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Inverse scattering for the multipoint potentials of Bethe–Peierls–Thomas–Fermi type

P.C. Kuo, R.G. Novikov

ABSTRACT. We consider the Schrödinger equation with a multipoint potential of the Bethe–Peierls–Thomas–Fermi type. We show that such a potential in dimension $d = 2$ or $d = 3$ is uniquely determined by its scattering amplitude at a fixed positive energy. Moreover, we show that there is no non-zero potential of this type with zero scattering amplitude at a fixed positive energy and a fixed incident direction. Nevertheless, we also show that a multipoint potential of this type is not uniquely determined by its scattering amplitude at a positive energy E and a fixed incident direction. Our proofs also contribute to the theory of inverse source problem for the Helmholtz equation with multipoint source.

Keywords: Schrödinger equation, Helmholtz equation, multipoint scatterers, multipoint sources, inverse scattering, inverse source problem
MSC2020: 35J05; 35J10; 35R30; 81U40

1. INTRODUCTION

We consider the stationary Schrödinger equation in dimensions $d = 1, 2, 3$,

$$(1.1) \quad -\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^d, \quad E > 0,$$

with a multipoint potential (scatterer) of Bethe–Peierls–Thomas–Fermi type:

$$(1.2) \quad v(x) = \sum_{j=1}^n \delta_{\alpha_j}(x - y_j), \quad \alpha_j \in \mathbb{C}, \quad y_j \in \mathbb{R}^d, \quad y_j \neq y_{j'} \quad \forall j \neq j'.$$

It is known that point scatterers $\delta_{\alpha}(x)$ of this type are only defined for $x \in \mathbb{R}^d$ with $d = 1, 2, 3$. For $d = 1$, $\delta_{\alpha}(x) = \varepsilon\delta(x)$, where $\varepsilon = -1/\alpha$ and δ denotes the standard Dirac delta function. For $d = 2$ and $d = 3$, the precise definition of "renormalized" delta function δ_{α} is more subtle and is discussed, in particular, in [2], [9], [17], and in references therein. Historically, the aforementioned point scatterers in dimension $d = 3$ were first introduced to describe the interaction between neutrons and protons by Bethe, Peierls [6], Thomas [29] and Fermi [13]. The mathematical theory of these point scatterers in dimensions $d = 2$ and $d = 3$ goes back to Zeldovich [30], Berezin and Faddeev[5]. Similar point scatterers also arise, in particular, in acoustics; see [3], [10] and references therein.

The simplest way to define a local solution ψ of equation (1.1) with potential v in (1.2) is as follows (see, for example, [17]):

$$(1.3) \quad -\Delta\psi(x) = E\psi(x), \quad \forall x \in \mathbb{R}^d \setminus \{y_1, \dots, y_n\},$$

and, in addition:

if $d = 1$, then $\psi(x)$ is continuous at $x = y_j$, but its first derivative has a jump

$$(1.4) \quad -\alpha_j [\psi'(y_j + 0) - \psi'(y_j - 0)] = \psi(y_j);$$

if $d = 2$, then

$$(1.5) \quad \psi(x) = \psi_{j,-1} \ln|x - y_j| + \psi_{j,0} + O(|x - y_j|) \quad \text{as } x \rightarrow y_j,$$

with

$$\psi_{j,0} = (-2\pi\alpha_j - \ln 2 + \gamma);$$

if $d = 3$, then

$$(1.6) \quad \psi(x) = \frac{\Psi_{j,-1}}{|x-y_j|} + \psi_{j,0} + O(|x-y_j|) \quad \text{as } x \rightarrow y_j,$$

with

$$4\pi\alpha_j\psi_{j,-1} = \psi_{j,0}.$$

We consider scattering and inverse scattering for model (1.1), (1.2). The direct scattering and spectral theory for this model is well-developed, at least, for real parameters α_j 's in (1.2); see, e.g., [2], [5], [9], [18], [19], [21] and references therein. An important point is that the direct scattering problem for model (1.1), (1.2) is exactly solvable. Some formulas of this theory are recalled in Section 2. These include formulas for the scattering amplitude f^+ ; see formulas (2.4), (2.7)-(2.9).

