



**HAL**  
open science

# Local Conservation Laws and Entropy Inequality for Kinetic Models with Delocalized Collision Integrals

Frédérique Charles, Zhe Chen, François Golse

► **To cite this version:**

Frédérique Charles, Zhe Chen, François Golse. Local Conservation Laws and Entropy Inequality for Kinetic Models with Delocalized Collision Integrals. 2024. hal-04852993

**HAL Id: hal-04852993**

**<https://polytechnique.hal.science/hal-04852993v1>**

Preprint submitted on 21 Dec 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

**LOCAL CONSERVATION LAWS AND ENTROPY INEQUALITY  
FOR KINETIC MODELS  
WITH DELOCALIZED COLLISION INTEGRALS**

FRÉDÉRIQUE CHARLES, ZHE CHEN, AND FRANÇOIS GOLSE

ABSTRACT. This article presents a common setting for the collision integrals  $St$  appearing in the kinetic theory of dense gases. It includes the collision integrals of the Enskog equation, of (a variant of) the Povzner equation, and of a model for soft sphere collisions proposed by Cercignani [Comm. Pure Appl. Math. **36** (1983), 479–494]. All these collision integrals are “delocalized”, in the sense that they involve products of the distribution functions of gas molecules evaluated at positions whose distance is of the order of the molecular radius. Our first main result is to express these collision integrals as the divergence in  $v$  of some mass current, where  $v$  is the velocity variable, while  $v_i St$  and  $|v|^2 St$  are expressed as the phase space divergence (i.e divergence in both position and velocity) of appropriate momentum and energy currents. This extends to the case of dense gases an earlier result by Villani [Math. Modelling Numer. Anal. M2AN **33** (1999), 209–227] in the case of the classical Boltzmann equation (where the collision integral involves products of the distribution function of gas molecules evaluated at different velocities, but at the same position). Applications of this conservative formulation of delocalized collision integrals include the possibility of obtaining the local conservation laws of momentum and energy starting from this kinetic theory of dense gases. Similarly a local variant of the Boltzmann H Theorem, involving some kind of free energy instead of Boltzmann’s H function, can be obtained in the form of an expression for the entropy production in terms of the phase space divergence of some phase space current, and of a nonpositive term.

INTRODUCTION

We are concerned in this paper with the basic structure of some kinetic equations used to model non perfect monatomic gases. The classical kinetic theory of gases of Maxwell (1866) and Boltzmann (1872) is limited to perfect monatomic gases. It has been later generalized, in particular by Enskog (1922), to treat the case of dense gases (see chapter 16 of [12], for instance). Further generalizations of Enskog’s theory — most notably Povzner’s kinetic equation (1962), and Cercignani’s model for the kinetic theory of soft spheres (1980) — are presented in section 1.

The classical kinetic theory of gases is intimately related to the Euler equations of (inviscid) gas dynamics through the local conservation laws of mass, momentum and energy, satisfied by classical solutions of the Boltzmann equation (see sections 3.1 and 3.3 in [11], and the remarks in section 2). At the time of this writing, it would seem that these *local* conservation laws of mass, momentum and energy are

---

2020 *Mathematics Subject Classification.* 35Q20, 82C40, 76P05.

*Key words and phrases.* Kinetic theory of gases; Dense gases; Boltzmann-Enskog equation; Povzner equation; Local conservation laws; Boltzmann’s H Theorem.

not known to be verified by classical solutions of the Enskog equation or its variants presented in section 1.

For this reason, we propose in this work a systematic method for obtaining local conservation laws in the theory of kinetic models of Enskog type. Our arguments are based on writing the collision integrals of Enskog type as the divergence in the position and velocity variables of mass, momentum and energy currents, following a little-known idea of Landau on the Boltzmann collision integral (see for instance §41 in [20] and [29]).

In the case of the Boltzmann equation, the Friedrichs-Lax entropy condition used in the theory of the Euler system of (inviscid) gas dynamics (see sections 3.2, 4.5 and chapter V of [13]) can be deduced from the local form of the Boltzmann H theorem [3]. For this reason, we also discuss the possibility of obtaining *local* entropy inequalities for classical solutions of Enskog-type kinetic equations.

The outline of this paper is as follows: various examples of kinetic models with delocalized collision integrals analogous to Enskog's are presented in section 1. Local conservation laws and its applications to fluid dynamics are briefly recalled in section 2. Our main results are presented in section 3. In particular, a general formulation of delocalized collision integrals is presented in section 3.1, and the corresponding global conservation laws of mass, momentum and energy in 3.2 are briefly recalled for these generalizations of Enskog's equation. The core of this paper is Theorem 3.3, in section 3.3, providing an analogue of Landau's representation of Boltzmann's collision integral as the divergence in  $v$  of a mass current, in the more complex case of delocalized collision integral. Our second main result is the local entropy inequality for a subclass of delocalized collision integrals presented in Theorem 3.5 of section 3.4. Applications of our results to the continuum mechanics of dense gases are discussed in section 4.

## 1. EXAMPLES OF DELOCALIZED COLLISION INTEGRALS IN THE KINETIC THEORY OF GASES

All collisional kinetic models take the form

$$(1) \quad (\partial_t + v \cdot \nabla_x) f(t, x, v) = \text{St}[f](t, x, v),$$

where  $f \equiv f(t, x, v) \geq 0$  is the velocity distribution function, i.e. the number density of gas molecules at position  $x \in \mathbf{R}^3$  and with velocity  $v \in \mathbf{R}^3$  at time  $t$ , while  $\text{St}$  is the collision operator, usually a nonlinear integral operator acting on the velocity, and sometimes the position variables in  $f$ .

In the classical kinetic theory of gases due to Maxwell (1866) and Boltzmann (1872), the collision integral computes the variation in the number of gas molecules with velocity  $v \in \mathbf{R}^3$  due to binary, elastic collisions with gas molecules at the same position with a velocity different from  $v$ . Specifically, if the gas molecules are hard spheres of radius  $r > 0$ , the Boltzmann collision integral is<sup>1</sup>

$$(2) \quad \mathcal{B}[f, f](v) := \iint_{\mathbf{R}^3 \times \mathbf{S}^2} (f(v')f(w') - f(v)f(w))(2r)^2(v - w|n)_+ dw dn,$$

where  $f \geq 0$  is the velocity distribution function, while  $(u|v)$  is the inner product of  $u, v \in \mathbf{R}^3$ , and  $z_+ := \max(z, 0)$  for all  $z \in \mathbf{R}$  (see section 2.5 in [11]). In the

---

<sup>1</sup>See in [26] the paragraph following equations (1.7) in chapter 1, especially footnote 6 on p. 4 in that chapter.

integrand above,

$$(3) \quad v' \equiv v'(v, w, n) := v - (v - w|n)n, \quad w' \equiv w'(v, w, n) := w + (v - w|n)n.$$

Notice that these transformations are invariant under the substitution  $n \mapsto -n$ .

One easily checks that  $v'(v, w, n), w'(v, w, n)$  runs through the set of all the solutions of the system of equations

$$(4) \quad v' + w' = v + w, \quad |v'|^2 + |w'|^2 = |v|^2 + |w|^2$$

as  $n$  runs through the unit sphere  $\mathbf{S}^2$  of  $\mathbf{R}^3$ . The first equality above is the conservation of momentum for a pair of point particles with mass  $m > 0$ , while the second equality is the conservation of kinetic energy for the same particles in any elastic collision involving such particles located at the same position in  $\mathbf{R}^3$ , with velocities  $v, w$  immediately after the collision, and  $v', w'$  immediately before the collision. Without entering the details of the derivation of the Boltzmann collision integral, suffices it to say that, in that integral, one should think of the term  $-f(v)f(w)$  as measuring the loss of gas molecules with velocity  $v$  due to an impending collision with another gas molecule at the same position with velocity  $w$ . Similarly, one should think of the term  $f(v')f(w')$  as measuring the creation of gas molecules with velocity  $v$  resulting from the collision of a gas molecule with velocity  $v'$  with a gas molecule with velocity  $w'$  immediately before the collision. Thus all the pair of velocities involved in the Boltzmann collision integrand are always velocities of gas molecules *about to collide* (and never of gas molecules having just collided). This “justifies” the assumption of Maxwell and Boltzmann that such particle velocities are statistically independent, so that their joint distribution is  $f(v')f(w')$  in the latter case, and  $f(v)f(w)$  in the former.

