepsilon/2))^{k-r+2} - (\gamma + \log(\varepsilon/2))^{k-r+2}]. \end{aligned}$$

Substituting $m = k - i + 1$, $a = 2\pi i$, and $b = \gamma + \log \varepsilon/2$ into the identity (7.10), one obtains

$$\begin{aligned} \sum_{r=i}^k (-2)^{r-i-1} \theta_{k-i+1}(r-i) \int_{\tilde{\gamma}_{r,\varepsilon}} e^{-\tilde{f}_k} \tilde{\omega}_0 \\ \sim_{\varepsilon \rightarrow 0^+} -(-2)^{k-i+1} (2\pi i)^{k+1} \tilde{\theta}_{k-i+1} + \frac{(-1)^{k-i} 2^{k+1} (\pi i)^i}{k-i+1} (\gamma + \log(\varepsilon/2))^{k-i+1}. \end{aligned}$$

Letting $\varepsilon \rightarrow 0^+$ leads to $\mathrm{P}^{\mathrm{rd},\mathrm{mod}}(\tilde{\alpha}_i, \tilde{\omega}_0) = (-1)^{k-i} \mathrm{BM}_k^{\mathrm{reg}}(i, -1)$.

Finally, if $1 \leq i, j \leq k'$, it is straightforward to verify that

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\tilde{\alpha}'_{i,\varepsilon}} e^{-\tilde{f}_k} \tilde{\omega}_j = (-1)^{k-i} \mathrm{BM}_k(i, 2j - 1) \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0^+} \int_{\tilde{\gamma}_{i,\varepsilon}} e^{-\tilde{f}_k} \tilde{\omega}_j = 0$$

(similar to the proof of Proposition 5.4), and the remaining case follows. □

Starting from the $\tilde{\alpha}_i$, we obtain the following cycles in $\mathrm{H}_{k+1}^{\mathrm{rd}}(U, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}}$:

$$\tau_0 = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k \times \mu_2} \mathrm{sgn}(\sigma) \cdot \sigma(\tilde{\alpha}_0), \quad \tau_i = \frac{(-1)^{k-i}}{2k!} \sum_{\sigma \in \mathfrak{S}_k \times \mu_2} \mathrm{sgn}(\sigma) \cdot \sigma(\tilde{\alpha}_i) \quad (1 \leq i \leq k').$$

For each i such that $1 \leq i \leq k'$, let Γ_i denote the image of τ_i in $\mathrm{H}_{k+1}^{\mathrm{mod}}(U, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}}$.

Corollary 7.12. (1) *The family $(\tau_i)_{0 \leq i \leq k'}$ forms a basis of $\mathrm{H}_{k+1}^{\mathrm{rd}}(U, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}}$.*

(2) *The period realizations*

- of $\mathrm{H}_c^1(\mathbb{G}_m, \mathrm{Sym}^k \mathrm{Kl}_2)$ and of $\mathrm{H}_c^{k+1}(U_0, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}}$,
- as well as those of $\mathrm{H}^1(\mathbb{G}_m, \mathrm{Sym}^k \mathrm{Kl}_2)$ and $\mathrm{H}^{k+1}(U_0, \tilde{f}_k)^{\mathfrak{S}_k \times \mu_2, \mathrm{sgn}}$,

are isomorphic.

(3) *If k is not a multiple of 4, the family $(\Gamma_i)_{1 \leq i \leq k'}$ forms a basis of $(\mathrm{M}_k)_{\mathbb{B}}$. If k is a multiple of 4, the \mathbb{Q} -linear relation*

$$\sum_{i=0}^{\lfloor (k-2)/4 \rfloor} \binom{k/2}{2i+1} \Gamma_{2i+1} = 0$$

holds and $(\Gamma_i)_{2 \leq i \leq k'}$ forms a basis of $(\mathrm{M}_k)_{\mathbb{B}}$. In particular, the period realization of the motive M_k is isomorphic to

$$(\mathrm{H}_{\mathrm{dR},\mathrm{mid}}^1(\mathbb{G}_m, \mathrm{Sym}^k \mathrm{Kl}_2)_{\mathbb{Q}}, \mathrm{H}_1^{\mathrm{mid}}(\mathbb{G}_m, \mathrm{Sym}^k \mathrm{Kl}_2)_{\mathbb{Q}}, \mathrm{P}_k^{\mathrm{mid}}).$$

(4) *The involution F_{∞} acts as $F_{\infty}(\Gamma_i) = (-1)^i \Gamma_i$ for all i such that $1 \leq i \leq k'$.*

Proof. For each i such that $0 \leq i \leq k'$, equation (5.1), Propositions 5.4, 6.2 and 7.11 yield the equality of periods $\mathrm{P}^{\mathrm{rd},\mathrm{mod}}(\tau_i, w_j) = \mathrm{P}_k^{\mathrm{rd},\mathrm{mod}}(\alpha_i, \omega_j)$ for all j such that $0 \leq j \leq k'$ (with w_j defined by (7.5)), from which (1) follows, as indicated in Step 2 of the summary. The proof of (2) has been explained in Step 3 and Step 4 of the summary.

In view of Theorem 4.7(3) and Proposition 6.10, to prove (3), it suffices to show that $\mathrm{P}^{\mathrm{mid}}(\Gamma_i, w_j) = \mathrm{P}_k^{\mathrm{mid}}(\beta_i, \omega_j)$ for all i, j such that $1 \leq i, j \leq k'$. Since Γ_i and β_i are the images of the rapid decay cycles τ_i and α_i respectively and the pairing $\mathrm{P}_k^{\mathrm{mid}}$ is induced by $\mathrm{P}^{\mathrm{rd},\mathrm{mod}}$, this amounts to $\mathrm{P}^{\mathrm{rd},\mathrm{mod}}(\tau_i, w_j) = \mathrm{P}_k^{\mathrm{rd},\mathrm{mod}}(\alpha_i, \omega_j)$, which we just checked.

Finally, the statement about F_∞ follows from the fact that the period pairing Prd.mod is compatible with complex conjugation in that (recall that $\iota^* w_j = w_j$ and $\text{conj}^* w_j = w_j$),

$$\text{Prd.mod}(\text{conj}_* \tau_i, w_j) = \int_{F_\infty \tau_i} e^{-\tilde{f}_k} w_j = \int_{\text{conj}_* \tau_i} \text{conj}^*(e^{-\tilde{f}_k} w_j) = \overline{\text{Prd.mod}(\tau_i, w_j)} \quad \text{for all } i, j,$$

along with the equality $\overline{\text{Prd.mod}(\tau_i, w_j)} = (-1)^i \text{Prd.mod}(\tau_i, w_j)$ if $1 \leq i, j \leq k'$, since $\text{BM}_k(i, 2j - 1)$ is real for even i and purely imaginary for odd i . □

8. L-functions of Kloosterman sums and Bessel moments

In this section, we make precise the statement of Deligne’s conjecture for the motive M_k . Being a classical motive, M_k has an associated L -function $L(\text{M}_k, s)$, which is identified in [Fresán et al. 2022, Theorems 5.8 and 5.17] with the L -function $L_k(s)$ of symmetric power moments of Kloosterman sums, and its Betti realization $(\text{M}_k)_\mathbb{B}$ carries a pure Hodge structure of weight $k + 1$. The factor at infinity was computed in [loc. cit., Corollary 5.30]:

$$L_\infty(\text{M}_k, s) = \pi^{-ms/2} \prod_{j=1}^m \Gamma\left(\frac{s-j}{2}\right), \tag{8.1}$$

where $m = k'$ if k is not a multiple of 4 and $m = k' - 1$ otherwise. This also follows from Corollary 7.12(4): indeed, according to [loc. cit., Theorem 1.8], the only nontrivial Hodge numbers of M_k are

$$h^{p,q} = \begin{cases} 1 & \text{for } p = 2, \dots, k - 1 \text{ if } k \text{ is odd} \\ 1 & \text{for } \min\{p, q\} = 2, \dots, 2\lfloor(k - 1/4)\rfloor \text{ if } k \text{ is even.} \end{cases}$$

The ordered basis $\mathcal{B}_{k,\text{mid}}$ is adapted to the Hodge filtration by Corollary 3.21, and F_∞ acts as -1 on $H^{p,p}$ for $k \equiv 3 \pmod{4}$, which is what we need for the conclusion.

By [loc. cit., Theorems 1.2 and 1.3], the L -function $L(\text{M}_k, s)$ extends meromorphically to the complex plane and the completed L -function

$$\Lambda(\text{M}_k, s) = L(\text{M}_k, s)L_\infty(\text{M}_k, s)$$

satisfies a functional equation relating its values at s and $k + 2 - s$. The *critical integers* are the integral values of s at which neither $L_\infty(\text{M}_k, s)$ nor $L_\infty(\text{M}_k, k + 2 - s)$ has a pole and the *critical values* are the values of $L(\text{M}_k, s)$ at critical integers.