For other exactly solvable models in quantum mechanics, see, e.g., [15], [18], [20], [27], [28] and references therein.

In the present work we continue studies on inverse scattering for model (1.1), (1.2) in dimensions $d = 2$ and $d = 3$. In connection with results given in the literature in this direction, see [1], [3], [10], [14], [16], [17], [22], [24], [26] and references therein.

We show that for model (1.1), (1.2), the scattering amplitude f^+ at a fixed positive energy E uniquely determines the potential v in (1.2); see Theorem 3.1 in Section 3. Moreover, we show that there is no non-zero potential v in (1.2) with zero scattering amplitude f^+ at a fixed positive energy E and a fixed incident direction; see Theorem 3.2 in Section 3. Nevertheless, we also show that a multipoint potential v in (1.2) is not uniquely determined by its scattering amplitude f^+ at a positive energy E and a fixed incident direction; see Theorem 3.3 in Section 3. In particular, we continue recent studies of [17], [19] on possibilities for transparency or partial transparency for potential v in (1.2) (that is vanishing or partial vanishing the scattering amplitude f^+).

In addition, in connection with inverse scattering for the Schrödinger equation (1.1) with regular potential v , see [7], [8], [19], [25], [23] and references therein.

Note that proofs of the present work also contribute to the theory of inverse source problem for the Helmholtz equation:

$$(1.7) \quad \Delta\psi(x) + E\psi(x) = S(x), \quad x \in \mathbb{R}^d, \quad d \geq 2, \quad E > 0,$$

at fixed E , with S of the form

$$(1.8) \quad S(x) = \sum_{j=1}^n c_j \delta(x-y_j), \quad c_j \in \mathbb{C}, \quad y_j \in \mathbb{R}^d,$$

where δ is the standard Dirac function. The related inverse source problem from far-field data consists in finding S from the far-field amplitude a^+ for model (1.7), (1.8) at fixed E . Some formulas for a are recalled in Section 2; see formulas (2.14), (2.15). Our results on the aforementioned inverse source problem consist in Proposition 3.1 in Section 3 and its proof in Section 4 via some complex analysis.

In connection with results given in the literature on the inverse source problem from near-field data for model (1.7), (1.8), see, for example, [4], [11], [12] and references therein. In addition, the inverse source problem for the simplest discrete Helmholtz equation is recently studied in [26].

Further structure of the paper is as follows. Some preliminaries are recalled in Section 2. The main results of the present work are stated in Section 3. The proofs are given in Sections 4 and 5.

2. PRELIMINARIES

Let

$$(2.1) \quad \mathbb{S}_r^{d-1} = \{x \in \mathbb{R}^d; |x| = r\}, \quad r > 0; \quad \mathbb{S}^{d-1} = \mathbb{S}_1^{d-1}.$$

For equation (1.1) with potential v in (1.2), we consider the scattering eigenfunctions ψ^+ such that

$$(2.2) \quad \psi^+(x, k) = \psi_0 + \psi^{sc}(x, k), \quad x \in \mathbb{R}^d, \quad k \in \mathbb{S}_\kappa^{d-1}$$

where $\kappa = \sqrt{E}$, $\psi_0(x, k) = e^{ik \cdot x}$ and $\psi^{sc}(x, k)$ satisfies the Sommerfeld radiation condition:

$$(2.3) \quad |x|^{\frac{d-1}{2}} \left(\frac{\partial}{\partial |x|} - i\kappa \right) \psi^{sc}(x, k) \xrightarrow[|x| \rightarrow \infty]{} 0 \quad \text{uniformly in all directions } \frac{x}{|x|}.$$

Then,

$$(2.4) \quad \psi^{sc}(x, k) = \frac{e^{i\kappa|x|}}{|x|^{(d-1)/2}} f^+(k, \frac{\kappa}{|x|}x) + O\left(\frac{1}{|x|^{(d+1)/2}}\right) \quad \text{as } |x| \rightarrow \infty.$$