Therefore, in the Boltzmann equation for the hard sphere gas, the molecular radius appears only as a scaling factor of the collision differential cross-section  $(2r)^2(\cos(\widehat{v-w}, n))_+$ , and never in the distribution function itself in the collision integral. This has been justified (for instance by Lanford [21], see also chapters 2 and 4 in [11] for a streamlined presentation of Lanford’s arguments) in the regime known as the “Boltzmann-Grad limit” of the classical mechanics of  $N$ -particle system, where  $N \gg 1$  while  $r \ll 1$  satisfy  $Nr^2 \sim 1$ . In particular  $Nr^3 = O(r) \rightarrow 0$  in that limit, so that there is no excluded volume, and the resulting equations of state for the pressure and the internal energy correspond to a perfect gas.

Situations where the molecular radius is larger may fail to satisfy the Boltzmann-Grad assumption. Alternatively, one might also seek corrections in  $r$  to the perfect gas equation of state for the pressure and the internal energy. Whatever the motivation, one might be led to consider collision integrals involving a *delocalized* collision integrand, i.e. a collision integrand where gas molecules are not considered as point particles located at the same point, but are hard spheres of radius  $r > 0$ .

Perhaps the best known kinetic model with such delocalized collision integral is the Enskog-Boltzmann equation (see [1, 28]). In the hard sphere case, the Enskog-Boltzmann collision integral takes the form

$$(5) \quad \mathcal{E}[f, f](x, v) := \iint_{\mathbf{R}^3 \times \mathbf{S}^2} (f(x, v'(v, w, n))f(x - 2rn, w'(v, w, n)) - f(x, v)f(x + 2rn, w))4r^2(v - w|n)_+ dw dn.$$

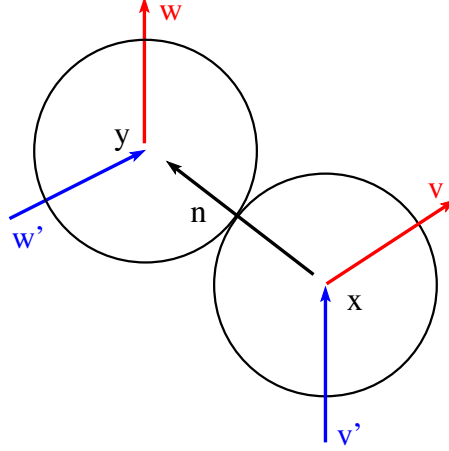


FIGURE 1. Two hard spheres centered at  $x$  and  $y = x + rn$ , with pre-collision velocities  $v'$  and  $w'$ , and post-collision velocities  $v$  and  $w$ . Observe that the post-collision velocities  $v$  and  $w$  satisfy the condition  $(w - v|n) = (w - v|\frac{y-x}{|y-x|}) > 0$ , while the pre-collision velocities satisfy  $(w' - v'|n) = (w' - v'|\frac{y-x}{|y-x|}) < 0$ .

This collision integral can be recast as

$$\begin{aligned} \mathcal{E}[f, f](x, v) := & \iint_{(\mathbf{R}^3)^2} (f(x, v'(v, w, n))f(x - z, w'(v, w, n)) \\ & - f(x, v)f(x + z, w))(v - w|\frac{z}{|z|})_+ \delta_{2r}(|z|) dw dz , \end{aligned}$$

where  $\delta_{2r}$  is the Dirac “function” at  $2r > 0$  on  $\mathbf{R}$ .

A first obvious generalization is based on the last formula for the Enskog-Boltzmann collision integral. Pick  $\zeta \in C(\mathbf{R})$  satisfying

$$\zeta \geq 0, \quad \text{supp}(\zeta) = [-1, 1], \quad \int_{\mathbf{R}} \zeta(s) ds = 1 ,$$

and set

$$\zeta_\epsilon(s) := \frac{1}{\epsilon} \zeta\left(\frac{s}{\epsilon}\right), \quad \theta_{\epsilon, r}(|z|) = \zeta_\epsilon(|z| - 2r) ,$$

so that  $\theta_{\epsilon, r} \rightarrow \delta_{2r}$  in  $\mathcal{D}'(\mathbf{R})$  as  $\epsilon \rightarrow 0^+$ . If one replaces the Dirac measure  $\delta_{2r}(|z|)$  with its regularized variant  $\theta_{\epsilon, r}(|z|)$ , one arrives at the expression

$$(6) \quad \begin{aligned} \mathcal{C}[f, f](x, v) := & \iint_{(\mathbf{R}^3)^2} \left( f(x, v'(v, w, \frac{z}{|z|}))f(x - z, w'(v, w, \frac{z}{|z|})) \right. \\ & \left. - f(x, v)f(x + z, w) \right) \theta_{\epsilon, r}(|z|) \left( v - w|\frac{z}{|z|} \right)_+ dw dz . \end{aligned}$$

One can think of this as a collision integral for “soft” spheres — soft meaning that the radius of the colliding balls is slightly distributed around  $r > 0$ , instead of being exactly equal to  $r$ ; otherwise the collision transformation of  $(v, v_*) \mapsto (v', v'_*)$  is the same as above. In other words, these collisions are elastic. (It is unlikely that collisions between pairs of soft spherical particles really behave in this manner, which is why the word soft is put between quotes.) See [10] for a presentation of this model, which is derived from a large particle system in the style of Lanford’s

argument for the Boltzmann equation. In particular, Cercignani proposed a specific formula for  $\theta_{\epsilon,r}$ : see equations (2.1)-(2.5) on p. 481 in [10].

The Cercignani soft-sphere collision integral can be recast as

$$\begin{aligned} \mathcal{C}[f, f](x, v) := & \iint_{(\mathbf{R}^3)^2} \left( f(x, v'(v, w, \frac{x-y}{|x-y|})) f(y, w'(v, w, \frac{x-y}{|x-y|})) \mathbf{1}_{(v-w|x-y|)>0} \right. \\ & \left. - f(x, v) f(y, w) \mathbf{1}_{(v-w|x-y|)<0} \right) \theta_{\epsilon,r}(|x-y|) \left| \left( v-w \frac{x-y}{|x-y|} \right) \right| dw dy. \end{aligned}$$

A last generalization consists in forgetting the specifics of the distribution  $\theta_{\epsilon,r}$ . This suggests considering instead the Povzner type collision integral

$$(7) \quad \begin{aligned} \mathcal{P}[f, f](x, v) := & \iint_{\mathbf{R}^3 \times \mathbf{R}^3} \left( f(x, v'(v, w, \frac{x-y}{|x-y|})) f(y, w'(v, w, \frac{x-y}{|x-y|})) \mathbf{1}_{(v-w|x-y|)>0} \right. \\ & \left. - f(x, v) f(y, w) \mathbf{1}_{(v-w|x-y|)<0} \right) b(|x-y|) \left| \left( v-w \frac{x-y}{|x-y|} \right) \right| dw dy \end{aligned}$$

where  $b \in C_c([0, +\infty))$  is a nonnegative function. See [24], where this model<sup>2</sup> is introduced for the first time.

Along with the Enskog-Boltzmann collision integral above, one also finds in the literature the genuine Enskog equation (see chapter 16 of [12] or [7]) and a modified, or revised Enskog collision integral (see [25], or [4] and [5, 6]):

$$(8) \quad \begin{aligned} \tilde{\mathcal{E}}[f, f](x, v) := & \iint_{\mathbf{R}^3 \times \mathbf{S}^2} \left( Y(\rho_f(x), \rho_f(x-2rn)) f(x, v'(v, w, n)) f(x-2rn, w'(v, w, n)) \right. \\ & \left. - Y(\rho_f(x), \rho_f(x+2rn)) f(x, v) f(x+2rn, w) \right) 4r^2 (v-w|n)_+ dw dn, \end{aligned}$$

where the factor  $Y : [0, +\infty)^2 \rightarrow [0, +\infty)$  is used to enhance the collision frequency, and where

$$(9) \quad \rho_f(x) := \int_{\mathbf{R}^3} f(x, v) dv.$$

---

<sup>2</sup>Povzner's original collision integral in [24] is

$$\begin{aligned} \tilde{\mathcal{P}}[f, f](x, v) := & \iint_{(\mathbf{R}^3)^2} \left( f(x, v'(v, w, \frac{x-y}{|x-y|})) f(y, w'(v, w, \frac{x-y}{|x-y|})) \right. \\ & \left. - f(x, v) f(y, w) \right) b(|x-y|) \left| \left( v-w \frac{x-y}{|x-y|} \right) \right| dw dy. \end{aligned}$$

Since the transformation  $(v, w) \mapsto (v'(v, w, n), w'(v, w, n))$  is invariant under the central symmetry  $n \mapsto -n$ ,

$$\mathcal{B}[f, f](v) = \iint_{\mathbf{R}^3 \times \mathbf{S}^2} (f(v')f(w') - f(v)f(w)) 2r^2 |v-w|n| dw dn.$$

Povzner's original definition of his delocalized collision integral is based on this last form of the Boltzmann integral instead of (2). Since  $(v' - w'|x-y) = -(v-w|x-y) < 0$  for the gain part  $f(x, v')f(y, w')$  and  $(v-w|x-y) < 0$  for the loss part  $f(x, v)f(y, w)$  of the collision integral  $\mathcal{P}$ , only particles about to collide are uncorrelated in (7), as in the case of (2). That this important feature of Boltzmann's theory is absent from [24] may be an oversight. For this reason, we prefer considering  $\mathcal{P}$  instead of  $\tilde{\mathcal{P}}$ .