Deligne’s conjecture [1979, Section 1] predicts that critical values agree, up to a rational factor, with the determinants of certain minors of the period matrix, which are defined as follows. Let a be an integer, and let $\text{M}_k(a)_\mathbb{B}^+$ and $\text{M}_k(a)_\mathbb{B}^-$ denote respectively the invariants and antiinvariants of F_∞ acting on $\text{M}_k(a)_\mathbb{B}$. As F_∞ exchanges the subspaces $H^{p-a, q-a}$ and $H^{q-a, p-a}$ in the Hodge decomposition and acts as -1 on $H^{p-a, p-a}$, the eigenspaces for F_∞ have dimensions either $\sum_{p>q} h^{p-a, q-a}$ or $\sum_{p\geq q} h^{p-a, q-a}$, and there exists unique steps $F^\pm \text{M}_k(a)_{\text{dR}}$ of the Hodge filtration with $\dim F^\pm \text{M}_k(a)_{\text{dR}} = \dim \text{M}_k(a)_\mathbb{B}^\pm$. If n is a

k	rank	critical integers n	c_n
3	1	$2 - 2a$ $2a + 3$	1 $\pi^{2a} D_{3,\text{odd}}$
$4r + 1$	$2r$	$2r + 1$ $2r + 2$	$\pi^{-r(r+1)} D_{k,\text{odd}}$ $\pi^{-r(r-1)} D_{k,\text{even}}$
$4r + 2$	$2r$	$2r + 1$ $2r + 2$ $2r + 3$	$\pi^{-r(r+1)} D_{k,\text{even}}$ $\pi^{-r(r+1)} D_{k,\text{odd}}$ $\pi^{-r(r-1)} D_{k,\text{even}}$
$4r + 3$	$2r + 1$	$2r + 2$ $2r + 3$	$\pi^{-r(r+1)} D_{k,\text{even}}$ $\pi^{-r(r+1)} D_{k,\text{odd}}$
$4r + 4$	$2r$	$2r + 1$ $2r + 2$ $2r + 3$ $2r + 4$ $2r + 5$	$\pi^{-r(r+3)} D'_{k,\text{even}}$ $\pi^{-r(r+3)} D'_{k,\text{odd}}$ $\pi^{-r(r+1)} D'_{k,\text{even}}$ $\pi^{-r(r+1)} D'_{k,\text{odd}}$ $\pi^{-r(r-1)} D'_{k,\text{even}}$

Table 1. Critical integers n and values of c_n ($r \geq 1$).

critical integer, Deligne defines

$$c_n = \det(\mathbb{P}_k^{\text{mid}}(\sigma_i, v_i)) \in \mathbb{C}^\times / \mathbb{Q}^\times$$

where σ_i runs through any basis of the \mathbb{Q} -linear dual of $\mathbb{M}_k(k + 1 - n)_\mathbb{B}^+$ and v_i runs through any basis of $F^+ \mathbb{M}_k(k + 1 - n)_{\text{dR}}$, see the paragraph before [Deligne 1979, Conjecture 1.8] taking the duality of pure Hodge structures $(\mathbb{M}_k)_\mathbb{H}^\vee \cong (\mathbb{M}_k)_\mathbb{H}(k + 1)$ from [Fresán et al. 2022, (3.4)] into account. He then conjectures that $L(\mathbb{M}_k, n)$ is a rational multiple of c_n .

Notation 8.2. For each $k \geq 3$, the determinants of Bessel moments $D_{k,\text{odd}}$ and $D_{k,\text{even}}$ are defined by the following formulas:

$$\begin{aligned}
 D_{k,\text{odd}} &= \det \left(\int_0^\infty I_0(t)^{2i-1} K_0(t)^{k+1-2i} t^{2j-1} dt \right)_{1 \leq i, j \leq \lfloor (k+1)/4 \rfloor} \\
 D_{k,\text{even}} &= \det \left(\int_0^\infty I_0(t)^{2i} K_0(t)^{k-2i} t^{2j-1} dt \right)_{1 \leq i, j \leq \lfloor k/4 \rfloor}
 \end{aligned}
 \tag{8.3}$$

In other words, $D_{k,\text{odd}}$, resp. $D_{k,\text{even}}$, is obtained by extracting from $\mathbb{P}_k^{\text{mid}}$ the entries that belong to odd, resp. even, lines and to the first $\lfloor (k + 1/4) \rfloor$, resp. $\lfloor k/4 \rfloor$, columns. If k is a multiple of 4, we let $D'_{k,\text{odd}}$ and $D'_{k,\text{even}}$ be the determinants of the same matrices except that we remove the last row and column, i.e., those indexed by $i = j = k/4$.

Theorem 8.4. For $k = 3$ or $k \geq 5$, the critical integers n for $L(\mathbb{M}_k, s)$ and the corresponding values c_n are given by Table 1.

Proof. The case $k = 3$ is exceptional in that there exist infinitely many critical integers, namely all even integers ≤ 2 and all odd integers ≥ 3 . For each $k \geq 5$, a straightforward computation using (8.1) yields the list of critical integers. To compute c_n , we first exhibit a basis of the \mathbb{Q} -linear dual of $M_k(a)_B^+$ by means of Corollary 7.12. If k is not a multiple of 4, a basis is given by $\{(2\pi i)^{-a} \Gamma_{2i}\}_{1 \leq i \leq \lfloor k'/2 \rfloor}$ if a is even and by $\{(2\pi i)^{-a} \Gamma_{2i-1}\}_{1 \leq i \leq \lfloor (k'+1)/2 \rfloor}$ if a is odd. If k is a multiple of 4, say $k = 4r + 4$, then the \mathbb{Q} -linear dual of $M_k(a)_B^+$ has basis $\{(2\pi i)^{-a} \Gamma_{2i}\}_{1 \leq i \leq r}$ if a is even and $\{(2\pi i)^{-a} \Gamma_{2i+1}\}_{1 \leq i \leq r}$ if a is odd, which shows that $M_k(a)_B^+$ always has dimension r .

Let us treat the case $k = 4r + 3$ in detail. For $n = 2r + 2$, the eigenspace $M_k(2r + 2)_B^+$ has dimension r , and hence $F^+ M_k(2r + 2)_{dR}$ is spanned by $\{w_j\}_{1 \leq j \leq r}$. Therefore, c_{2r+2} is the determinant of the matrix with entries

$$(2\pi i)^{-(2r+2)} \text{Prd,mod}(\Gamma_{2i}, w_j) \sim_{\mathbb{Q}^\times} \pi^{2(i-r-1)} \int_0^\infty I_0(t)^{2i} K_0(t)^{k-2i} t^{2j-1} dt,$$

from which we get $c_{2r+2} = \pi^{-r(r+1)} D_{k,\text{even}}$. For $n = 2r + 3$, the eigenspace $M_k(2r + 1)_B^+$ has dimension $r + 1$, and hence $F^+ M_k(2r + 1)_{dR}$ is spanned by $\{w_j\}_{1 \leq j \leq r+1}$. With respect to these bases, the matrix defining c_{2r+3} has entries

$$(2\pi i)^{-(2r+1)} \text{Prd,mod}(\Gamma_{2i-1}, w_j) \sim_{\mathbb{Q}^\times} \pi^{2(i-r-1)} \int_0^\infty I_0(t)^{2i-1} K_0(t)^{k+1-2i} t^{2j-1} dt,$$

which gives $c_{2r+3} = \pi^{-r(r+1)} D_{k,\text{odd}}$. The cases $k = 4r + 1$ and $k = 4r + 2$ are completely parallel.

For $k = 4r + 4$ and any critical value n , the eigenspace $M_k(4r + 5 - n)_B^+$ has dimension r , and hence $F^+ M_k(4r + 5 - n)_{dR}$ is spanned by $\{w_j\}_{1 \leq j \leq r}$ (note that $r < k/4$, so that we do not need to modify the ω). If n is odd, c_n is the determinant of the matrix with entries

$$(2\pi i)^{-(4r+5-n)} \text{Prd,mod}(\Gamma_{2i}, w_j) \sim_{\mathbb{Q}^\times} \pi^{2i+n-4r-5} \int_0^\infty I_0(t)^{2i} K_0(t)^{k-2i} t^{2j-1} dt,$$

thus yielding

$$c_{2r+1} = \pi^{-r(r+3)} D'_{k,\text{even}}, \quad c_{2r+3} = \pi^{-r(r+1)} D'_{k,\text{even}}, \quad \text{and} \quad c_{2r+5} = \pi^{-r(r-1)} D'_{k,\text{even}}.$$

If n is even, the matrix has entries

$$(2\pi i)^{-(4r+5-n)} \text{Prd,mod}(\Gamma_{2i+1}, w_j).$$

Thanks to the linear relation from Corollary 7.12(3), the determinant of this matrix agrees up to a rational number with that of

$$\pi^{2i+n-4r-6} \int_0^\infty I_0(t)^{2i-1} K_0(t)^{k+1-2i} t^{2j-1} dt,$$

which gives the remaining values

$$c_{2r+2} = \pi^{-r(r+3)} D'_{k,\text{odd}} \quad \text{and} \quad c_{2r+4} = \pi^{-r(r+1)} D'_{k,\text{odd}}. \quad \square$$

Remark 8.5. Besides the case $k = 3$, in which $L_3(s) = L(\chi_3, s - 2)$ is a shifted Dirichlet L -function, Deligne’s conjecture holds for $k = 5, 6, 8$. In all three cases, $L_k(s)$ is the L -function of a modular form evaluated at $s - 2$ (see [Fresán et al. 2022, Table 1]) and the matrices in (8.3) have size one. The critical values have been expressed in terms of Bessel moments in [Bloch et al. 2015; Zhou 2018a; 2019b], thus confirming the conjecture. The first case where a true determinant is expected to occur is $L_7(5)$, which is a critical value of the L -function of the symmetric square of a modular form. Other numerical confirmations of Deligne’s conjecture for $k \leq 24$ appear in [Broadhurst and Roberts 2019].

Remark 8.6. Broadhurst and Roberts conjecture that, for $k \geq 12$ a multiple of 4, the L -function $L(M_k, s)$ always vanishes at the central point $s = \frac{1}{2}(k+2)$, even when the expected sign of the functional equation (see [Fresán et al. 2022, below Theorem 1.3]) is positive. According to Beilinson’s conjecture, this vanishing should be explained by the existence of certain nontrivial extension of M_k by $\mathbb{Q}(-k/2)$. It seems possible to construct such an extension by considering the quotient of the cohomology with compact support of $\text{Sym}^k \text{Kl}_2$ by its weight zero piece and to relate its nonsplitting to the shape of the full period matrix $P_k^{\text{mod,rd}}$ from Section 6. Indeed, suppose $4 \mid k$. The Nori motive $\tilde{M}_k = (H_c^{k-1}(\mathcal{X}_0)/W_0)^{\text{S}_k \times \mu_2, \text{sgn}(-1)}$, where W_0 denotes the weight zero part, is an extension of M_k by a rank-one motive E of weight k . It has been shown in the proof of [loc. cit., Theorem 5.17] (see especially the discussion of the terms $(E_\infty^{1,k-2})_{\mathbb{F}_p}^{G,\chi}$ and $\ker(\beta)^{G,\chi}/W_0$) that as a representation of $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$, the ℓ -adic étale realization of E is isomorphic to $\mathbb{Q}_\ell(-k/2)$ for all primes $p \geq 3$ and $\ell \neq p$. Since the Frobenius elements of characteristic $p \geq 3$ are dense in $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, one obtains an isomorphism $E \simeq \mathbb{Q}(-k/2)$. From the viewpoint of the period realization, the nonsplitting of the extension \tilde{M}_k amounts to the claim that the entries of the last row of the period matrix $P_k^{\text{mod,rd}}$ span a \mathbb{Q} -space of dimension at least two in \mathbb{C} by Proposition 6.5(2). Concretely, the latter holds if and only if the moments $\text{BM}_k(k/2, 2j - 1)$ and $\text{BM}_k^{\text{reg}}(k/2, k' + 2j)$, for $1 \leq j < k/4$, are not all in $(2\pi i)^{k/2} \mathbb{Q}$ (rational multiples of the entry $P_{k;k/2,k/4}^{\text{mod,rd}}$). Such nonsplitting has been verified numerically for the period matrix P_k of $\tilde{M}_k^\vee(-k - 1)$ by Broadhurst and Roberts; see [Roberts 2017, Section 2].