The function f^+ arising in (2.4) is defined on $\mathbb{S}_\kappa^{d-1} \times \mathbb{S}_\kappa^{d-1}$, and is the scattering amplitude for model (1.1),(1.2). It is convenient to present f^+ as follows:

$$(2.5) \quad f^+(k, l) = c(d, \kappa) f(k, l), \quad k, l \in \mathbb{S}_\kappa^{d-1}; \quad c(d, \kappa) = -\pi i (\sqrt{2\pi} e^{-i\pi/4})^{(d-1)} \kappa^{(d-3)/2}.$$

Crucially, when v is the multipoint potential of Bethe–Peierls–Thomas–Fermi type (1.2), we have the following explicit formulas for the scattering eigenfunction ψ^+ and scattering amplitude f^+ (see [17] and references therein):

$$(2.6) \quad \psi^+(x, k) = e^{ik \cdot x} + \sum_{j=1}^n q_j(k) G^+(x - y_j, \kappa), \quad x \in \mathbb{R}^d, \quad k \in \mathbb{S}_\kappa^{d-1},$$

$$(2.7) \quad f(k, l) = \frac{1}{(2\pi)^d} \sum_{j=1}^n q_j(k) e^{-il \cdot y_j}, \quad k, l \in \mathbb{S}_\kappa^{d-1},$$

where $q(k) = (q_1(k), \dots, q_n(k))^t$ satisfies

$$(2.8) \quad A(\kappa)q(k) = b(k),$$

$$(2.9) \quad A_{j,j}(\kappa) = \begin{cases} \alpha_j + (2i\kappa)^{-1} & , \text{if } d = 1 \\ \alpha_j - (4\pi)^{-1}(\pi i - 2\ln(\kappa)) & , \text{if } d = 2 \\ \alpha_j - i(4\pi)^{-1}\kappa & , \text{if } d = 3 \end{cases}$$

$$A_{j,j'}(\kappa) = G^+(y_j - y_{j'}, \kappa), \quad j \neq j',$$

$$b(k) = -\left(e^{ik \cdot y_1}, e^{ik \cdot y_2}, \dots, e^{ik \cdot y_n}\right)^t,$$

and G^+ is the Green function with the Sommerfeld radiation condition for the operator $\Delta + E$, given by

$$(2.10) \quad G^+(x, \kappa) = -(2\pi)^{-d} \int_{\mathbb{R}^d} \frac{e^{i\xi \cdot x} d\xi}{|\xi|^2 - \kappa^2 - i \cdot 0}$$

$$= \begin{cases} \frac{e^{i\kappa|x|}}{2i\kappa} & , \text{if } d = 1 \\ -\frac{i}{4} H_0^1(|x|\kappa) & , \text{if } d = 2 \\ -\frac{e^{i\kappa|x|}}{4\pi|x|} & , \text{if } d = 3 \end{cases}$$

where H_0^1 denotes the zeroth-order Hankel function of the first kind. Note that $G^+(x, \kappa)$ only depends on $|x|$ when $E = \kappa^2$ is fixed.

In connection with singularity, the following formulas hold:

$$\begin{aligned} -\frac{1}{4\pi}q_j &= \psi_{j,-1}, \quad d = 3 \\ \frac{1}{2\pi}q_j &= \psi_{j,-1}, \quad d = 2 \\ q_j &= \psi'(y_j+0) - \psi'(y_j-0), \quad d = 1 \end{aligned}$$

Note that we assume the condition

$$(2.11) \quad \det(A(\kappa)) \neq 0,$$

which is automatically true when α_j 's are all real.

For equation (1.7) with the source function S in (1.8), we consider the solution ψ satisfying the Sommerfeld radiation condition (2.3), where $\kappa = \sqrt{E} > 0$. Then

$$(2.12) \quad \psi(x, \kappa) = \frac{e^{i\kappa|x|}}{|x|^{(d-1)/2}} a^+(\frac{\kappa}{|x|}x) + O\left(\frac{1}{|x|^{(d+1)/2}}\right) \quad \text{as } |x| \rightarrow \infty.$$