## 2. LOCAL CONSERVATION LAWS FOR THE BOLTZMANN COLLISION INTEGRAL

Returning to the Boltzmann collision integral (2), it is well known that, for all  $f \in L^1(\mathbf{R}^3, (1 + |v|)^3 dv)$ ,

$$(10) \quad \int_{\mathbf{R}^3} \mathcal{B}[f, f](v) \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ |v|^2 \end{pmatrix} dv = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

see for instance sections 3.1 and 3.3 in [11]. These five identities correspond to the local conservation of mass (or particle number), of the three components of momentum, and of energy in the collision process described by the Boltzmann collision integral.

As a consequence of these identities, if  $f$  is a classical solution of the Boltzmann equation

$$(\partial_t + v \cdot \nabla_x) f(t, x, v) = \mathcal{B}[f, f](t, x, v) (= \mathcal{B}[f(t, x, \cdot), f(t, x, \cdot)](v))$$

such that  $f(t, x, \cdot), \partial_t f(t, x, \cdot)$  and  $\nabla_x f(t, x, \cdot) \in L^1(\mathbf{R}^3, (1 + |v|)^3 dv)$ , then

$$(11) \quad \partial_t \int_{\mathbf{R}^3} f(t, x, v) \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ \frac{1}{2}|v|^2 \end{pmatrix} dv + \nabla_x \cdot \int_{\mathbf{R}^3} v f(t, x, v) \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ \frac{1}{2}|v|^2 \end{pmatrix} dv = 0.$$

These five differential identities are of considerable importance in connecting the kinetic description of gases with continuum mechanics. Indeed, if one calls

$$(12) \quad \rho(t, x) := \int_{\mathbf{R}^3} f(t, x, v) dv \geq 0, \quad u(t, x) = \frac{\mathbf{1}_{\rho(t, x) > 0}}{\rho(t, x)} \int_{\mathbf{R}^3} v f(t, x, v) dv \in \mathbf{R}^3,$$

the first identity above takes the form of the equation of continuity of fluid mechanics

$$(13) \quad \partial_t \rho(t, x) + \nabla_x \cdot (\rho(t, x) u(t, x)) = 0,$$

while the second identity becomes

$$(14) \quad \partial_t (\rho(t, x) u(t, x)) + \nabla_x \cdot (\rho(t, x) u(t, x)^{\otimes 2}) + \nabla_x \cdot P(t, x) = 0,$$

where

$$(15) \quad P(t, x) := \int_{\mathbf{R}^3} (v - u(t, x))^{\otimes 2} f(t, x, v) dv.$$

Finally, the third identity takes the form

$$(16) \quad \partial_t \left( \frac{1}{2} \rho(t, x) |u(t, x)|^2 + \frac{1}{2} \text{tr} P(t, x) \right) + \nabla_x \cdot \left( u(t, x) \left( \frac{1}{2} \rho(t, x) |u(t, x)|^2 + \frac{1}{2} \text{tr} P(t, x) \right) \right) + \nabla_x \cdot (P(t, x) \cdot u(t, x) + Q(t, x)) = 0,$$

where  $Q$  is the time-dependent vector field defined by

$$(17) \quad Q(t, x) := \frac{1}{2} \int_{\mathbf{R}^3} (v - u(t, x)) |v - u(t, x)|^2 f(t, x, v) dv.$$

If  $f(t, x, v)$  is an even function of  $v - u(t, x)$  one easily checks (by a parity argument) that

$$Q(t, x) = 0,$$

These identities are obviously reminiscent of the Euler system of equations for compressible fluids (see (3.3.17) in chapter III of [13]), and the tensor field  $P(t, x)$  plays the role of the pressure. Specifically if  $f$  is a radial function of  $v - u(t, x)$  — for instance if  $f$  is a local Maxwellian equilibrium in  $v$  centered at  $u(t, x)$  — one finds that

$$P(t, x) = p(t, x)I, \quad \text{and hence} \quad \text{tr}P(t, x) = 3p(t, x)I.$$

Hence

$$(18) \quad \partial_t(\rho(t, x)u(t, x)) + \nabla_x \cdot (\rho(t, x)u(t, x)^{\otimes 2}) + \nabla_x p(t, x) = 0,$$

while

$$(19) \quad \partial_t(\frac{1}{2}\rho(t, x)|u(t, x)|^2 + \frac{3}{2}p(t, x)) + \nabla_x \cdot (u(t, x)(\frac{1}{2}\rho(t, x)|u(t, x)|^2 + \frac{5}{2}p(t, x))) = 0.$$

Setting  $\theta(t, x) = p(t, x)/\rho(t, x)$ , one finds that these are the Euler equation and local energy conservation law for a perfect gas with adiabatic index

$$\gamma = (5/2)/(3/2) = 5/3.$$

It is well known that the adiabatic index of a perfect gas is  $\gamma = 1 + 2/n$ , where  $n$  is the number of degrees of freedom of the gas molecule (see formulas (44.1-2) in [19]). For a monatomic gas, each molecule has  $n = 3$  degrees of freedom (corresponding to translations in the directions of the coordinate axis), so that  $\gamma = 5/3$  is the adiabatic index of a perfect monatomic gas. Thus the collision integral (2) can describe only perfect monatomic gases.

All these considerations suggest studying the following question: compute

$$\int_{\mathbf{R}^3} \text{St}[f](t, x, v) \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ |v|^2 \end{pmatrix} dv,$$

where  $\text{St}[f]$  is any one of the delocalized collision integrals presented above, i.e.  $\mathcal{E}[f, f]$  (Enskog-Boltzmann) or  $\tilde{\mathcal{E}}[f, f]$  (revised Enskog), or  $\mathcal{C}[f, f]$  (Cercignani's soft-sphere collision integral), or  $\mathcal{P}[f, f]$  (Povzner-type collision integral).

In general, delocalization may prevent some of these quantities to be identically 0. However, it is certainly interesting to study these expressions in view of applications to fluid mechanics.

### 3. MAIN RESULTS: DELOCALIZED COLLISION INTEGRALS IN CONSERVATIVE FORM AND APPLICATIONS

In order to approach the question of local conservation laws for delocalized collision integrals, we shall make a small detour. While it may seem unnecessary, it sheds some light on the structure of these collision integrals.

Our approach finds its origin in a seemingly little-known property of the Boltzmann collision integral. The fact that the Boltzmann collision integral satisfies the local conservation of mass suggests seeking a mass current  $J[f, f](v)$  such that

$$\mathcal{B}[f, f](v) = -\nabla_v \cdot J[f, f](v)$$



for all  $f \in \mathcal{S}(\mathbf{R}^3)$  (the Schwartz space of smooth functions with rapidly decaying derivatives of all orders). This is the starting point of the derivation of the Landau collision integral from the Boltzmann collision integral in the grazing collisions regime (see §41 in [20]). A more systematic (and rigorous) discussion of this property can be found in [29].

**3.1. Delocalized Collision Integrals: a General Setting.** In this section, we propose a common setting for the delocalized collision integrals  $\mathcal{E}, \mathcal{C}, \mathcal{P}, \tilde{\mathcal{E}}$  presented in section 1.