Appendix: Period realization of an exponential mixed Hodge structure

by Claude Sabbah

This appendix can be regarded as a complement, with respect to period structures, to the appendix in [Fresán et al. 2022]. We consider the abelian category Per of *period structures*, whose objects consist of pairs $(V^\mathbb{C}, V_\mathbb{Q})$ consisting of a finite dimensional \mathbb{C} -vector space $V^\mathbb{C}$, a finite dimensional \mathbb{Q} -vector space $V_\mathbb{Q}$, together with an isomorphism

$$\text{comp} : \mathbb{C} \otimes_{\mathbb{Q}} V_\mathbb{Q} \xrightarrow{\sim} V^\mathbb{C}$$

and whose morphisms are the natural ones. There is a natural forgetful functor $\text{Per} : \text{MHS} \rightarrow \text{Per}$ from the category of \mathbb{Q} -mixed Hodge structures to that of period structures.

A1. The fiber period realization of an exponential mixed Hodge structure. Let X be a complex smooth quasiprojective variety and let N^H be an object of the abelian category $\text{MHM}(X)$ of \mathbb{Q} -mixed Hodge modules on X . It consists of a triple $((N, F^\bullet N), (\mathcal{F}_\mathbb{Q}, W_\bullet \mathcal{F}_\mathbb{Q}), \text{comp}_X)$, where

- $(N, F^\bullet N)$ is a holonomic \mathcal{D}_X -module endowed with a coherent filtration,
- $(\mathcal{F}_\mathbb{Q}, W_\bullet \mathcal{F}_\mathbb{Q})$ is a \mathbb{Q} -perverse sheaf on X^{an} endowed with an increasing filtration by \mathbb{Q} -perverse subsheaves,
- and the *comparison isomorphism* comp_X is an isomorphism $\mathbb{C} \otimes_{\mathbb{Q}} \mathcal{F}_\mathbb{Q} \xrightarrow{\sim} {}^p\text{DR } N$, where ${}^p\text{DR } N$ is the shifted de Rham complex $\text{DR } N[\dim X]$.

These data are subject to various compatibility relations that we do not make explicit here, referring to [Saito 1990; 2017] for details. From the mixed Hodge module N^H we only retain the triple $\text{Per}(N^H) := (N, \mathcal{F}_\mathbb{Q}, \text{comp}_X)$ by forgetting the Hodge and weight filtrations.

For example, let us consider the pure Hodge module ${}^p\mathbb{Q}_X^H$, with underlying \mathcal{D}_X -module equal to \mathcal{O}_X and underlying perverse sheaf ${}^p\mathbb{Q}_X = \mathbb{Q}_X[\dim X]$. The comparison isomorphism is induced by the isomorphism $\mathbb{C}_X = \mathcal{H}^0 \text{DR } \mathcal{O}_X \xrightarrow{\sim} \text{DR } \mathcal{O}_X$.

Let \mathbb{A}_θ^1 be the affine line with coordinate θ . The \mathbb{Q} -linear neutral Tannakian category EMHS (*exponential mixed Hodge structures*), as defined in [Kontsevich and Soibelman 2011, Section 4], is the full subcategory of $\text{MHM}(\mathbb{A}_\theta^1)$ consisting of objects N^H whose underlying perverse sheaf has vanishing global cohomology, with tensor structure given by the additive convolution \star . Denoting by $j : \mathbb{G}_m \rightarrow \mathbb{A}^1$ the inclusion, one defines a projector

$$\Pi : \text{MHM}(\mathbb{A}^1) \rightarrow \text{EMHS}, \quad N^H \mapsto N^H \star_{\mathbb{H}!} j^! {}^p\mathbb{Q}_{\mathbb{G}_m}^H,$$

consisting in neglecting constant mixed Hodge modules on \mathbb{A}^1 .

For an object N^H in EMHS, its *de Rham fiber* is the \mathbb{C} -vector space defined as

$$H_{\text{dR}}^1(\mathbb{A}_\theta^1, N \otimes E^\theta) = H^0 a_{\mathbb{A}_\theta^1, +} (N \otimes E^\theta),$$

where E^θ is the connection $(\mathcal{O}_{\mathbb{A}_\theta^1}, d + d\theta)$ and $a_{\mathbb{A}_\theta^1}$ denote the structure morphism of \mathbb{A}_θ^1 . This vector space is endowed with a filtration, called the irregular Hodge filtration; see [Fresán et al. 2022, Section A.3].

In order to define the Betti fiber functor, we consider the real oriented blowing-up $\varpi : \tilde{\mathbb{P}}^1 \rightarrow \mathbb{P}^1$ of \mathbb{P}^1 at ∞ and the open subset $\tilde{\mathbb{P}}_{\text{mod}}^1 = \mathbb{A}^{1, \text{an}} \cup \partial_{\text{mod}} \tilde{\mathbb{P}}^1$ in the neighborhood of which $e^{-\theta}$ has moderate growth, equivalently rapid decay, i.e., defined by $\text{Re}(\theta) > 0$. Let us denote the open inclusions $\mathbb{A}^{1, \text{an}} \hookrightarrow \tilde{\mathbb{P}}_{\text{mod}}^1$ and $\tilde{\mathbb{P}}_{\text{mod}}^1 \hookrightarrow \tilde{\mathbb{P}}^1$ respectively by α and β . The *Betti fiber* of N^H is defined as

$$H^0(\tilde{\mathbb{P}}^1, \beta_! R\alpha_* \mathcal{F}_\mathbb{Q}) = H_c^0(\tilde{\mathbb{P}}_{\text{mod}}^1, R\alpha_* \mathcal{F}_\mathbb{Q}).$$

Let us notice that these definitions can be extended to all objects N^H of $\text{MHM}(\mathbb{A}_\theta^1)$ and then the corresponding vector spaces only depend on $\Pi(N^H)$ since $H_{\text{dR}}^r(\mathbb{A}_\theta^1, E^\theta) = 0$ for all r .

In order to define the comparison isomorphism, we first recall that the de Rham fiber $H_{\text{dR}}^1(\mathbb{A}_\theta^1, N \otimes E^\theta)$ can be computed as the hypercohomology of the moderate de Rham complex $\text{DR}^{\text{mod}}(N \otimes E^\theta)$ on $\tilde{\mathbb{P}}^1$,

that there exists a canonical functorial isomorphism (see e.g., [Fresán et al. 2023, Section 2.e])

$$H_{\text{dR}}^1(\mathbb{A}_\theta^1, N \otimes E^\theta) \simeq H^0(\tilde{\mathbb{P}}^1, {}^p\text{DR}^{\text{mod}}(N \otimes E^\theta)). \quad (\text{A.1})$$

Furthermore:

Lemma A.2. *There exists a **unique** isomorphism*

$$\beta_! R\alpha_* \text{DR}^{\text{an}}(N) \simeq \text{DR}^{\text{mod}}(N \otimes E^\theta)$$

extending the identity on the analytic complex line $\mathbb{A}^{1\text{an}}$.

Proof. Indeed, let us recall (see [loc. cit.]) that $\text{DR}^{\text{mod}}(N \otimes E^\theta)$ has cohomology in degree zero only. Then one can easily show that its \mathcal{H}^0 is zero on the complement of $\tilde{\mathbb{P}}_{\text{mod}}^1$, so that the natural morphism

$$\beta_! \beta^{-1} \text{DR}^{\text{mod}}(N \otimes E^\theta) \rightarrow \text{DR}^{\text{mod}}(N \otimes E^\theta)$$

is an isomorphism. One then shows in the same way that

$$\beta^{-1} \text{DR}^{\text{mod}}(N \otimes E^\theta) = R\alpha_* \alpha^{-1} \beta^{-1} \text{DR}^{\text{mod}}(N \otimes E^\theta).$$

Uniqueness follows from the adjunction formulas [Kashiwara and Schapira 1990, (2.3.6) and (3.0.1)]. \square

On the other hand, termwise multiplication by $e^{-\theta}$ induces an isomorphism

$${}^p\text{DR}^{\text{an}}(N) \xrightarrow{\sim} {}^p\text{DR}^{\text{an}}(N \otimes E^\theta).$$

As a consequence, there exists a unique isomorphism

$$\text{comp}_{\tilde{\mathbb{P}}^1} : \beta_! R\alpha_* \mathcal{F}_{\mathbb{C}} \rightarrow {}^p\text{DR}^{\text{mod}}(N \otimes E^\theta)$$

extending $e^{-\theta} \circ \text{comp}_{\mathbb{A}^{1\text{an}}}$. This morphism is thus functorial with respect to N^{H} .

Definition A.3 (fiber period structure $\text{FPer}(N^{\text{H}})$). Let $N^{\text{H}} = (N, \mathcal{F}_{\mathbb{Q}}, \text{comp}_{\mathbb{A}^{1\text{an}}})$ be an object of EMHS:

(1) The *fiber comparison isomorphism*

$$\text{comp} : \mathbb{C} \otimes H^0(\tilde{\mathbb{P}}^1, \beta_! R\alpha_* \mathcal{F}_{\mathbb{Q}}) \rightarrow H_{\text{dR}}^1(\mathbb{A}_\theta^1, N \otimes E^\theta)$$

is the composition of $H^0(\tilde{\mathbb{P}}^1, \text{comp}_{\tilde{\mathbb{P}}^1})$ with that given by (A.1).