The function a^+ arising in (2.12) is defined on \mathbb{S}_κ^{d-1} and is the far-field amplitude for model (1.7), (1.8). In a similar way with f^+ , it is convenient to present a^+ as follows:

$$(2.13) \quad a^+(l) = c(d, \kappa)a(l), \quad l \in \mathbb{S}_\kappa^{d-1}; \quad c(d, \kappa) = -\pi i (\sqrt{2\pi} e^{-i\pi/4})^{(d-1)} \kappa^{(d-3)/2}.$$

We have the following formulas for ψ and a :

$$(2.14) \quad \psi(x, \kappa) = \sum_{j=1}^n c_j G^+(x - y_j, \kappa), \quad x \in \mathbb{R}^d,$$

$$(2.15) \quad a(l) = \frac{1}{(2\pi)^d} \sum_{j=1}^n c_j e^{-il \cdot y_j}, \quad l \in \mathbb{S}_\kappa^{d-1}.$$

One can see a similarity between formulas (2.6), (2.7) for model (1.1), (1.2) and formulas (2.14), (2.15) for model (1.7), (1.8).

3. MAIN RESULTS

We start with the following global uniqueness result on inverse scattering for model (1.1), (1.2) at a fixed energy $E > 0$ in dimension $d = 2$ or $d = 3$.

Theorem 3.1. *A multipoint potential v of the form (1.2), for $d = 2$ or $d = 3$ under condition (2.11), is uniquely determined by its scattering amplitude f^+ at a fixed energy $E > 0$.*

Prototypes of Theorem 3.1 for regular potentials v are given in [7] and [25]. This theorem is proved in Section 5. In turn, this proof uses the following result of independent interest.

Proposition 3.1. *Let $y_1, y_2, \dots, y_n \in \mathbb{R}^d$, with $d \geq 2$, be mutually distinct and $c_1, c_2, \dots, c_n \in \mathbb{C} \setminus \{0\}$, where $n \in \mathbb{N}$. Let $u(\theta) = \sum_{j=1}^n c_j e^{iy_j \cdot \theta}$, $\theta \in \mathbb{S}^{d-1}$. Then y_j 's and c_j 's are uniquely determined by u on Θ , where Θ is a non-empty open subset of \mathbb{S}^{d-1} .*

Proposition 3.1 is proved in Section 4 using some complex analysis. Of course, this proposition for real y_j 's can also be proved by using relations between far-field and near-field data and results of [4], [13], [29]. However, the later proof looks much more complicated.

Note that Proposition 3.1 also holds for $y_j \in \mathbb{C}^d$.

In virtue of (2.12), (2.13), (2.15), Proposition 3.1 implies the global uniqueness for the inverse source problem from far-field data for model (1.7), (1.8) at fixed E . Note that Proposition 3.1 is not valid even for real y_j 's for some other convex and real-analytic surfaces in place of \mathbb{S}^{d-1} . In particular, article [26] gives related examples of non-uniqueness in the case of the surfaces

$$\Gamma(\lambda) = \{k \in [-\pi, \pi]^d; \sum_{i=1}^n 2 \cos(k_i) = \lambda\}, \text{ where } |\lambda| \in (2d-4, 2d).$$

Due to Theorem 3.1, there are no non-zero potentials v in (1.2) which are transparent at a fixed positive energy E in dimensions $d = 2, 3$ under assumption (2.11), that is with $f^+ \equiv 0$ at fixed E . Moreover, the following much stronger result in this direction also holds.

Theorem 3.2. *Let v be a multipoint potential of the form (1.2), with $d = 2$ or $d = 3$ under condition (2.11), and f^+ be its scattering amplitude. Suppose that $f^+(k, l) \equiv 0$ at a fixed energy $E > 0$ and a fixed incident vector k , then $v \equiv 0$.*

In the framework of formula (1.2) and related formulas in Sections 1 and 2, the case of $v \equiv 0$ can be considered as the case of $n = 0$ or, in other words, as the case of no scatterers. The case of $v \equiv 0$ can also be considered as the case when $\alpha_j = \infty$ for all j .

In spite of Theorem 3.2, the following non-uniqueness result also holds.

Theorem 3.3. *A multipoint potential v of the form (1.2), for $d = 2$ or $d = 3$ under condition (2.11), is not uniquely determined, in general, by its scattering amplitude $f^+(k, l)$ at a positive energy E and a fixed incident vector k .*

Theorem 3.2 and Theorem 3.3 are proved in Section 5.