The general form of delocalized collision integrals considered in this paper is (20)

$$\begin{aligned} \text{St}[f, g](x, v) := & \int_{(\mathbf{R}^3)^2} (f(x, v'(v, w, n_{x,y}))g(y, w'(v, w, n_{x,y}))\mathbf{1}_{(v-w|n_{x,y})>0} \\ & - f(x, v)g(y, w)\mathbf{1}_{(v-w|n_{x,y})<0})B[\rho_f, \rho_g](x, y, v-w)dydw, \end{aligned}$$

where

$$\rho_f(x) := \int_{\mathbf{R}^3} f(x, v)dv, \quad \rho_g(y) := \int_{\mathbf{R}^3} g(y, w)dw$$

are the macroscopic densities of the distribution functions  $f$  and  $g$ , while

$$n_{x,y} := \frac{x-y}{|x-y|}, \quad x \neq y.$$

The collision kernel  $B[\rho_f, \rho_g](x, y, v-w) \geq 0$  is in general a functional of the macroscopic densities  $\rho_f$  and  $\rho_g$ , and a function — or more generally a distribution — of the positions  $x, y$ , and of the relative velocity  $v-w$ . In addition, it satisfies

$$\begin{cases} (i) & B[\varpi, \rho](y, x, w-v) = B[\rho, \varpi](x, y, v-w), \\ (ii) & B[\rho, \varpi](x, y, v-w) = B[\rho, \varpi](x, y, v'(v, w, n_{x,y}) - w'(v, w, n_{x,y})), \\ (iii) & B[\rho, \varpi](x, y, v-w) = 0 \text{ if } |x-y| > R, \end{cases}$$

for some  $R > 0$ , a real parameter which can be thought of as being the radius of the sphere of influence of a gas molecule<sup>3</sup> of the gas molecules, together with the bound

$$(iv) \quad \sup_{v,w \in \mathbf{R}^3} \frac{1}{1+|v-w|} \int_{(\mathbf{R}^3)^2} B[\rho, \varpi](x, y, v-w)\mathbf{1}_{|x| \leq A} dx dy < \infty.$$

Let us check that the examples of delocalized collision integrals considered in section 1 can be put in the form (20).

(1) Enskog-Boltzmann equation: in that case

$$B[\rho, \varpi](x, y, v-w) = (v-w|n_{x,y})|\delta_{2r}(|x-y|),$$

(and of course, the collision kernel is independent of the macroscopic densities  $\rho, \varpi$ );

(2) Soft spheres: in that case

$$B[\rho, \varpi](x, y, v-w) = \theta_{\epsilon,r}(|x-y|)|(v-w|n_{x,y})|$$

(and of course, the collision kernel is independent of the macroscopic densities  $\rho, \varpi$ );

(3) Povzner-type equation: in that case

$$B[\rho, \varpi](x, y, v-w) = b(|x-y|)|(v-w|n_{x,y})|$$

<sup>3</sup>In other words,  $R$  is the distance beyond which a point particles is not affected by the presence of one gas molecule. Assuming short range molecular interactions, one can think of  $R$  as being of the order of a few van der Waals radii of the gas molecules.

(and of course, the collision kernel is independent of the macroscopic densities  $\rho, \varpi$ ). This type of collision kernel will be used below in section 3.4 in connection with the H Theorem;

(4) Enskog equation: in that case

$$B[\rho, \varpi](x, y, v - w) = Y[\rho(x), \varpi(y)]|(v - w|n_{x,y})|\delta_{2r}(|x - y|).$$

**3.2. Global Conservation Laws for Delocalized Collision Integrals.** The following result is well-known in the case of the Enskog equation [23, 15]. We briefly sketch its proof in the more general setting considered here.

**Proposition 3.1.** *For all measurable, rapidly decaying  $f, g$  defined a.e. on  $\mathbf{R}^3 \times \mathbf{R}^3$ , the collision integral  $\text{St}$  satisfies the local mass conservation identity*

$$\int_{\mathbf{R}^3} \text{St}[f, g](x, v) dv = 0.$$

*It also satisfies the global momentum and energy conservation identities*

$$\int_{\mathbf{R}^3 \times \mathbf{R}^3} \text{St}[f, f](x, v) v_j dx dv = \int_{\mathbf{R}^3 \times \mathbf{R}^3} \text{St}[f, f](x, v) |v|^2 dx dv = 0, \quad j = 1, 2, 3.$$

*Proof.* For all measurable, rapidly decaying  $f, g$  defined a.e. on  $\mathbf{R}^3 \times \mathbf{R}^3$ ,

$$\begin{aligned} & \int_{\mathbf{R}^3} \text{St}[f, g](x, v) dv = \int_{(\mathbf{R}^3)^3} (f(x, v'(v, w, n))g(y, w'(v, w, n))\mathbf{1}_{(v-w|n_{x,y})>0} \\ & \quad - f(x, v)f(y, w)\mathbf{1}_{(v-w|n_{x,y})<0})B[\rho_f, \rho_g](x, y, v - w)dv dw dy \\ & = \int_{(\mathbf{R}^3)^3} (B[\rho_f, \rho_g](x, y, v'(v, w, n_{x,y}) - w'(v, w, n_{x,y}))\mathbf{1}_{(v'(v, w, n_{x,y}) - w'(v, w, n_{x,y})|n_{x,y})>0} \\ & \quad - B[\rho_f, \rho_g](x, y, v - w)\mathbf{1}_{(v-w|n_{x,y})<0})f(x, v)g(y, w)dv dw dy \\ & \quad \text{(with the change of variables } (v, w) \mapsto (v'(v, w, n), w'(v, w, n)) \text{ for each } n \\ & = \int_{(\mathbf{R}^3)^3} (B[\rho_f, \rho_g](x, y, v'(v, w, n_{x,y}) - w'(v, w, n_{x,y})) - B[\rho_f, \rho_g](x, y, v - w)) \\ & \quad \times \mathbf{1}_{(v-w|n_{x,y})<0} f(x, v)g(y, w)dv dw dy = 0. \\ & \quad \text{(since } (v'(v, w, n) - w'(v, w, n)|n) = -(v - w|n)) \end{aligned}$$

by assumption (ii) on  $B$ . Similarly

$$\begin{aligned}
& \int_{(\mathbf{R}^3)^2} \text{St}[f, f](x, v) v dv dx \\
&= \int_{(\mathbf{R}^3)^4} (f(x, v'(v, w, n_{x,y})) f(y, w'(v, w, n_{x,y})) \mathbf{1}_{(v-w|n_{x,y})>0} \\
&\quad - f(x, v) f(y, w) \mathbf{1}_{(v-w|n_{x,y})<0}) B[\rho_f, \rho_f](x, y, v-w) v dx dv dy dw \\
&= \int_{(\mathbf{R}^3)^4} (B[\rho_f, \rho_f](x, y, v'(v, w, n_{x,y}) - w'(v, w, n_{x,y})) v'(v, w, n_{x,y}) \\
&\quad - B[\rho_f, \rho_f](x, y, v-w) v) \mathbf{1}_{(v-w|n_{x,y})<0} f(x, v) f(y, w) dx dv dy dw \\
&\quad \text{(with the change of variables } (v, w) \mapsto (v'(v, w, n), w'(v, w, n)) \text{ for each } n \\
&\quad \text{since } (v'(v, w, n) - w'(v, w, n)|n) = -(v-w|n)) \\
&= \int_{(\mathbf{R}^3)^4} (B[\rho_f, \rho_f](y, x, v'(w, v, n_{y,x}) - w'(w, v, n_{y,x})) v'(w, v, n_{y,x}) \\
&\quad - B[\rho_f, \rho_f](y, x, w, v) w) \mathbf{1}_{(w-v|n_{y,x})<0} f(x, v) f(y, w) dx dv dy dw \\
&\quad \text{(exchanging the molecule at } x \text{ with velocity } v \\
&\quad \text{and the molecule at } y \text{ with velocity } w \text{ by assumption (i) on } B) \\
&= \int_{(\mathbf{R}^3)^4} (B[\rho_f, \rho_f](y, x, w'(v, w, n_{x,y}) - v'(v, w, n_{x,y})) w'(v, w, n_{x,y}) \\
&\quad - B[\rho_f, \rho_f](y, x, w, v) w) \mathbf{1}_{(w-v|n_{y,x})<0} f(x, v) f(y, w) dx dv dy dw \\
&\quad \text{(since } v'(w, v, n) = w'(v, w, n) = w'(v, w, -n) \\
&\quad \text{and } w'(w, v, n) = v'(v, w, n) = v'(v, w, -n)) \\
&= \int_{(\mathbf{R}^3)^4} (B[\rho_f, \rho_f](x, y, v'(v, w, n_{x,y}) - w'(v, w, n_{x,y})) w'(v, w, n_{x,y}) \\
&\quad - B[\rho_f, \rho_f](x, y, v, w) w) \mathbf{1}_{(v-w|n_{x,y})<0} f(x, v) f(y, w) dx dv dy dw. \\
&\quad \text{(using assumption (i) on } B)
\end{aligned}$$

Summarizing, the first left-hand side is equal to the right-hand side of either the second or the sixth equality, and therefore to the mean of these expressions, i.e.

$$\begin{aligned}
& \int_{\mathbf{R}^3 \times \mathbf{R}^3} \text{St}[f, f](x, v) v dv dx \\
&= \int_{(\mathbf{R}^3)^4} (B[\rho_f, \rho_f](x, y, v'(v, w, n_{x,y}) - w'(v, w, n_{x,y})) \frac{v'(v, w, n_{x,y}) + w'(v, w, n_{x,y})}{2} \\
&\quad - B[\rho_f, \rho_f](x, y, v-w) \frac{v+w}{2}) \mathbf{1}_{(v-w|n_{x,y})<0} f(x, v) f(y, w) dx dv dy dw = 0
\end{aligned}$$

by assumption (ii) on  $B$ , and because of the first identity in (4).