(2) The *fiber period structure* $\text{FPer}(N^{\text{H}})$ of an exponential mixed Hodge structure N^{H} is the following object of the category Per :

$$\text{FPer}(N^{\text{H}}) = (H_{\text{dR}}^1(\mathbb{A}_\theta^1, N \otimes E^\theta), H^0(\tilde{\mathbb{P}}^1, \beta_! R\alpha_* \mathcal{F}_{\mathbb{Q}}), \text{comp}).$$

In the following, we also regard FPer as a functor defined on $\text{MHM}(\mathbb{A}_\theta^1)$ by factorizing through EMHS by Π . The following lemma is then obvious.

Lemma A.4. *The assignment FPer defines an exact functor $\text{MHM}(\mathbb{A}_\theta^1) \rightarrow \text{Per}$ factoring through Π , and there is an isomorphism of functors to the category Per :*

$$\text{FPer}(N^{\text{H}}) \simeq (H^0(\tilde{\mathbb{P}}^1, \beta_! R\alpha_* {}^p\text{DR}^{\text{an}}(N \otimes E^\theta)), H^0(\tilde{\mathbb{P}}^1, \beta_! R\alpha_* \mathcal{F}_{\mathbb{Q}}), H^0(\tilde{\mathbb{P}}^1, \beta_! R\alpha_*(e^{-\theta} \circ \text{comp}_{\mathbb{A}^{1\text{an}}}))).$$

On the other hand, let $i_0 : \{0\} \hookrightarrow \mathbb{A}^1$ be the closed embedding. The abelian category $\text{MHS} = \text{MHM}(\{0\})$ of mixed Hodge structures is identified with a full subcategory of $\text{MHM}(\mathbb{A}^1)$ and of EMHS via the pushforward ${}_{\text{H}}i_{0!}$ of mixed Hodge modules and the composition $\Pi \circ {}_{\text{H}}i_{0!}$, respectively.

Proposition A.5. *There is a commutative diagram of functors:*

$$\begin{array}{ccc}
 \text{MHS} & \xrightarrow{\Pi \circ {}_{\text{H}}i_{0!}} & \text{EMHS} \\
 \text{Per} \downarrow & & \swarrow \text{FPer} \\
 & & \text{Per}
 \end{array}$$

Proof. This follows from the tautological identification for a vector space V over either \mathbb{Q} or \mathbb{C}

$$\text{H}_{\mathbb{C}}^0(\widetilde{\mathbb{P}}_{\text{mod}}^1, R\alpha_* Ri_{0!}V) = V,$$

which commutes with change of base field. □

A2. Computation of the fiber comparison isomorphism for a Gauss–Manin exponential mixed Hodge structure. Let $f : X \rightarrow \mathbb{A}_{\theta}^1$ be a projective morphism from a smooth quasiprojective variety X to the affine line \mathbb{A}_{θ}^1 . We denote the pushforward functor $\text{D}^b(\text{MHM}(X)) \rightarrow \text{D}^b(\text{MHM}(\mathbb{A}_{\theta}^1))$ by ${}_{\text{H}}f_*$; see [Saito 1990, Theorem 4.3]. For each object N^{H} of $\text{MHM}(X)$ and each integer r , we consider the mixed Hodge module $\mathcal{H}^r({}_{\text{H}}f_* N^{\text{H}})$ that we simply denote by ${}_{\text{H}}f_*^r N^{\text{H}}$. We will compute its fiber period structure by means of cohomology on a suitable real blow-up space.

For that purpose, consider a smooth projective compactification $\bar{f} : \bar{X} \rightarrow \mathbb{P}^1$ of X and f giving rise to a commutative diagram

$$\begin{array}{ccc}
 X & \xrightarrow{j} & \bar{X} \\
 f \downarrow & & \downarrow \bar{f} \\
 \mathbb{A}_{\theta}^1 & \xrightarrow{\quad} & \mathbb{P}^1
 \end{array}$$

such that the pole divisor $P = \bar{f}^{-1}(\infty)$ is a strict normal crossing divisor in \bar{X} . We denote by $\varpi : \tilde{X}(P) \rightarrow \bar{X}$ the real-oriented blowing-up of \bar{X} along the irreducible components of P . Then \bar{f} lifts as a real-analytic morphism $\tilde{f} : \tilde{X}(P) \rightarrow \tilde{\mathbb{P}}^1$. In this section, we set $\tilde{X} = \tilde{X}(P)$. We define the open subset $\partial_{\text{mod}}\tilde{X}$ of $\partial\tilde{X} = \varpi^{-1}(P)$ as consisting of points of $\varpi^{-1}(P)$ in the neighborhood of which e^{-f} has moderate growth — equivalently, rapid decay — so that $\partial_{\text{mod}}\tilde{X} = \tilde{f}^{-1}(\partial_{\text{mod}}\tilde{\mathbb{P}}^1)$. The subset

$$\tilde{X}_{\text{mod}} := X \cup \partial_{\text{mod}}\tilde{X} = \tilde{f}^{-1}(\tilde{\mathbb{P}}_{\text{mod}}^1)$$

is open in \tilde{X} . We also denote by α, β the respective open inclusions

$$X \xrightarrow{\alpha} \tilde{X}_{\text{mod}} \xrightarrow{\beta} \tilde{X}.$$

As in [Section 2.1](#), we set $E^f = (\mathcal{O}_X, d + df)$ and, for a \mathcal{D}_X -module N regarded as an \mathcal{O}_X -module with flat connection ∇ , we set $N \otimes E^f = (N, \nabla + df)$. To a mixed Hodge module N^H on X we associate a period structure in Per as follows. The vector spaces are (the perverse convention is used here)

$$\begin{aligned} {}^p\text{H}_{\text{dR}}^r(X, N \otimes E^f) &:= \text{H}^r(X, {}^p\text{DR}_X(N \otimes E^f)), \\ {}^p\text{H}_{\text{B}}^r(X, \mathcal{F}_{\mathbb{Q}} \otimes E^f) &:= \text{H}^r(\tilde{X}, \beta_! R\alpha_* \mathcal{F}_{\mathbb{Q}}) = \text{H}_{\mathbb{C}}^r(\tilde{X}_{\text{mod}}, R\alpha_* \mathcal{F}_{\mathbb{Q}}). \end{aligned}$$

In order to make the comparison isomorphism explicit, we need the following proposition. We note that termwise multiplication by the holomorphic function e^{-f} induces an isomorphism ${}^p\text{DR}_X^{\text{an}} N \xrightarrow{\sim} {}^p\text{DR}_X^{\text{an}}(N \otimes E^f)$. On the other hand, we endow \tilde{X} with the sheaf $\mathcal{S}_{\tilde{X}}^{\text{mod}}$ of holomorphic functions on X having moderate growth along $\partial\tilde{X}$. Then any \mathcal{D}_X -module M has a (shifted) moderate de Rham complex ${}^p\text{DR}_{\tilde{X}}^{\text{mod}} M$ on \tilde{X} ; see e.g., [[Fresán et al. 2023](#), Section 2.e]. We also denote by $j : X \rightarrow \tilde{X}$ the inclusion.

Proposition A.6. *For a regular holonomic \mathcal{D}_X -module N , the following two natural morphisms*

$$\begin{aligned} \beta_! \beta^{-1} {}^p\text{DR}_{\tilde{X}}^{\text{mod}} j_+(N \otimes E^f) &\rightarrow {}^p\text{DR}_{\tilde{X}}^{\text{mod}} j_+(N \otimes E^f), \\ \beta^{-1} {}^p\text{DR}_{\tilde{X}}^{\text{mod}} j_+(N \otimes E^f) &\rightarrow R\alpha_* \alpha^{-1} \beta^{-1} {}^p\text{DR}_{\tilde{X}}^{\text{mod}} j_+(N \otimes E^f) \end{aligned}$$

are quasiisomorphisms.

Sketch of proof. A similar result is proved in [[Sabbah 1996](#)] (proof of Theorem 5.1) but the definition of the sheaf of functions with moderate growth used there is more restrictive than the one needed for our purposes. Instead, one uses computations similar to those of [[Hien 2007](#), Proposition 3.3] (see also [[Hien 2009](#), Proposition 1]) together with [[Mochizuki 2014](#), Corollary 4.7.3]. \square

We conclude that there is a natural isomorphism

$$\beta_! R\alpha_* {}^p\text{DR}_X^{\text{an}}(N \otimes E^f) \xrightarrow{\sim} {}^p\text{DR}_{\tilde{X}}^{\text{mod}} j_+(N \otimes E^f) \quad (\text{A.7})$$

in the derived category $\text{D}^b(\mathbb{C}_{\tilde{X}})$, which is functorial with respect to N . Moreover, by the arguments of adjunction already used in the proof of [Lemma A.2](#), this is the unique isomorphism extending the identity on X .

Corollary A.8. *For a mixed Hodge module $N^H \in \text{MHM}(X)$ there exists a unique isomorphism*

$$\text{comp}_{\tilde{X}} : \beta_! R\alpha_* \mathcal{F}_{\mathbb{C}} \rightarrow {}^p\text{DR}_{\tilde{X}}^{\text{mod}} j_+(N \otimes E^f)$$

which extends $e^{-f} \circ \text{comp}_X : \mathcal{F}_{\mathbb{C}} \rightarrow {}^p\text{DR}_X^{\text{an}}(N \otimes E^f)$. We have

$$\text{comp}_{\tilde{X}} = (\text{A.7}) \circ \beta_! R\alpha_*(e^{-f} \circ \text{comp}_X).$$

This morphism is functorial on $\text{MHM}(X)$. \square

In a way similar to [\(A.1\)](#), one shows that, for any r , there exists a canonical isomorphism

$$\begin{aligned} \text{can}_r : \text{H}^r(\tilde{X}, {}^p\text{DR}_{\tilde{X}}^{\text{mod}} j_+(N \otimes E^f)) &\simeq \text{H}^r(\bar{X}, {}^p\text{DR}_{\bar{X}} j_+(N \otimes E^f)) \\ &= \text{H}^r(X, {}^p\text{DR}_X(N \otimes E^f)) =: {}^p\text{H}_{\text{dR}}^r(X, N \otimes E^f). \end{aligned}$$