In particular, by the results of Theorem 3.1-3.3 we continue recent studies of [17], [19] on possibilities of transparency or partial transparency for potential v in (1.2).

4. PROOF OF PROPOSITION 3.1

In addition to u on \mathbb{S}^{d-1} , we also consider

$$(4.1) \quad u(\theta) = \sum_{j=1}^n c_j e^{iy_j \cdot \theta}, \quad \theta \in \Sigma^{d-1},$$

where

$$(4.2) \quad \Sigma^{d-1} = \left\{ \theta \in \mathbb{C}^d; \theta^2 = \theta_1^2 + \dots + \theta_d^2 = 1 \right\}.$$

Note that $\Sigma^{d-1} \cap \mathbb{R}^d = \mathbb{S}^{d-1}$.

As u is holomorphic on Σ^{d-1} , the restriction $u|_{\mathbb{S}^{d-1}}$ uniquely determines u on Σ^{d-1} and on \mathbb{S}^{d-1} via analytic continuation.

Consider the complex vector-valued function

$$(4.3) \quad \theta(\tau, e_1, e_2) = \sqrt{1 + \tau^2} e_1 - i\tau e_2, \quad \tau \in \mathbb{R}, \quad (e_1, e_2) \in U\mathbb{S}^{d-1},$$

where $U\mathbb{S}^{d-1} = \{(e_1, e_2) \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}; e_1 \cdot e_2 = 0\}$ denotes the unit tangent bundle on \mathbb{S}^{d-1} .

One can see that θ defined in (4.3) takes values in Σ^{d-1} .

Then

$$(4.4) \quad u(\theta(\tau, e_1, e_2)) = \sum_{i=1}^n c_i e^{iy_j \cdot \theta(\tau, e_1, e_2)} = \sum_{i=1}^n c_i e^{\tau e_2 \cdot y_j} e^{-i\sqrt{1+\tau^2} e_1 \cdot y_j}, \quad \tau \in \mathbb{R}, \quad (e_1, e_2) \in U\mathbb{S}^{d-1},$$

where the modulus of each term is

$$(4.5) \quad |c_j e^{iy_j \cdot \theta(\tau, e_1, e_2)}| = |c_j| e^{\tau e_2 \cdot y_j}.$$

Now we propose the following recursive determination of all y_j 's and c_j 's. The recursive determination is based on the behavior of $\tilde{u} := u \circ \theta$ on $\mathbb{R} \times U\mathbb{S}^{d-1}$, which splits into the following three cases:

- (i) $\tilde{u} \equiv 0$. In view of (4.4), this implies $n = 0$ and $u \equiv 0$. The determination is completed.
- (ii) $\tilde{u} \not\equiv 0$ but \tilde{u} is bounded on $\mathbb{R} \times U\mathbb{S}^{d-1}$. In view of (4.4), this implies $n = 1$, $y_1 = 0$, and $\tilde{u} \equiv c_1$. The determination is completed.
- (iii) \tilde{u} is unbounded. In view of (4.4), we can take $e_2 \in \mathbb{S}^{d-1}$ such that \tilde{u} has maximal exponential growth as τ tends to $+\infty$. This implies that $e_2 = \frac{y_l}{|y_l|}$ for some y_l such that $|y_l| = \max_{1 \leq k \leq n} |y_k| \neq 0$.

In this way, for any fixed e_1 such that $(e_1, e_2) \in U\mathbb{S}^{d-1}$, we have $e_1 \cdot y_l = 0$ and, in addition, $e_2 \cdot y_l = |y_l| > |e_2 \cdot y_k|$ for all $k \neq l$. Therefore,

$$(4.6) \quad \tilde{u}(\tau, e_1, e_2) \sim c_l e^{\tau |y_l|} \quad \text{as } \tau \rightarrow \infty.$$

Now we reconstruct y_l and c_l using (4.6) and using that $e_2 = \frac{y_l}{|y_l|}$. We complete case (iii) by setting

$$u_{new}(\theta) := u(\theta) - c_l e^{iy_l \cdot \theta} = \sum_{1 \leq j \leq n, j \neq l} c_j e^{iy_j \cdot \theta}, \quad \theta \in \Sigma^{d-1}.$$

Then we proceed in the same way based on the behavior of $\tilde{u}_{new} = u_{new} \circ \theta$.