Proceeding in exactly the same manner, we see that

$$\begin{aligned}
& \int_{\mathbf{R}^3 \times \mathbf{R}^3} \text{St}[f, f](x, v) |v|^2 dv dx \\
&= \int_{(\mathbf{R}^3)^4} (B[\rho_f, \rho_f](x, y, v'(v, w, n_{x,y}) - w'(v, w, n_{x,y})) \frac{|v'(v, w, n_{x,y})|^2 + |w'(v, w, n_{x,y})|^2}{2} \\
&\quad - B[\rho_f, \rho_f](x, y, v-w) \frac{|v|^2 + |w|^2}{2}) \mathbf{1}_{(v-w|n_{x,y})<0} f(x, v) f(y, w) dx dv dy dw = 0
\end{aligned}$$

again by assumption (ii) on  $B$ , and because of the second identity in (4). This concludes the proof.  $\square$

**3.3. Landau Currents for Delocalized Collision Integrals.** By analogy with the arguments in §41 in [20] and in [29] leading to a representation of the Boltzmann collision integral as the divergence of a ‘‘Landau’’ mass current in the velocity variable, our first main result in this paper is to represent delocalized collision integrals (20) in terms of divergences of Landau mass, momentum and energy currents. The main difference with the Boltzmann case is that only the global conservation laws of momentum and energy are known to be satisfied by delocalized collision integrals (20). As a result, one should seek representations in terms of divergences of Landau momentum and energy currents *in both the position and velocity variables*.

**Definition 3.2.** *For each rapidly decaying  $f, g \in C(\mathbf{R}^3 \times \mathbf{R}^3)$ , and each rapidly decaying  $\rho, \varpi \in C(\mathbf{R}^3)$ , set*

$$\begin{aligned} \mathfrak{J}_0[\rho, \varpi, f, g](x, y, v) := & \int_{\mathbf{R}^3 \times \mathbf{R}^3} B[\rho, \varpi](x, y, v - w) \mathbf{1}_{0 < s < (v-w)|n_{y,x}} \\ & \times f(x, v + sn_{y,x}) g(y, w + sn_{y,x}) n_{y,x} dw ds. \end{aligned}$$

Our first main result in this paper is the following theorem, whose proof can be found in section 5.

**Theorem 3.3.** *For each rapidly decaying  $f \in C(\mathbf{R}^3 \times \mathbf{R}^3)$ ,*

$$(21) \quad \text{St}[f, f] = \nabla_v \cdot \mathcal{J}_0[f], \quad \text{where } \mathcal{J}_0[f](x, v) := \int_{\mathbf{R}^3} \mathfrak{J}_0[\rho_f, \rho_f, f, f](x, y, v) dy.$$

For  $i = 1, 2, 3$ ,

$$(22) \quad \text{St}[f, f]v_i = \nabla_x \cdot \mathcal{I}_i[f] + \nabla_v \cdot \mathcal{J}_i[f],$$

where

$$\begin{aligned} \mathcal{I}_i[f](x, v) := & -\frac{1}{2} \int_{(\mathbf{R}^3)^2} \int_0^{\frac{\pi}{2}} B[\rho_f, \rho_f](x_{-\theta}, y_{-\theta}, v_{-\theta} - w_{-\theta}) (v_{-\theta} - w_{-\theta} | n_{y_{-\theta}, x_{-\theta}}) + \\ & \times f(x_{-\theta}, v_{-\theta}) f(y_{-\theta}, w_{-\theta}) (n_{y_{-\theta}, x_{-\theta}})_i y d\theta dy dw \end{aligned}$$

where

$$\begin{pmatrix} x_\theta \\ y_\theta \end{pmatrix} := \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_\theta \\ w_\theta \end{pmatrix} := \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix},$$

and

$$\begin{aligned} \mathcal{J}_i[f, f](x, v) := & \int_{\mathbf{R}^3} \mathfrak{J}_0[\rho_f, \rho_f, (v_i - (v|n_{x,y})(n_{x,y})_i) f, f](x, y, v) dy \\ & + \int_{\mathbf{R}^3} \mathfrak{J}_0[\rho_f, \rho_f, f, (w|n_{x,y})(n_{x,y})_i f](x, y, v) dy \\ & - \frac{1}{2} \int_{(\mathbf{R}^3)^4} \int_0^{\frac{\pi}{2}} B[\rho_f, \rho_f](x_{-\theta}, y_{-\theta}, v_{-\theta} - w_{-\theta}) (v_{-\theta} - w_{-\theta} | n_{y_{-\theta}, x_{-\theta}}) + \\ & \times f(x_{-\theta}, v_{-\theta}) f(y_{-\theta}, w_{-\theta}) (n_{y_{-\theta}, x_{-\theta}})_i w d\theta dy dw. \end{aligned}$$

Finally

$$(23) \quad \text{St}[f, f]|v|^2 = \nabla_x \cdot \mathcal{I}_4[f] + \nabla_v \cdot \mathcal{J}_4[f],$$

where

$$\begin{aligned} \mathcal{I}_4[f](x, v) := & -\frac{1}{2} \int_{(\mathbf{R}^3)^2} \int_0^{\frac{\pi}{2}} B[\rho_f, \rho_f](x_{-\theta}, y_{-\theta}, v_{-\theta} - w_{-\theta}) (v_{-\theta} - w_{-\theta} |n_{y_{-\theta}, x_{-\theta}})_+ \\ & \times (v_{-\theta} + w_{-\theta} |n_{y_{-\theta}, x_{-\theta}}) f(x_{-\theta}, v_{-\theta}) f(y_{-\theta}, w_{-\theta}) y d\theta dy dw, \end{aligned}$$

and

$$\begin{aligned} \mathcal{J}_4[f](x, v) := & \int_{\mathbf{R}^3} \mathfrak{J}_0[\rho_f, \rho_f, (|v|^2 - (v|n_{x,y})^2) f, f](x, y, v) dy \\ & + \int_{\mathbf{R}^3} \mathfrak{J}_0[\rho_f, \rho_f, f, (w|n_{x,y})^2 f](x, y, v) dy \\ & - \frac{1}{2} \int_{(\mathbf{R}^3)^2} \int_0^{\frac{\pi}{2}} B[\rho_f, \rho_f](x_{-\theta}, y_{-\theta}, v_{-\theta} - w_{-\theta}) (v_{-\theta} - w_{-\theta} |n_{y_{-\theta}, x_{-\theta}})_+ \\ & \times (v_{-\theta} + w_{-\theta} |n_{y_{-\theta}, x_{-\theta}}) f(x_{-\theta}, v_{-\theta}) f(y_{-\theta}, w_{-\theta}) w d\theta dy dw. \end{aligned}$$

There is some arbitrariness in the choice of the currents  $\mathcal{J}_0$  and  $\mathcal{I}_i, \mathcal{J}_i$  for  $i = 1, \dots, 4$ . All these currents are obviously not unique, but defined up to arbitrary divergence-free vector fields. One can certainly try to eliminate this arbitrariness by means of some appropriate “gauge” condition, but the choice of such conditions is quite arbitrary, and we have chosen not to investigate this issue any further.

**3.4. H Theorem.** In general, delocalized collision integrals are not known to satisfy a property analogous to Boltzmann’s H Theorem, which is well known in the case of the Boltzmann collision integral (2). However, variants of the H Theorem are known in some cases: see [25, 2, 23, 8] for the Enskog case, or [10] for the soft sphere case.