Definition A.9. Let N^H be an object of $\text{MHM}(X)$. For each r (the perverse convention is used here), the fiber period structure $\text{FPer}^r(N^H \otimes E^f)$ is defined as

$$\text{FPer}^r(N^H \otimes E^f) = ({}^p\text{H}_{\text{dR}}^r(X, N \otimes E^f), \text{H}_{\mathbb{B}}^r(X, \mathcal{F}_{\mathbb{Q}} \otimes E^f), \text{can}_r \circ {}^p\text{H}^r(\tilde{X}, \text{comp}_{\tilde{X}})).$$

Lemma A.10 (expression of $\text{FPer}^r(N^H \otimes E^f)$ on the real blow-up). *For each r , there is an isomorphism in Per :*

$$\text{FPer}^r(N^H \otimes E^f) \simeq (\text{H}^r(\tilde{X}, \beta_! R\alpha_* {}^p\text{DR}_X^{\text{an}} j_+(N \otimes E^f)v), \text{H}^r(\tilde{X}, \beta_! R\alpha_* \mathcal{F}_{\mathbb{Q}}), \text{H}^r(\tilde{X}, \beta_! R\alpha_*(e^{-f} \circ \text{comp}_X))).$$

Proof. The isomorphism is the identity on the middle term and the isomorphism on the left term is given by $\text{can}_r \circ \text{H}^r(\tilde{X}, (\text{A.7}))$. □

We then deduce that the fiber period structure $\text{FPer}^r(N^H \otimes E^f)$ is isomorphic to that attached to the exponential mixed Hodge structure $\Pi({}_H f_*^r N^H)$:

Proposition A.11. *For each r , there is a functorial isomorphism of fiber period structures:*

$$\text{FPer}({}_H f_*^r N^H) \simeq \text{FPer}^r(N^H \otimes E^f).$$

Proof. By definition, the comparison isomorphism $\text{comp}_{\mathbb{A}_\theta^1}$ for ${}_H f_*^j N^H$ is obtained by taking the j -th perverse cohomology of the composed isomorphism

$$Rf_* \mathcal{F}_{\mathbb{C}} \xrightarrow{Rf_* \text{comp}_X} Rf_* {}^p\text{DR}^{\text{an}} N \xrightarrow{\sim} {}^p\text{DR}^{\text{an}} f_+ N,$$

where the second morphism is the standard functorial isomorphism from \mathcal{D} -module theory. We also have a commutative diagram:

$$\begin{array}{ccc} Rf_* {}^p\text{DR}^{\text{an}} N & \longrightarrow & {}^p\text{DR}^{\text{an}} f_+ N \\ Rf_* e^{-f} \downarrow & & \downarrow e^{-\theta} \\ Rf_* {}^p\text{DR}^{\text{an}}(N \otimes E^f) & \longrightarrow & {}^p\text{DR}^{\text{an}}((f_+ N) \otimes E^\theta) \end{array}$$

We now take the models of Lemmas A.4 and A.10 and the desired isomorphism is obtained (after applying $R\Gamma(\tilde{\mathbb{P}}^1, \cdot)$ and taking cohomology) from the canonical isomorphism of functors

$$R\tilde{f}_* \circ \beta_! R\alpha_* \simeq \beta_! R\alpha_* Rf_*$$

from $\text{D}^b(\mathbb{C}_X)$ to $\text{D}^b(\mathbb{C}_{\tilde{\mathbb{P}}^1})$, that we recall now. We consider the two commutative squares:

$$\begin{array}{ccccc} X & \xrightarrow{\alpha} & \tilde{X}_{\text{mod}} & \xrightarrow{\beta} & \tilde{X} \\ f \downarrow & & \downarrow \tilde{f}_{\text{mod}} & & \downarrow \tilde{f} \\ \mathbb{A}_\theta^1 & \xrightarrow{\alpha} & \tilde{\mathbb{P}}_{\text{mod}}^1 & \xrightarrow{\beta} & \tilde{\mathbb{P}}^1 \end{array}$$

Since \tilde{f} is proper, we have a canonical isomorphism of functors $R\tilde{f}_* \circ \beta_! \simeq \beta_! \circ R\tilde{f}_{\text{mod}*}$, and on the other hand we have $R\tilde{f}_{\text{mod}*} \circ R\alpha_* \simeq R\alpha_* \circ R\tilde{f}_*$. □

A3. Fiber period realization of a pair (U, f) .

Application of the results of Section A2. Let U be a smooth complex quasiprojective variety of dimension d and let $f : U \rightarrow \mathbb{A}_\theta^1$ be a regular function. Starting with the mixed Hodge module ${}^p\mathbb{Q}_U^H$, we aim at giving a formula for the fiber period structure of the exponential mixed Hodge structures associated with the pushforward mixed Hodge modules ${}_{\mathbb{H}}f_*^r {}^p\mathbb{Q}_U^H$ and ${}_{\mathbb{H}}f_!^r {}^p\mathbb{Q}_U^H$ ($r \in \mathbb{Z}$). For that purpose, it is convenient to choose a partial completion $\kappa : U \hookrightarrow X$ of U as a smooth quasiprojective variety X so that $H = X \setminus U$ is a strict normal crossing divisor and that f extends as a projective morphism $f : X \rightarrow \mathbb{A}_\theta^1$. The result will be independent of such a choice. The commutative diagram used in Section A2 is thus completed on the left as follows:

$$\begin{array}{ccccc} U & \xhookrightarrow{\kappa} & X & \xhookrightarrow{j} & \bar{X} \\ f \downarrow & & \downarrow f & & \downarrow \bar{f} \\ \mathbb{A}_\theta^1 & \xlongequal{\quad} & \mathbb{A}_\theta^1 & \xhookrightarrow{\quad} & \mathbb{P}^1 \end{array}$$

We consider the objects N^H of $\text{MHM}(X)$ defined as $N^H = {}_{\mathbb{H}}\kappa_* {}^p\mathbb{Q}_U^H$ or $N^H = {}_{\mathbb{H}}\kappa_! {}^p\mathbb{Q}_U^H$. Correspondingly, we write $N \otimes E^f$ as $E^f(*H)$ or $E^f(!H)$ and we have $\mathcal{F}_{\mathbb{Q}} = R\kappa_* {}^p\mathbb{Q}_U$ or $R\kappa_! {}^p\mathbb{Q}_U$. In this way, we are in the setting of Section A2.

We denote respectively by $H^r(U, f)$ and $H_c^r(U, f)$ the exponential mixed Hodge structure $\Pi({}_{\mathbb{H}}f_*^{r-d} {}^p\mathbb{Q}_U^H)$ and $\Pi({}_{\mathbb{H}}f_!^{r-d} {}^p\mathbb{Q}_U^H)$, and

$$H_{\text{mid}}^r(U, f) = \text{im}[H_c^r(U, f) \rightarrow H^r(U, f)]. \tag{A.12}$$

We denote respectively the associated de Rham fibers by $H_{\text{dR}}^r(U, f)$ and $H_{\text{dR},c}^r(U, f)$, and the Betti fibers by $H_{\mathbb{B}}^r(U, f)$ and $H_{\mathbb{B},c}^r(U, f)$. We then have

$$H_{\text{dR}}^r(U, f) \simeq H^r(X, \text{DR}(E^f(*H))), \quad H_{\text{dR},c}^r(U, f) \simeq H^r(X, \text{DR}(E^f(!H))).$$

Keeping the notation of Section A2 for α, β (and emphasizing now the divisor P), we have by Proposition A.11

$$H_{\mathbb{B}}^r(U, f) = H^r(\tilde{X}(P), \beta_! R\alpha_* R\kappa_* \mathbb{Q}_U), \quad H_{\mathbb{B},c}^r(U, f) = H^r(\tilde{X}(P), \beta_! R\alpha_* R\kappa_! \mathbb{Q}_U).$$

We set $\tilde{U}_{\text{mod}}(P) = \tilde{X}_{\text{mod}}(P) \setminus \varpi^{-1}(H)$ and we denote by Φ the family of closed subsets of $\tilde{U}_{\text{mod}}(P)$ whose closure in \tilde{X} is contained in the open subset $\tilde{X}_{\text{mod}}(P)$.

Proposition A.13. *We have*

$$H_{\mathbb{B}}^r(U, f) \simeq H_{\Phi}^r(\tilde{U}_{\text{mod}}(P), \mathbb{Q}) \quad \text{and} \quad H_{\mathbb{B},c}^r(U, f) \simeq H_c^r(\tilde{U}_{\text{mod}}(P), \mathbb{Q}),$$

and the natural morphism $H_{\mathbb{B},c}^r(U, f) \rightarrow H_{\mathbb{B}}^r(U, f)$ is induced by the inclusion of the families of supports.

Remark A.14. Setting $\tilde{U}(P) = \tilde{X}(P) \setminus \varpi^{-1}(H)$ and denoting by $\partial_{\text{exp}} \tilde{X}(P)$, respectively $\partial_{\text{exp}} \tilde{U}(P)$, the closed subset complement to $\tilde{X}_{\text{mod}}(P)$ in $\tilde{X}(P)$, respectively to $\tilde{U}_{\text{mod}}(P)$ in $\tilde{U}(P)$, the spaces $H_{\mathbb{B}}^r(U, f)$

and $H_{B,c}^r(U, f)$ also read in terms of relative cohomology

$$H_{B,c}^r(U, f) \simeq H^r(\tilde{U}(P), \partial_{\text{exp}} \tilde{U}(P), \mathbb{Q}), \quad H_{B,c}^r(U, f) \simeq H^r(\tilde{X}(P), \partial_{\text{exp}} \tilde{X}(P) \cup \varpi^{-1}(H), \mathbb{Q}),$$

and the natural morphism between both is induced by the inclusion of pairs $(\tilde{U}(P), \partial_{\text{exp}} \tilde{U}(P)) \hookrightarrow (\tilde{X}(P), \partial_{\text{exp}} \tilde{X}(P))$, so that $H_{B,\text{mid}}^r(U, f)$ is the corresponding image, according to (A.12).

Proof of Proposition A.13. In the proof, we simply set $\tilde{X} = \tilde{X}(P)$. With the notation above, we consider the commutative diagram

$$\begin{array}{ccccccc} U & \xrightarrow{\alpha_U} & \tilde{U}_{\text{mod}} & \xrightarrow{\beta_U} & \tilde{U} & \xleftarrow{\gamma_U} & \partial_{\text{exp}} \tilde{U} \\ \downarrow \kappa & & \downarrow \tilde{\kappa}' & & \downarrow \tilde{\kappa} & & \downarrow \\ X & \xrightarrow{\alpha} & \tilde{X}_{\text{mod}} & \xrightarrow{\beta} & \tilde{X} & \xleftarrow{\gamma} & \partial_{\text{exp}} \tilde{X} \end{array}$$

where the first line is obtained from the second one by deleting $\varpi^{-1}(H)$.