This recursive determination will terminate within n or $n + 1$ iterations and gives all y_j 's and c_j 's.

5. PROOFS OF THEOREM 3.1, 3.2

5.1. **Nontriviality of q_j .** We start with the following lemma.

Lemma 5.1. *Under the assumption of Theorem 3.1, for each $j = 1, \dots, n$, the coefficient q_j in formula (2.7) is an analytic function on \mathbb{S}_κ^{d-1} and q_j is not identically zero.*

By relation (2.8) and condition (2.11), for all $1 \leq j \leq n$, we have

$$(5.1) \quad q_j(k) = \sum_{l=1}^n (A(\kappa))_{j,l}^{-1} e^{ik \cdot y_l}, \quad k \in \mathbb{S}_\kappa^{d-1}.$$

For a fixed j , not all $(A(\kappa))_{j,1}^{-1}, \dots, (A(\kappa))_{j,n}^{-1}$ are zero since $\det A^{-1}(\kappa) = (\det A(\kappa))^{-1} \neq 0$. Therefore, Lemma 5.1 follows from formula (5.1) and from Proposition 3.1 for $u(\theta) = q_j(\kappa \theta)$.

Note that in view of analyticity of q_j , we also have that $q_j(k) \neq 0$ almost everywhere on \mathbb{S}_κ^{d-1} .

5.2. **Proof of Theorem 3.1.** Applying Proposition 3.1 and Lemma 5.1 to $u(\theta) = f(k, \kappa \theta)$ given by formula (2.7), one can see that f at fixed $E > 0$ uniquely determines q_j on \mathbb{S}_κ^{d-1} and y_j for all $1 \leq j \leq n$.

Next, when $q_j(k) \neq 0$, the diagonal entry $A_{j,j}(\kappa)$ is determined by the

$$(5.2) \quad A_{j,j}(\kappa) q_j(k) = -e^{ik \cdot y_j} - \sum_{j' \neq j} A_{j,j'}(\kappa) q_{j'}(k), \quad 1 \leq j \leq n.$$

Here, equation (5.2) follows from formula (2.8). Finally, $A_{j,j}(\kappa)$ determines α_j via formula (2.9) for all $1 \leq j \leq n$. This completes the proof of Theorem 3.1.

5.3. **Proof of Theorem 3.2.** Let $E > 0$ and $k \in \mathbb{S}_\kappa^{d-1}$. Suppose that $f^+(k, l) = 0$ for all $l \in \mathbb{S}_\kappa^{d-1}$. By formula (2.7) and Proposition 3.1, $q_j(k) = 0$ for all $1 \leq j \leq n$. It follows from relation (2.8) that $b(k) = 0$. However, $b_j(k) = -e^{ik \cdot y_j} \neq 0$ for all $1 \leq j \leq n$. This implies that $v \equiv 0$.

6. PROOFS OF THEOREM 3.3

In this section, we provide two approaches to constructing counterexamples. The first one is based on fitting parameters of two-point scatterers. The second one consists of adding a point scatterer at a special position which corresponds to a zero of the wave function ψ^+ in (2.6).

6.1. Proof of Theorem 3.3 by fitting parameters. Consider v and \tilde{v} with parameters n , α_j , y_j and \tilde{n} , $\tilde{\alpha}_j$, \tilde{y}_j respectively, where $n = \tilde{n} = 2$, $y_1 = \tilde{y}_1 = 0$, $y_2 = -\tilde{y}_2 \neq 0$, $\kappa y_2 = 0$.