In this section, we shall assume that the collision kernel is of the form

$$(24) \quad B[\rho, \varpi](x, y, v - w) = b(|x - y|) |(v - w|n_{x,y})|$$

where

$$b \in C([0, +\infty)) \text{ satisfies } g \geq 0 \text{ on } [0, +\infty) \text{ and } \text{supp } b \subset [0, R].$$

We begin with a global Boltzmann type H Theorem — except that the analogue of Boltzmann’s H function is modified to take into account delocalization effects in the collision integral.

**Theorem 3.4.** *Assume that  $f(t, x, v)$  is a classical solution of*

$$(\partial_t + v \cdot \nabla_x) f(t, x, v) = \text{St}[f, f](t, x, v),$$

*with  $(x, v) \mapsto f(t, x, v)$  rapidly decaying at infinity, while  $(x, v) \mapsto \ln f(t, x, v)$  has at most polynomial growth at infinity. Then*

$$\frac{d}{dt} (\mathbf{H}_B[f] + \mathbf{H}_\beta[\rho_f]) = -\Lambda[f] \leq 0,$$

*where  $\mathbf{H}_B[f]$  is the Boltzmann H functional*

$$\mathbf{H}_B[f](t) := \int_{(\mathbf{R}^3)^2} f(t, x, v) \ln f(t, x, v) dx dv,$$

*and  $\mathbf{H}_\beta[\rho]$  the potential interaction energy*

$$\mathbf{H}_\beta[\rho](t) := \frac{1}{2} \int_{(\mathbf{R}^3)^2} \rho_f(t, x) \rho_f(t, y) \beta(|x - y|) dx dy,$$

while

$$\Lambda[f](t) := \frac{1}{2} \int_{(\mathbf{R}^3)^4} f(t, x, v) f(t, y, w) \ell \left( \frac{f(t, x, v'(v, w, n_{y,x})) f(t, y, w'(v, w, n_{y,x}))}{f(t, x, v) f(t, y, w)} \right) \\ \times b(|x - y|)(v - w|n_{y,x})_+ dx dv dy dw$$

is the entropy production term. In these formulas, we have used the notations

$$\beta(r) := \int_r^\infty b(s) ds, \quad \ell(z) = z - 1 - \ln z \geq 0 \text{ for all } z > 0.$$

This result is not really original: it is a modest generalization of the inequality (2.9) in [2]. However, this generalization, especially the potential part  $\mathbf{H}_\beta$ , may have some important implications, to be discussed later.

The idea of conservative formulation of delocalized collision integral can be used in the context of the H Theorem. The idea is that the

**Theorem 3.5.** *Assume that  $f(t, x, v)$  is a classical solution of*

$$(\partial_t + v \cdot \nabla_x) f(t, x, v) = \text{St}[f, f](t, x, v),$$

with  $(x, v) \mapsto f(t, x, v)$  rapidly decaying at infinity, while  $(x, v) \mapsto \ln f(t, x, v)$  has at most polynomial growth at infinity. Then

$$(25) \quad \text{St}[f, f](t, x, v) \ln f(t, x, v) \leq -\nabla_x \cdot \mathcal{K}[f](t, x, v) - \nabla_v \cdot \mathcal{L}[f](t, x, v) \\ - \frac{1}{2} (\partial_t + v \cdot \nabla_x) (f(t, x, v) \rho_f(t, \cdot) \star \beta(|\cdot|)(x)),$$

in the sense of distributions. In this inequality, the notation  $\star$  designates the convolution (in the variable  $x$ ), viz.

$$(26) \quad \rho_f(t, \cdot) \star \beta(|\cdot|)(x) := \int_{\mathbf{R}^3} \rho_f(t, x - y) \beta(|y|) dy,$$

while

$$\mathcal{K}[f](t, x, v) := \frac{1}{2} \int_{(\mathbf{R}^3)^2} \int_0^{\pi/2} (f(t, x_{-\theta}, v'(v, w, n_{y,x})_{-\theta}) f(t, y_{-\theta}, w'(v, w, n_{y,x})_{-\theta}) \\ - f(t, x_{-\theta}, v_{-\theta}) f(t, y_{-\theta}, w_{-\theta})) \ln f(t, y_{-\theta}, w_{-\theta}) b(|x_{-\theta} - y_{-\theta}|) \\ \times (v_{-\theta} - w_{-\theta} |n_{x_{-\theta}, y_{-\theta}})_+ y d\theta dy dw,$$

while

$$\mathcal{L}[f](t, x, v) := \frac{1}{2} \mathcal{J}_0[f](t, x, v) (1 - \rho_f \star \beta(|\cdot|)(t, x)) \\ + \frac{1}{2} (\mathcal{J}_0[f \ln f, f](t, x, v) + \mathcal{J}_0[f, f \ln f](t, x, v)) \\ + \frac{1}{2} \int_{(\mathbf{R}^3)^2} \int_0^{\pi/2} (f(t, x_{-\theta}, v'(v, w, n_{y,x})_{-\theta}) f(t, y_{-\theta}, w'(v, w, n_{y,x})_{-\theta}) \\ - f(t, x_{-\theta}, v_{-\theta}) f(t, y_{-\theta}, w_{-\theta})) \ln f(t, y_{-\theta}, w_{-\theta}) b(|x_{-\theta} - y_{-\theta}|) \\ \times (v_{-\theta} - w_{-\theta} |n_{x_{-\theta}, y_{-\theta}})_+ w d\theta dy dw.$$

In this equality, the notation  $\mathcal{J}_0[f, g]$  designates the current

$$\mathcal{J}_0[f, g](t, x, v) := \int_{(\mathbf{R}^3)^2 \times \mathbf{R}} b(|x - y|)(v - w|n_{y,x}) \mathbf{1}_{0 < s < (v-w|n_{y,x})} \\ \times f(t, x, v + sn_{y,x}) g(t, y, w + sn_{y,x}) n_{y,x} dy dw ds.$$

In other words

$$\mathcal{J}_0[f, g](t, x, v) = \int_{\mathbf{R}^3} \mathfrak{J}_0[\rho_f, \rho_g, f, g](t, x, y, v) dy$$

with  $\mathfrak{J}_0[\rho_f, \rho_g, f, g]$  as in Definition 3.2. However, here, the situation is much simpler, since

$$B[\rho_f, \rho_g](x, y, v - w) := b(|x - y|)(v - w|n_{x,y}|)$$

is independent of  $\rho_f$  and  $\rho_g$ , so that  $\mathfrak{J}_0[\rho_f, \rho_g, f, g](t, x, y, v)$  is also independent of  $\rho_f$  and  $\rho_g$ .

The proofs of Theorems 3.4 and 3.5 can be found in section 6.

#### 4. APPLICATIONS TO THE MACROSCOPIC DESCRIPTION OF DENSE GASES

**4.1. Local Conservation of Mass, Momentum and Energy.** The discussion in section 2 makes it very clear that the Boltzmann collision integral does not contribute to the stress tensor in (14), i.e. the pressure tensor  $P$ . This may seem somewhat disturbing, since one would expect that collisions between gas molecules should account for the acceleration driving the fluid.

This last point is of course not entirely true. Indeed, even in a gas of noninteracting point particles, i.e. if the distribution function satisfies

$$(\partial_t + v \cdot \nabla_x) f(t, x, v) = 0,$$

the pressure tensor  $P$  defined in (15) can be computed explicitly, and it is neither equal to zero (unless  $f(t, x, v) = \rho(t, x)\delta(v - u(t, x))$ ), nor a constant in general. It can be computed explicitly for instance in the case where

$$f(0, x, v) = \exp(-(|x|^2 + |v|^2)).$$

One finds indeed that

$$f(t, x, v) = \exp\left(-\left(\frac{|x|^2}{1+t^2} + (1+t^2)|v - \frac{t}{1+t^2}x|^2\right)\right),$$

so that

$$P(t, x) = \frac{\pi^{3/2}}{2(1+t^2)^{5/2}} \exp\left(-\frac{|x|^2}{1+t^2}\right) I.$$

This elementary example shows that the force deriving from the pressure may be non trivial even in the absence of collisions.

In the case of the Boltzmann equation, the contribution of collisions to the pressure tensor is indirect: in the highly collisional regime, collisions induce a relaxation of the distribution function  $f(t, x, v)$  to a local Maxwellian distribution, a radial function of  $v - u(t, x)$ , so that the current  $Q$  vanishes identically, and  $\rho, u$  defined in (12) and  $p := \text{tr}P$  is a solution of the system of equations (13)-(18)-(19), i.e. the Euler system of gas dynamics for a perfect gas with adiabatic index  $\gamma = 5/3$ , as explained at the end of section 2. This follows from the approach of the hydrodynamic limit of the Boltzmann equation involving Hilbert's expansion, as in [9] or chapter 11 of [11] — see also [3].