For the identification of $H_{B,c}^r(U, f)$, we need the next lemma.

Lemma A.15. *For $\star = *$ or $\star = !$, there is an isomorphism in $D^b(\tilde{X}, \mathbb{Q})$*

$$R\tilde{\kappa}_\star \beta_{U!} R\alpha_{U*} \mathbb{Q}_U \simeq \beta_! R\alpha_* R\kappa_\star \mathbb{Q}_U.$$

Proof. We can replace $R\alpha_* R\kappa_\star$ with $R\tilde{\kappa}'_\star R\alpha_{U*}$. Furthermore, a local computation shows that

$$R\alpha_{U*} \mathbb{Q}_U = \mathbb{Q}_{\tilde{U}_{\text{mod}}} = \beta_U^{-1} \mathbb{Q}_{\tilde{U}} \quad \text{and} \quad (R\beta_U \circ R\alpha_U)_* \mathbb{Q}_U = \mathbb{Q}_{\tilde{U}}.$$

We are thus reduced to finding an isomorphism $R\tilde{\kappa}_\star \beta_{U!} \mathbb{Q}_{\tilde{U}_{\text{mod}}} \simeq \beta_! R\tilde{\kappa}'_\star \mathbb{Q}_{\tilde{U}_{\text{mod}}}$. Let us first construct a morphism. There is a natural morphism

$$R\tilde{\kappa}_\star \beta_{U!} \mathbb{Q}_{\tilde{U}_{\text{mod}}} \rightarrow R\tilde{\kappa}_\star R\beta_{U*} \mathbb{Q}_{\tilde{U}_{\text{mod}}} \simeq R\beta_* R\tilde{\kappa}'_\star \mathbb{Q}_{\tilde{U}_{\text{mod}}}$$

and this morphism can be lifted as a morphism to $\beta_! R\tilde{\kappa}'_\star \mathbb{Q}_{\tilde{U}_{\text{mod}}}$ if and only if its restriction by γ is zero. Clearly, $\gamma^{-1} R\tilde{\kappa}_\star \beta_{U!} \mathbb{Q}_{\tilde{U}_{\text{mod}}}$ is zero on $\partial_{\text{exp}} \tilde{U}$ and we need to check that the same property holds true on $\varpi^{-1}(H) \cap \partial_{\text{exp}} \tilde{X}$. The question reduces to a local computation in the neighborhood of each point of $P \cap H$ in \tilde{X} . We thus work in an adapted coordinate neighborhood Δ^d of such a point. We can write $\Delta^d = \Delta^\ell \times \Delta^{d-\ell}$, with $P \cap \Delta^d = P' \times \Delta^{d-\ell}$ defined by the vanishing of the product of coordinates in Δ^ℓ and $H \cap \Delta^d = \Delta^\ell \times H''$ defined by the vanishing of the product of some coordinates in $\Delta^{d-\ell}$. In this model, the real blowing-up $\varpi : \tilde{\Delta}^\ell \times \Delta^{d-\ell} \rightarrow \Delta^\ell \times \Delta^{d-\ell}$ is induced by the real blowing-up of Δ^ℓ along its coordinates hyperplanes. In restriction to this chart we have $U = \Delta^\ell \times (\Delta^{d-\ell} \setminus H'')$ and

$$\begin{aligned} \tilde{X}_{\text{mod}} &= (\tilde{\Delta}^\ell)_{\text{mod}} \times \Delta^{d-\ell}, & \tilde{U}_{\text{mod}} &= (\tilde{\Delta}^\ell)_{\text{mod}} \times (\Delta^{d-\ell} \setminus H''), \\ \partial_{\text{exp}} \tilde{X} &= \partial_{\text{exp}}(\tilde{\Delta}^\ell) \times \Delta^{d-\ell}, & \partial_{\text{exp}} \tilde{U} &= \partial_{\text{exp}}(\tilde{\Delta}^\ell) \times (\Delta^{d-\ell} \setminus H''). \end{aligned} \tag{A.16}$$

The assertion is then clear since the morphisms β_U and $\tilde{\kappa}$ act on disjoint sets of variables. With the same local computation, one checks that the morphism thus obtained is an isomorphism. \square

We can now conclude the proof for $H_B^r(U, f)$. From the previous lemma with $\star = *$ we deduce

$$H^r(\tilde{X}, \beta_! R\alpha_* R\kappa_* \mathbb{Q}_U) \simeq H^r(\tilde{X}, R\tilde{\kappa}_* \beta_{U!} R\alpha_{U*} \mathbb{Q}_U) = H^r(\tilde{U}, \beta_{U!} \mathbb{Q}_{\tilde{\text{mod}}}),$$

and the assertion is then clear. On the other hand, the distinguished triangle in $D^b(\tilde{U}, \mathbb{Q})$

$$\beta_{U!} \beta_U^{-1} \mathbb{Q}_{\tilde{U}} \rightarrow \mathbb{Q}_{\tilde{U}} \rightarrow R\gamma_{U*} \gamma_U^{-1} \mathbb{Q}_{\tilde{U}} \xrightarrow{+1}$$

gives the expression of $H_B^r(U, f)$ in terms of relative cohomology as asserted in [Remark A.14](#).

For $H_{B,c}^r(U, f)$, the previous lemma with $\star = !$ gives similarly

$$H^r(\tilde{X}, \beta_! R\alpha_* \kappa_! \mathbb{Q}_U) \simeq H^r(\tilde{X}, (\tilde{\kappa} \circ \beta_U)_! R\alpha_{U*} \mathbb{Q}_U) = H_c^r(\tilde{\text{mod}}, \mathbb{Q}). \quad \square$$

Remark A.17. Let $Z \subset U$ be a divisor on which f vanishes, let a_Z denote the structure morphism, let $i_Z: Z \hookrightarrow U$ denote the closed inclusion and $j_Z: U \setminus Z \hookrightarrow U$ the complementary open inclusion. We set ${}^p\mathbb{Q}_Z^H = {}_H a_Z^* \mathbb{Q}_{\text{Spec } \mathbb{C}}^H[\dim Z]$, so that there is an isomorphism

$${}_{H^!} {}^* {}^p\mathbb{Q}_U^H = \mathcal{H}^0 {}_{H^!} {}^* {}^p\mathbb{Q}_U^H \simeq {}^p\mathbb{Q}_Z^H$$

and an exact sequence

$$0 \rightarrow {}_{H^!} i_{Z*} {}^p\mathbb{Q}_Z^H \rightarrow {}_{H^!} j_{Z!} {}^* {}^p\mathbb{Q}_U^H \rightarrow {}^p\mathbb{Q}_U^H \rightarrow 0,$$

giving rise to an exact sequence in EMHS, see [\[Fresán et al. 2022, \(A.21\)\]](#),

$$\dots \rightarrow H_c^{r-1}(Z) \rightarrow H_c^r(U \setminus Z, f) \rightarrow H_c^r(U, f) \rightarrow H_c^r(Z) \rightarrow \dots$$

If $H_c^r(U \setminus Z, f) = 0$ for each r , the exponential mixed Hodge structure $H_c^r(U, f)$ is isomorphic to the mixed Hodge structure $H_c^r(Z)$ and, correspondingly, the fiber period structure $\text{FPer}(H_c^r(U, f))$ is isomorphic to $\text{Per}(H_c^r(Z))$. We will make explicit this exact sequence for the Betti fibers. Since $\bar{f}: \bar{X} \rightarrow \mathbb{P}^1$ is a morphism, we have $\bar{Z} \cap P = \emptyset$. We have a distinguished triangle

$$\beta_! R\alpha_* Ri_{Z*} \mathbb{Q}_Z \rightarrow \beta_! R\alpha_* j_{Z!} \mathbb{Q}_{U \setminus Z} \rightarrow \beta_! R\alpha_* \mathbb{Q}_U \xrightarrow{+1}$$

and since the closure of Z in $\tilde{X}(P)$ does not intersect $\partial \tilde{X}(P)$, we find

$$\beta_! R\alpha_* Ri_{Z*} \mathbb{Q}_Z = \beta_! R\alpha_* Ri_{Z*} \mathbb{Q}_Z \quad \text{and} \quad \beta_! R\alpha_* j_{Z!} \mathbb{Q}_{U \setminus Z} = \beta_{Z!} \mathbb{Q}_{\tilde{\text{mod}}(P) \setminus Z},$$

where β_Z is the inclusion $\tilde{\text{mod}}(P) \setminus Z \hookrightarrow \tilde{X}(P)$. The Betti exact sequence reduces then to

$$\dots \rightarrow H_c^{r-1}(Z, \mathbb{Q}) \rightarrow H_c^r(\tilde{\text{mod}}(P) \setminus Z, \mathbb{Q}) \rightarrow H_c^r(\tilde{\text{mod}}(P), \mathbb{Q}) \rightarrow H_c^r(Z, \mathbb{Q}) \rightarrow \dots \quad (\text{A.18})$$

Computation with the total real blow-up. In order to use results of [\[Fresán et al. 2023\]](#), we consider the real blowing-up $\pi: \tilde{X}(D) \rightarrow \bar{X}$ of the irreducible components of $D = P \cup H$ in \bar{X} . There is a natural morphism $\tilde{\omega}: \tilde{X}(D) \rightarrow \tilde{X}(P)$, so that $\pi = \tilde{\omega} \circ \tilde{\omega}$. In a local chart where formulas [\(A.16\)](#) hold, $\tilde{\omega}$ is the blowing-up map of the components of H'' in $\Delta^{d-\ell}$:

$$\tilde{X}(D) = \tilde{\Delta}^\ell \times \tilde{\Delta}^{d-\ell} \rightarrow \tilde{\Delta}^\ell \times \Delta^{d-\ell} = \tilde{X}(P).$$

We consider the open subsets $\tilde{U}_{\text{mod}}(D) = U \cup \partial_{\text{mod}}\tilde{X}(D)$ and $\tilde{U}_{\text{rd}}(D) = U \cup \partial_{\text{rd}}\tilde{X}(D)$, where:

- $\partial_{\text{mod}}\tilde{X}(D)$ is the open subset of $\pi^{-1}(D)$ in the neighborhood of which e^{-f} has moderate growth (it contains $\pi^{-1}(D \setminus P)$).
- $\partial_{\text{rd}}\tilde{X}(D)$ is the open subset of $\pi^{-1}(P)$ in the neighborhood of which e^{-f} has moderate growth, equivalently, rapid decay.