$$\alpha_1 = \tilde{\alpha}_1 = \begin{cases} \frac{1}{4\pi}(\pi i - 2\ln(\kappa)) - \frac{i}{4}H_0^1(|y_2|\kappa) & , d = 2, \\ \frac{i\kappa}{4\pi} - \frac{e^{i\kappa|y_2|}}{4\pi|y_2|} & , d = 3, \end{cases}$$

and α_2 and $\tilde{\alpha}_2$ are arbitrarily complex numbers other than α_1 . By (2.9), (2.10) and our parameters chosen, we have

$$A = \begin{pmatrix} g & g \\ g & h \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} g & \tilde{g} \\ g & \tilde{h} \end{pmatrix}, \quad \text{and } b = \tilde{b} = \begin{pmatrix} -1 \\ -1 \end{pmatrix},$$

where $g = G^+(y_2, \kappa) \neq 0$, $h \neq g$, and $\tilde{h} \neq g$, ensuring that $\det A \neq 0$ and $\det \tilde{A} \neq 0$.

Then equation (2.8) implies

$$q(k) = \tilde{q}(k) = \frac{-1}{g} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Therefore, by formula (2.7),

$$f^+(k, l) = \frac{-1}{g(2\pi)^d} = \tilde{f}^+(k, l), \quad \forall l \in \mathbb{S}_\kappa^{d-1}.$$

Nevertheless, $v \neq \tilde{v}$ as $y_2 \neq \tilde{y}_2$.

Thus, Theorem 3.3 is proved. However, by this fitting approach, we did not find counterexamples when all α_j 's and $\tilde{\alpha}_j$'s are real.

6.2. Proof of Theorem 3.3 by adding invisible point scatterers. We use the notations v , n , y_j , α_j and \tilde{v} , \tilde{n} , \tilde{y}_j , $\tilde{\alpha}_j$ in a similar way with Subsection 6.1. Let v be a multipoint potential of (1.2). Suppose there exists $\tilde{y}_{n+1} \in \mathbb{R}^d \setminus \{y_1, \dots, y_n\}$ such that $\psi^+(y_{n+1}, k) = 0$, where ψ^+ is given by formula (2.6); that is,

$$(6.1) \quad e^{ik\tilde{y}_{n+1}} + \sum_{j=1}^n q_j(k)G^+(\tilde{y}_{n+1} - y_j, \kappa) = 0, \quad \tilde{y}_{n+1} \in \mathbb{R}^d \setminus \{y_1, \dots, y_n\}.$$

Let $y_j = \tilde{y}_j$, $1 \leq j \leq n$. Next, choose $\tilde{\alpha}_{n+1} \in \mathbb{C}$ so that $\det \tilde{A} \neq 0$. This is possible since $\det A \neq 0$ and by formula (2.9), we have

$$(6.2) \quad \tilde{A} = \begin{pmatrix} A & g^t \\ g & \tilde{A}_{n+1, n+1} \end{pmatrix},$$

where

$$\tilde{A}_{n+1, n+1} = \begin{cases} \tilde{\alpha}_{n+1} - (4\pi)^{-1}(\pi i - 2\ln(\kappa)) & , \text{if } d = 2, \\ \tilde{\alpha}_{n+1} - i(4\pi)^{-1}\kappa & , \text{if } d = 3, \end{cases}$$

and

$$g = (G^+(\tilde{y}_1 - y_{n+1}, \kappa), \dots, G^+(\tilde{y}_n - y_{n+1}, \kappa)).$$

It follows from (2.8), (6.1), and (6.2) that

$$\tilde{q} = \begin{pmatrix} q \\ 0 \end{pmatrix}.$$

Therefore, by (2.7), we have $\tilde{f}^+(k, l) = f^+(k, l)$ for all $l \in \mathbb{S}_\kappa^{d-1}$. This completes the schema of proof of Theorem 3.3 by adding an invisible point scatterer at a zero of ψ^+ .

In addition, by (2.6), we also have that $\tilde{\psi}^+(x, k) = \psi^+(x, k)$ for all $x \in \mathbb{R}^d$. Therefore, we can continue adding invisible point scatterers if ψ^+ has other zeros; i.e., equation (6.1) has other solutions.

In general, \tilde{y}_{n+1} satisfying (6.1) may not exist. However, there are already the following examples with $n = 1$ when such \tilde{y}_{n+1} exists.