In the case of dense gases, the situation is completely different. The collision integral *directly* contributes to the stress tensor and energy flux in the local balance of momentum and energy in the gas, and this contribution is easily computed in terms of the momentum and energy currents in Theorem 3.3. See for instance [22, 27] for other approaches to this issue.

**Corollary 4.1.** *Let  $f$  be a classical solution of the kinetic equation (1) with delocalized collision integral (20). Assume that the collision kernel  $B$  in (20) satisfies (i)-(iv), and that  $(x, v) \mapsto f(t, x, v)$  is continuous on  $\mathbf{R}^3 \times \mathbf{R}^3$  and rapidly decaying at infinity. Then*

$$\begin{cases} \partial_t \rho_f(t, x) + \nabla_x \cdot \int_{\mathbf{R}^3} v f(t, x, v) dv = 0, \\ \partial_t \int_{\mathbf{R}^3} v_i f(t, x, v) dv + \nabla_x \cdot \int_{\mathbf{R}^3} (v v_i f(t, x, v) - \mathcal{I}_i[f](t, x, v)) dv = 0, & i = 1, 2, 3, \\ \partial_t \int_{\mathbf{R}^3} \frac{1}{2} |v|^2 f(t, x, v) dv + \nabla_x \cdot \int_{\mathbf{R}^3} (v \frac{1}{2} |v|^2 f(t, x, v) - \frac{1}{2} \mathcal{I}_4[f](t, x, v)) dv = 0, \end{cases}$$

where  $\mathcal{I}_i[f]$  and  $\mathcal{I}_4[f]$  are respectively the projections of the momentum and energy phase-space currents involved in (22) and (23) on the space of positions.

With the definitions (9) for  $\rho_f$  and

$$u_f(t, x) := \frac{\mathbf{1}_{\rho_f(t, x) > 0}}{\rho_f(t, x)} \int_{\mathbf{R}^3} v f(t, x, v) dv,$$

and

$$P_f(t, x) := \int_{\mathbf{R}^3} (v - u_f(t, x))^{\otimes 2} f(t, x, v) dv,$$

while

$$Q_f(t, x) := \frac{1}{2} \int_{\mathbf{R}^3} (v - u_f(t, x)) |v - u_f(t, x)|^2 f(t, x, v) dv,$$

as in section 2, the identities obtained in the corollary above are recast as

$$(27) \quad \begin{cases} \partial_t \rho_f + \nabla_x \cdot (\rho_f u_f) = 0, \\ \partial_t (\rho_f u_f) + \nabla_x \cdot (\rho_f u_f^{\otimes 2} + P_f - \mathbb{I}[f]) = 0, \\ \partial_t (\frac{1}{2} (\rho_f |u_f|^2 + \text{tr} P_f)) + \nabla_x \cdot (\frac{1}{2} (\rho_f |u_f|^2 + \text{tr} P_f) u_f + P_f \cdot u_f + Q_f - \mathcal{I}_4[f]) = 0, \end{cases}$$

where  $\mathbb{I}[f]$  is the square matrix whose rows are  $\mathcal{I}_i[f] \in \mathbf{R}^3$ , for  $i = 1, 2, 3$ , i.e.

$$\mathbb{I}[f] := \begin{bmatrix} \mathcal{I}_1[f] \\ \mathcal{I}_2[f] \\ \mathcal{I}_3[f] \end{bmatrix}.$$

Obviously the system (27) requires some closure relations — i.e. expressions of  $\mathbb{I}[f]$ , of  $Q_f$  and of  $\mathcal{I}_4[f]$  in terms of  $\rho_f, u_f$  and  $P_f$  — before it can be considered as a bona fide system of equations for the dynamics of dense gases.

**4.2. Local Entropy Inequality.** It is a well-known fact that systems of conservation laws such as (27) with some appropriate closure assumptions may have infinitely many weak solutions corresponding to the same initial data: see for instance section 4.4 in chapter IV of [13]. The example of loss of uniqueness given in section 4.4 in chapter IV of [13] involves unphysical shock waves, which dissipate, instead of producing entropy. For that reason, systems of conservation laws often come with a Friedrichs-Lax entropy inequality used to qualify weak solutions of such systems: see section 4.5 and chapter V in [13]. (Unfortunately this is in general not enough to guarantee uniqueness in space dimension 2 and higher: see [14].)



However, these considerations suggest seeking an entropy inequality for (27). By analogy with [3], we may think of using the local variant of the H Theorem reported in Theorem (3.5) to do so.

**Corollary 4.2.** *Let  $f$  be a classical solution of (1) with delocalized collision integral (20) and collision kernel of the form (24). Assume that  $(x, v) \mapsto f(t, x, v)$  is rapidly decaying at infinity, while  $(x, v) \mapsto \ln f(t, x, v)$  has at most polynomial growth at infinity. Then, the local conservation laws (27) are accompanied with the entropy inequality*

$$\begin{aligned} & \partial_t \left( \int_{\mathbf{R}^3} f \ln f(t, x, v) dv + \frac{1}{2} (\rho_f(t, \cdot) \star \beta(|\cdot|))(x) \rho_f(t, x) \right) \\ & + \nabla_x \cdot \left( \int_{\mathbf{R}^3} (f \ln f + \mathcal{K}[f])(t, x, v) dv + \frac{1}{2} (\rho_f(t, \cdot) \star \beta(|\cdot|))(x) \rho_f(t, x) u_f(t, x) \right) \leq 0 \end{aligned}$$

where

$$\rho_f(t, x) := \int_{\mathbf{R}^3} f(t, x, v) dv \geq 0, \quad u(t, x) = \frac{\mathbf{1}_{\rho(t, x) > 0}}{\rho(t, x)} \int_{\mathbf{R}^3} v f(t, x, v) dv \in \mathbf{R}^3,$$

while the functional  $\mathcal{K}$  is defined in Theorem 3.5.

This is a straightforward consequence of Theorem 3.5, obtained by integrating in  $v \in \mathbf{R}^3$  both sides of the inequality in that theorem. To be rigorous, one should observe that the inequality in Theorem 3.5, which is stated in the sense of distributions, holds true when applied to a more general class of test functions, including  $(x, v) \mapsto \psi(x)$  where  $\psi$  belongs to  $C_c^1(\mathbf{R}^3)$ : see the beginning of the proof of Theorem 3.5.

The differential inequality obtained in Corollary 4.2 is not exactly an entropy inequality in the sense of Friedrichs and Lax, since the entropy

$$\eta(t, x) := \int_{\mathbf{R}^3} f \ln f(t, x, v) dv + \frac{1}{2} (\rho_f(t, \cdot) \star \beta(|\cdot|))(x) \rho_f(t, x),$$

even after some closure assumption specifying the  $v$ -dependence in  $f$  (e.g. a local Maxwellian distribution), is not a *local* function of the conserved densities in (27), since it involves the term  $\frac{1}{2} (\rho_f(t, \cdot) \star \beta(|\cdot|))(x) \rho_f(t, x)$  which is nonlocal in  $\rho_f$ .

Therefore, the usual theory of Friedrichs-Lax extensions [16] of systems of conservation laws, and in particular the fact the convexity of  $\eta$  as a *local* function of the conserved density implies the hyperbolicity of the system of conservation laws, which can then be put in Godunov's form [17], is unfortunately not available for us to handle the system (27). Nevertheless, even if one cannot apply directly the Godunov structure and the Friedrichs-Lax theory, the local entropy inequality reported in Corollary 4.2 may be of independent interest for studying (27). For instance, it could be interesting to connect the discussion of the fluid dynamic limit of the Enskog-Boltzmann equation in [18] with the results presented in this paper. We hope to return to this question in more detail in the future.

## 5. PROOF OF THEOREM 3.3

The proof of Theorem 3.3 involves rather intricate computations, and will be split in several steps.

**5.1. Weak Forms of the Collision Integral.** A first step in the representation of delocalized collision integrals in terms of divergences in  $(x, v)$  of momentum and energy currents is the weak formulation of these collision integrals.