In the local chart as above, these sets read

$$\tilde{U}_{\text{mod}}(D) = (\tilde{\Delta}^\ell)_{\text{mod}} \times \tilde{\Delta}^{d-\ell}, \quad \tilde{U}_{\text{rd}}(D) = (\tilde{\Delta}^\ell)_{\text{mod}} \times (\Delta^{d-\ell} \setminus H'').$$

For the sake of simplicity, we denote by $\mathbb{Q}_{\tilde{U}_{\text{mod}}(D)}$ the sheaf on $\tilde{X}(D)$ which is the extension by zero of the constant sheaf on $\tilde{U}_{\text{mod}}(D)$ with stalk \mathbb{Q} (notation of [Kashiwara and Schapira 1990]), and similarly with rd. From the previous identifications with now $\alpha : X \hookrightarrow \tilde{X}_{\text{mod}}(D)$ and $\beta : \tilde{X}_{\text{mod}}(D) \hookrightarrow \tilde{X}(D)$, we obtain

$$R\tilde{\omega}_* \mathbb{Q}_{\tilde{U}_{\text{mod}}(D)} = \beta_! R\alpha_* Rk_* \mathbb{Q}_U, \quad R\tilde{\omega}_* \mathbb{Q}_{\tilde{U}_{\text{rd}}(D)} = \mathbb{Q}_{\tilde{U}_{\text{mod}}(P)}$$

(in fact $\tilde{\omega} : \tilde{U}_{\text{rd}}(D) \rightarrow \tilde{U}_{\text{mod}}(P)$ is an isomorphism). Therefore,

$$H_{\mathbb{B}}^r(U, f) \simeq H_c^r(\tilde{U}_{\text{mod}}(D), \mathbb{Q}) \quad \text{and} \quad H_{\mathbb{B},c}^r(U, f) \simeq H_c^r(\tilde{U}_{\text{rd}}(D), \mathbb{Q}). \quad (\text{A.19})$$

Let Φ_{rd} (resp. Φ_{mod}) denote the family of closed subsets F of U whose (compact) closure \bar{F} in $\tilde{X}(D)$ is contained in $\tilde{U}_{\text{rd}}(D)$ (resp. $\tilde{U}_{\text{mod}}(D)$). A closed set F of U belongs to Φ_{rd} (resp. Φ_{mod}) if and only if $|\exp(-f)|_F$ tends to zero faster than any positive power of $\text{dist}(x, x_o)$ (resp. is bounded by some negative power of $\text{dist}(x, x_o)$) when $x \in F$ tends to some $x_o \in D$. Then the right-hand sides in (A.19) read

$$H_c^r(\tilde{U}_{\text{mod}}(D), \mathbb{Q}) = H_{\Phi_{\text{mod}}}^r(U, \mathbb{Q}), \quad H_c^r(\tilde{U}_{\text{rd}}(D), \mathbb{Q}) = H_{\Phi_{\text{rd}}}^r(U, \mathbb{Q}), \quad (\text{A.20})$$

and, by considering the natural morphism induced by the inclusion of family of supports, we have

$$H_{\mathbb{B},\text{mid}}^r(U, f) = \text{im}[H_{\Phi_{\text{rd}}}^r(U, \mathbb{Q}) \rightarrow H_{\Phi_{\text{mod}}}^r(U, \mathbb{Q})]. \quad (\text{A.21})$$

Rapid decay and moderate growth homology spaces for the pair (U, f) . If $\mathbf{1}$ denotes the generator of $E^f = (\mathcal{O}_U, d + df)$, then $\exp(-f) \cdot \mathbf{1}$ is an analytic flat section of E^f . The moderate growth and the rapid decay homology spaces of the pair (U, f) , as defined in [Fresán et al. 2023], are the homology of the chain complexes consisting of singular chains in $\tilde{X}(D)$ with boundary in $\partial\tilde{X}(D)$ twisted by the flat section $\exp(-f) \cdot \mathbf{1}$, whose support is contained in $\tilde{U}_{\text{mod}}(D)$ and $\tilde{U}_{\text{rd}}(D)$, respectively. The flat section being fixed, we get identifications with relative homology spaces

$$H_r^{\text{mod}}(U, f) \simeq H_r(\tilde{U}_{\text{mod}}(D), \partial\tilde{U}_{\text{mod}}(D), \mathbb{Q}) \quad \text{and} \quad H_r^{\text{rd}}(U, f) \simeq H_r(\tilde{U}_{\text{rd}}(D), \partial\tilde{U}_{\text{rd}}(D), \mathbb{Q}). \quad (\text{A.22})$$

Notation A.23. For the sake of simplicity, we omit the flat section $\exp(-f) \cdot \mathbf{1}$ in the notation of such twisted chains, that we simply call respectively rapid decay and moderate chains (we will not make use of the latter).

We have a more explicit expression of the rapid decay homology as follows. By suitably lifting to $\tilde{X}(D)$ the radial vector field of length one centered at $\infty \in \mathbb{P}^1$ so that it remains tangent to $\pi^{-1}(D \setminus P)$, and by following its flow, we obtain for every large enough $R > 0$ a deformation retraction of the pair $(\tilde{U}_{\text{rd}}(D), \partial_R \tilde{U}_{\text{rd}}(D))$ to the pair $(\tilde{U}_{\text{rd}}(D), \partial \tilde{U}_{\text{rd}}(D))$, where the thickened boundary $\partial_R \tilde{U}_{\text{rd}}(D)$ is defined as $\tilde{U}_{\text{rd}}(D) \cap \{|f| \geq R\} \cap \{\text{Re}(f) > 0\}$. Setting

$$\partial_{\text{rd},R} U = U \cap \partial_R \tilde{U}_{\text{rd}}(D) = U \cap \{|f| \geq R\} \cap \{\text{Re}(f) > 0\}, \quad R \gg 0,$$

by excision of $\partial \tilde{U}_{\text{rd}}(D)$, we obtain (for $R \gg 0$)

$$H_r^{\text{rd}}(U, f) \simeq H_r(\tilde{U}_{\text{rd}}(D), \partial \tilde{U}_{\text{rd}}(D), \mathbb{Q}) \simeq H_r(\tilde{U}_{\text{rd}}(D), \partial_R \tilde{U}_{\text{rd}}(D), \mathbb{Q}) \simeq H_r(U, \partial_{\text{rd},R} U, \mathbb{Q}). \quad (\text{A.24})$$

Remark A.25 (period pairing and period realization). Working with the transposed period structures as in [Fresán et al. 2023, Proposition 2.28], and considering rapid decay and moderate growth homology, one can show that there exist isomorphisms of (the transposes of) period structures

$$\begin{aligned} \text{FPer } H^r(U, f) &\simeq (H'_{\text{dR}}(U, f), H_r^{\text{rd}}(U, f), P_r^{\text{rd,mod}}), \\ \text{FPer } H'_c(U, f) &\simeq (H'_{\text{dR},c}(U, f), H_r^{\text{mod}}(U, f), P_r^{\text{mod,rd}}), \\ \text{FPer } H'_{\text{mid}}(U, f) &\simeq (H'_{\text{dR,mid}}(U, f), H_r^{\text{mod}}(U, f), P_r^{\text{mid}}). \end{aligned}$$

A4. Period structures over the category of varieties and morphisms defined over \mathbb{Q} . In this section, we denote by U_0 a variety defined over \mathbb{Q} and by U the variety defined over \mathbb{C} after extension of scalars from \mathbb{Q} to \mathbb{C} . When working over varieties and morphism defined over \mathbb{Q} , that is, smooth separated schemes of finite type over \mathbb{Q} and separated morphisms (e.g., U_0 is \mathbb{A}^n or \mathbb{G}_m^n and f is a polynomial or a Laurent polynomial with rational coefficients), we are led to consider period structures over $\text{Spec } \mathbb{Q}$. Such a period structure consists of a pair of finite-dimensional \mathbb{Q} -vector spaces $(V_0, V_{\mathbb{Q}})$ together with a comparison isomorphism $\text{comp} : \mathbb{C} \otimes_{\mathbb{Q}} V_{\mathbb{Q}} \simeq \mathbb{C} \otimes_{\mathbb{Q}} V_0 = V^{\mathbb{C}}$.

\mathbb{Q} -structure on the de Rham cohomology. We fix a good compactification

$$j : (U_0, f) \hookrightarrow (\bar{X}_0, \bar{f}),$$

that is, such that $D_0 = X_0 \setminus U_0$ is a divisor with strict normal crossings (i.e., such that the irreducible components over $\bar{\mathbb{Q}}$ are smooth and intersect transversally). We work with the corresponding category of \mathcal{D} -modules and functors; see, e.g., [Laumon 1983, Sections 4 and 5]. The de Rham cohomology $H'_{\text{dR}}(U_0, f)$ is defined in a standard way as the de Rham cohomology of the \mathcal{D}_{U_0} -module $(\mathcal{O}_{U_0}, d + df)$, and the de Rham cohomology with compact support $H'_{\text{dR},c}(U_0, f)$ is the de Rham cohomology of the $\mathcal{D}_{\bar{X}_0}$ -module $j_{\dagger}(\mathcal{O}_{U_0}, d + df)$, with $j_{\dagger} = \mathbf{D} \circ j_+ \circ \mathbf{D}$ and \mathbf{D} is the duality functor of \mathcal{D} -modules. We denote by (U, f) the corresponding object obtained by extension of scalars from \mathbb{Q} to \mathbb{C} . We have:

Lemma A.26. *Extension of scalars is compatible with taking de Rham cohomology, that is,*

$$\mathbb{C} \otimes_{\mathbb{Q}} H'_{\text{dR}}(U_0, f) \simeq H'_{\text{dR}}(U, f), \quad \mathbb{C} \otimes_{\mathbb{Q}} H'_{\text{dR},c}(U_0, f) \simeq H'_{\text{dR},c}(U, f). \quad \square$$

As an immediate consequence we obtain that the same result holds for the middle de Rham cohomology $H_{\text{dR,mid}}^r(U_0, f)$.