Let $d = 2$, $n = 1$, $y_1 = 0$, and $\alpha_1 = \frac{\pi i - 2 \ln(\kappa)}{4\pi} - \frac{iH_0^1(1)}{4e^i}$. We have $A = \alpha_1 - \frac{\pi i - 2 \ln(\kappa)}{4\pi} = -\frac{iH_0^1(1)}{4e^i}$ by formula (2.9) and $q(k) = -A^{-1}e^{ik\tilde{y}_1} = \frac{4e^i}{iH_0^1(1)}$ by formula (2.8). Now, for $\tilde{y}_2 = \frac{k}{\kappa^2}$, we have that

$$e^{ik\tilde{y}_2} + q(k)G^+(\tilde{y}_2, \kappa) = e^i + \frac{4e^i}{iH_0^1(1)}G^+\left(\frac{k}{\kappa^2}, \kappa\right) = e^i + \frac{4e^i}{iH_0^1(1)}\frac{-i}{4}H_0^1(1) = 0.$$

Thus, this \tilde{y}_2 is a solution of (6.1).

Let $d = 3$, $n = 1$, $y_1 = 0$, and $\alpha_1 = \frac{i\kappa}{4\pi} - \frac{1}{4\pi\kappa}$. We have $A = \alpha_1 - \frac{i\kappa}{4\pi} = -\frac{1}{4\pi\kappa}$ by formula (2.9) and $q(k) = -A^{-1}e^{ik\tilde{y}_1} = 4\pi\kappa$ by formula (2.8). Now, for $\tilde{y}_2 = k$, we have that

$$e^{ik\tilde{y}_2} + q(k)G^+(\tilde{y}_2, \kappa) = e^{ik^2} + \frac{1}{4\pi\kappa}G^+(k, \kappa) = e^{ik^2} + 4\pi\kappa\frac{-e^{ik^2}}{4\pi\kappa} = 0.$$

Therefore, $\tilde{y}_2 = k$ is a solution of (6.1).

However, in these examples, α_1 is not real.

In this connection, in Section 7, we give an example with real α_j 's through numerical implementation.

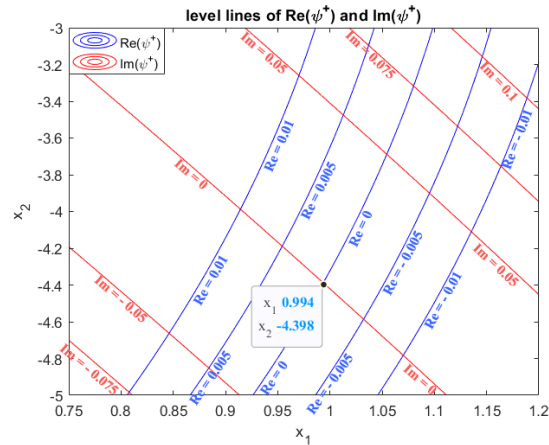
7. NUMERICAL EXAMPLE

This section aims to show solutions of equation (6.1) by numerical simulation for the case when α_j 's are all real. The purpose is to find a zero of the following function:

$$(7.1) \quad \psi^+(x) := e^{ik \cdot x} + \sum_{j=1}^n q_j(k)G^+(x - y_j, \kappa), \quad x \in \mathbb{R}^d.$$

Here, we are interested in the case when α 's are real since we have seen examples with complex α 's.

Let $d = 2$, $n = 2$, $\alpha_1 = 1$, $\alpha_2 = 1$, $k = \begin{pmatrix} 0.05 \\ 0.05 \end{pmatrix}$, $y_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and $y_2 = \begin{pmatrix} 0.1 \\ 0.15 \end{pmatrix}$. In this case, $q = \begin{pmatrix} 0.51 - 1.86i \\ 0.51 - 1.86i \end{pmatrix}$. In addition, we find a solution of (6.1) around the point $(0.994, -4.398)$ by tracing the level lines of the real part and imaginary part of ψ^+ . This is illustrated in the figure 1. This example show that the approach of adding invisible point scatterers in Subsection 6.2 also works for the case when all α_j 's are real.

FIGURE 1. Level sets of $\Re(\psi^+) = 0$ and $\Im(\psi^+) = 0$ intersect at a point around $(0.994, -4.398)$ 

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