Let us consider the setting of [Remark A.17](#) and let us assume that the triple (U, f, Z) is defined over \mathbb{Q} . Let us set $M = (\mathcal{O}_{U_0}, d + df)$. There is a natural exact sequence

$$0 = R^0\Gamma_{Z_0} M \rightarrow M \rightarrow j_{Z_0+} j_{Z_0}^+ M = M(*Z_0) \rightarrow R^1\Gamma_{Z_0} M \rightarrow 0$$

which identifies the complex $R\Gamma_{Z_0} M$ with $(\mathcal{O}_{U_0}(*Z_0)/\mathcal{O}_{U_0}, d + df)[-1]$. We note that

$$(\mathcal{O}_{U_0}(*Z_0)/\mathcal{O}_{U_0}, d + df) \simeq (\mathcal{O}_{U_0}(*Z_0)/\mathcal{O}_{U_0}, d).$$

Indeed, this amounts to showing that the exponential function $\exp(\pm f)$ is well-defined on $\mathcal{O}_{U_0}(*Z_0)/\mathcal{O}_{U_0}$, and this follows from the nilpotency of multiplication by f on each local section of $\mathcal{O}_{U_0}(*Z_0)/\mathcal{O}_{U_0}$. The long exact sequence in de Rham cohomology thus reads

$$\dots \rightarrow H_{\text{dR},Z_0}^r(U_0) \rightarrow H_{\text{dR}}^r(U_0, f) \rightarrow H_{\text{dR}}^r(U_0 \setminus Z_0, f) \rightarrow H_{\text{dR},Z_0}^{r+1}(U_0) \rightarrow \dots$$

Dually (in the sense of $\mathcal{D}_{\bar{X}_0}$ -modules, and up to changing f to $-f$), we obtain the exact sequence

$$\dots \rightarrow H_{\text{dR,c}}^{r-1}(Z_0) \rightarrow H_{\text{dR,c}}^r(U_0 \setminus Z_0, f) \rightarrow H_{\text{dR,c}}^r(U_0, f) \rightarrow H_{\text{dR,c}}^r(Z_0) \rightarrow \dots$$

Corollary A.27. *Assume moreover that the \mathbb{Q} -vector spaces $H_{\text{dR}}^r(U_0 \setminus Z_0, f)$ and $H_{\text{dR,c}}^r(U_0 \setminus Z_0, f)$ are zero for all r . Then the \mathbb{Q} -de Rham vector spaces $H_{\text{dR}}^r(U_0, f)$ and $H_{\text{dR},Z_0}^r(U_0)$, respectively $H_{\text{dR,c}}^r(U_0, f)$ and $H_{\text{dR,c}}^r(Z_0)$, coincide. \square*

Example A.28. We consider the setting of [\[Fresán et al. 2022, Example A.27\]](#) where the assumptions of [Corollary A.27](#) hold. We thus assume that $U_0 = \mathbb{A}_t^1 \times V_0$ for some smooth quasiprojective variety V_0 and $f = tg$ for some regular function g on V_0 . We set $\mathcal{X}_0 = g^{-1}(0)$ and $Z_0 = \mathbb{A}_t^1 \times \mathcal{X}_0$. [Corollary A.27](#) gives identifications of \mathbb{Q} -vector spaces

$$H_{\text{dR},Z_0}^r(U_0) \simeq H_{\text{dR}}^r(U_0, f) \quad \text{and} \quad H_{\text{dR,c}}^r(Z_0) \simeq H_{\text{dR,c}}^r(U_0, f).$$

Action of complex conjugation. We denote by $(U^{\mathbb{R}}, f^{\mathbb{R}})$ (or simply $U^{\mathbb{R}}, f^{\mathbb{R}}$) the real-analytic space and map associated with $(U(\mathbb{C}), f)$. Then the complex conjugation endows $U^{\mathbb{R}}$ with a real analytic involution conj which commutes with $f^{\mathbb{R}}$. Furthermore, one can find a compactification (\bar{X}_0, D_0) defined over \mathbb{Q} (since resolution of singularities holds in characteristic zero) so that conj extends in a unique way as a real analytic involution of $(\bar{X}^{\mathbb{R}}, D^{\mathbb{R}})$. Similarly, $\tilde{X}(D)^{\mathbb{R}}$, etc. belong to the semianalytic category and conj can then be lifted in a unique way as a semianalytic involution $\widetilde{\text{conj}}$ of $\tilde{X}(D)^{\mathbb{R}}$ that preserves $\partial\tilde{X}(D)^{\mathbb{R}}$. Lastly, since the moderate growth or rapid decay condition only involves $\text{Re}(f^{\mathbb{R}})$, the involution $\widetilde{\text{conj}}$ preserves the subsets $\tilde{U}_{\text{mod}}(D)^{\mathbb{R}}$ and $\tilde{U}_{\text{rd}}(D)^{\mathbb{R}}$.

Corollary A.29. *If (U, f) is obtained from (U_0, f) by extension of scalars, the \mathbb{Q} -Betti fibers $H_{\text{B}}^r(U, f)$ and $H_{\text{B,c}}^r(U, f)$ are naturally endowed with an involution conj^* , which is compatible with the natural morphism $H_{\text{B,c}}^r(U, f) \rightarrow H_{\text{B}}^r(U, f)$.*

Proof. Indeed, conj^* is induced by

$$\widetilde{\text{conj}}^* : H_c^r(\widetilde{U}_{\text{mod}}(D)^{\mathbb{R}}, \mathbb{Q}) \rightarrow H_c^r(\widetilde{U}_{\text{mod}}(D)^{\mathbb{R}}, \mathbb{Q}) \quad \text{or} \quad \widetilde{\text{conj}}^* : H_c^r(\widetilde{U}_{\text{rd}}(D)^{\mathbb{R}}, \mathbb{Q}) \rightarrow H_c^r(\widetilde{U}_{\text{rd}}(D)^{\mathbb{R}}, \mathbb{Q}). \quad \square$$

Corollary A.30. *In the setting of Remark A.17, assume that (U_0, f, Z_0) is defined over \mathbb{Q} . Then $\widetilde{\text{conj}}^*$ is compatible with the morphisms of the Betti exact sequence (A.18). In particular, if $H_c^r(U \setminus Z, f) = 0$ for all r , then the involutions conj^* on $H_{B,c}^r(U, f)$ and $H_c^r(Z^{\mathbb{R}}, \mathbb{Q})$ coincide. \square*

Example A.31. Let us keep the setting of Example A.28, that is, [Fresán et al. 2022, Example A.27]. It is proved in [loc. cit.] that, for all $r \in \mathbb{Z}$, we have a diagram of mixed Hodge structures

$$\begin{array}{ccc} H_c^r(\mathbb{A}_t^1 \times V, f) & \xrightarrow{\sim} & H_c^{r-2}(Z)(-1) \\ \downarrow & & \\ H^r(\mathbb{A}_t^1 \times V, f) & \xleftarrow{\sim} & H_Z^r(\mathbb{A}_t^1 \times V) \end{array} \tag{A.32}$$

where the vertical arrow is the natural one. Consider the case $r = d$, the dimension of $\mathbb{A}_t^1 \times V$. According to [loc. cit., Proposition A.19], the upper line is mixed of weights $\leq d$ and the lower line is mixed of weights $\geq d$. Furthermore, denoting by $H_{\text{mid}}^d(\mathbb{A}_t^1 \times V, f)$ the image of the vertical arrow, we have induced isomorphisms of pure Hodge structure:

$$\begin{array}{ccc} \text{gr}_d^W H_c^d(\mathbb{A}_t^1 \times V, f) & \xrightarrow{\sim} & (\text{gr}_{d-2}^W H_c^{d-2}(Z))(-1) \\ \downarrow \wr & & \\ H_{\text{mid}}^d(\mathbb{A}_t^1 \times V, f) & \xrightarrow{\sim} & W_d H^d(\mathbb{A}_t^1 \times V, f) \xleftarrow{\sim} W_d H_Z^d(\mathbb{A}_t^1 \times V) \end{array}$$

We assume that V_0, g are defined over \mathbb{Q} , making U_0, f also defined over \mathbb{Q} , as well as $\mathcal{X}_0 = g^{-1}(0)$ and $Z_0 = \mathbb{A}_t^1 \times \mathcal{X}$, so that $\mathcal{X}^{\mathbb{R}}$ and $Z^{\mathbb{R}}$ are preserved by conj . It follows from Corollary A.30 that the isomorphisms of (exponential) mixed Hodge structures (A.32) induce isomorphisms of Betti fibers which are compatible with conj^* .

Furthermore, the weight filtration $W_\bullet H_c^r(Z^{\mathbb{R}}, \mathbb{Q})$ is preserved by conj^* , since it comes from a filtration defined at the level of Nori motives; see [Huber and Müller-Stach 2017, Theorem 10.2.5]. One can argue that the weight filtration of $H_{B,c}^r(\mathbb{A}_t^1 \times V, f)$ is also preserved by conj^* by analyzing first the behavior of conj^* on $R^r f_* \mathbb{P}^1 \mathbb{Q}_U$. Nevertheless, it is enough for our purpose to check that conj^* induces an action on $H_{B,\text{mid}}^r(\mathbb{A}_t^1 \times V, f)$, a property that follows from interpreting the latter space as the image of the Betti vertical arrow (A.32).

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Added in proof

Recently, Y. Zhou also obtained [2021] the existence of quadratic relations as conjectured by Broadhurst and Roberts, with a different interpretation of the matrix D_k however. The methods are completely different from those of the present article and rely on the previous works of the author.

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javier.fresan@polytechnique.edu

Centre de Mathématiques Laurent Schwartz, CNRS, École polytechnique, Institut Polytechnique de Paris, France

claude.sabbah@polytechnique.edu

Centre de Mathématiques Laurent Schwartz, CNRS, École polytechnique, Institut Polytechnique de Paris, France

jdyu@ntu.edu.tw

Department of Mathematics, National Taiwan University, Taipei, Taiwan

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