**Lemma 5.1.** *Let  $f, g \in C(\mathbf{R}^3 \times \mathbf{R}^3)$ , rapidly decaying at infinity. For each test function  $\phi \in C(\mathbf{R}^3 \times \mathbf{R}^3)$  with (at most) polynomial growth at infinity,*

$$\int_{(\mathbf{R}^3)^2} \text{St}[f, g](x, v) \phi(x, v) dx dv = \int_{(\mathbf{R}^3)^4} (\phi(x, v'(v, w, n_{x,y})) - \phi(x, v)) f(x, v) g(y, w) \\ \times \mathbf{1}_{(v-w|n_{x,y}) < 0} B[\rho_f, \rho_g](x, y, v-w) dx dv dy dw.$$

*Proof.* By definition of the collision integral  $\text{St}[f, g]$ ,

$$\int_{(\mathbf{R}^3)^2} \text{St}[f, g](x, v) \phi(x, v) dx dv \\ = \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_g](x, y, v-w) (f(x, v'(v, w, n_{x,y})) g(y, w'(v, w, n_{x,y})) \mathbf{1}_{(v-w|n_{x,y}) > 0} \\ - f(x, v) g(y, w) \mathbf{1}_{(v-w|n_{x,y}) < 0}) \phi(x, v) dx dv dy dw.$$

The gain term (positive part) of this collision integral satisfies the following symmetries:

$$\int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_g](x, y, v-w) f(x, v'(v, w, n_{x,y})) g(y, w'(v, w, n_{x,y})) \\ \times \phi(x, v) \mathbf{1}_{(v-w|n_{x,y}) > 0} dx dv dy dw \\ = \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_g](x, y, v'(v, w, n_{x,y}) - w'(v, w, n_{x,y})) f(x, v) g(y, w) \\ \times \phi(x, v'(v, w, n_{x,y})) \mathbf{1}_{(v'(v, w, n_{x,y}) - w'(v, w, n_{x,y})|n_{x,y}) > 0} dx dv dy dw \\ \text{(integrating by substitution with } (v, w) \mapsto (v'(v, w, n_{x,y}), w'(v, w, n_{x,y}))\text{)} \\ = \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_g](x, y, v-w) f(x, v) g(y, v) \phi(x, v'(v, w, n_{x,y})) \mathbf{1}_{(v-w|n_{x,y}) < 0} dx dv dy dw \\ \text{(by (ii) and the equality } (v'(v, w, n_{x,y}) - w'(v, w, n_{x,y})|n_{x,y}) = -(v-w|n_{x,y})\text{)}.$$

Inserting this form of the gain term in the identity above implies the announced identity.  $\square$

The preceding lemma gives a weak formulation of  $\text{St}[f, g]$ , where  $f$  and  $g$  may be different. In particular, the proof of this lemma does not involve property (i) of the collision kernel  $B$ . The next lemma gives a weak formulation of  $\text{St}[f, f]$ . Since  $f = g$  in this case, property (i) of  $B$  is a key ingredient in the proof.

**Lemma 5.2.** *Let  $f \in C(\mathbf{R}^3 \times \mathbf{R}^3)$  be rapidly decaying at infinity. For each test function  $\phi \in C(\mathbf{R}^3 \times \mathbf{R}^3)$  with polynomial growth at infinity,*

$$\int_{(\mathbf{R}^3)^2} \text{St}[f, f](x, v) \phi(x, v) dx dv \\ = \frac{1}{2} \int_{(\mathbf{R}^3)^4} (\phi(x, v'(v, w, n_{x,y})) + \phi(y, w'(v, w, n_{x,y})) - \phi(x, v) - \phi(y, w)) \\ \times f(x, v) f(y, w) B[\rho_f, \rho_f](x, y, v-w) \mathbf{1}_{(v-w|n_{x,y}) < 0} dx dv dy dw.$$

*Proof.* Starting from the identity in the preceding lemma

$$(28) \quad \int_{(\mathbf{R}^3)^2} \text{St}[f, f](x, v) \phi(x, v) dx dv = \int_{(\mathbf{R}^3)^4} (\phi(x, v'(v, w, n_{x,y})) - \phi(x, v)) f(x, v) f(y, w) \\ \times B[\rho_f, \rho_f](x, y, v - w) \mathbf{1}_{(v-w|n_{x,y}) < 0} dx dv dy dw.$$

Integrating by substitution in the right-hand side with  $(x, v, y, w) \mapsto (y, w, x, v)$ ,

$$(29) \quad \int_{(\mathbf{R}^3)^2} \text{St}[f, f](x, v) \phi(x, v) dx dv \\ = \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](y, x, w - v) f(y, w) f(x, v) (\phi(y, v'(w, v, n_{y,x})) - \phi(y, w)) \\ \times \mathbf{1}_{(w-v|n_{y,x}) < 0} dx dv dy dw \\ = \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](x, y, v - w) f(x, v) f(y, w) (\phi(y, w'(v, w, n_{x,y})) - \phi(y, w)) \\ \times \mathbf{1}_{(v-w|n_{x,y}) < 0} dx dv dy dw,$$

where the last equality follows from (i), while

$$v'(w, v, -n) = v'(w, v, n) = w'(v, w, n).$$

Since the last right-hand sides of (28) and (29) are equal, they are both equal to their average, i.e.

$$\int_{(\mathbf{R}^3)^2} \text{St}[f, f] \phi(x, v) dx dv = \int_{(\mathbf{R}^3)^4} \frac{\phi(x, v'(v, w, n_{x,y})) - \phi(x, v) + \phi(y, w'(v, w, n_{x,y})) - \phi(y, w)}{2} \\ \times f(x, v) f(y, w) B[\rho_f, \rho_f](x, y, v - w) \mathbf{1}_{(v-w|n_{x,y}) < 0} dx dv dy dw,$$

which is precisely the announced identity.  $\square$

Lemma 5.1 will be used repeatedly in the proof of Theorem 3.3. Lemma 5.2, on the other hand, is used in the proofs of Theorems 3.4 and 3.5.

**5.2. Mass Current: Proof of (21).** Observe that

$$\phi(x, v'(v, w, n)) - \phi(x, v) = \int_0^{(v-w|n)} \frac{d}{ds} \phi(x, v - sn) ds \\ = - \int_0^{(v-w|n)} n \cdot \nabla_v \phi(x, v - sn) ds.$$

$$\int_{(\mathbf{R}^3)^2} \text{St}[f, g](x, v) \phi(x, v) dx dv = \int_{(\mathbf{R}^3)^4} (\phi(x, v'(v, w, n_{x,y})) - \phi(x, v)) f(x, v) g(y, w) \\ \times \mathbf{1}_{(v-w|n_{x,y}) < 0} B[\rho_f, \rho_g](x, y, v - w) dx dv dy dw$$

Inserting this expression in the right-hand side of the weak formulation of the collision integral  $\text{St}[f, g]$  in Lemma 5.1 shows that

$$\begin{aligned} \int_{(\mathbf{R}^3)^2} \text{St}[f, g](x, v) \phi(x, v) dx dv &= - \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_g](x, y, v - w) f(x, v) g(y, w) \\ &\quad \times \left( \int_0^{(v-w)|n_{y,x}} n_{y,x} \cdot \nabla_v \phi(x, v - sn_{y,x}) ds \right) \mathbf{1}_{(v-w)|n_{y,x} > 0} dx dv dy dw \\ &= - \int_{(\mathbf{R}^3)^4 \times \mathbf{R}} B[\rho_f, \rho_g](x, y, v - w) \mathbf{1}_{0 < s < (v-w)|n_{y,x}} f(x, v) g(y, w) \\ &\quad \times n_{y,x} \cdot \nabla_v \phi(x, v - sn_{y,x}) dx dv dy dw ds. \end{aligned}$$

The last right-hand side above is recast as

$$\begin{aligned} &\int_{(\mathbf{R}^3)^4 \times \mathbf{R}} B[\rho_f, \rho_g](x, y, v - w) \mathbf{1}_{0 < s < (v-w)|n_{y,x}} f(x, v) g(y, w) \\ &\quad \times n_{y,x} \cdot \nabla_v \phi(x, v - sn_{y,x}) dx dv dy dw ds \\ &= \int_{(\mathbf{R}^3)^4 \times \mathbf{R}} B[\rho_f, \rho_g](x, y, \bar{v} - \bar{w}) f(x, \bar{v} + sn_{y,x}) g(y, \bar{w} + sn_{y,x}) \\ &\quad \times \mathbf{1}_{0 < s < (\bar{v} - \bar{w})|n_{y,x}} n_{y,x} \cdot \nabla_v \phi(x, \bar{v}) dx d\bar{v} dy d\bar{w} ds, \end{aligned}$$

integrating by substitution with  $\bar{v} := v - sn_{y,x}$  and  $\bar{w} := w - sn_{y,x}$ . Therefore

$$\int_{(\mathbf{R}^3)^2} \text{St}[f, g](x, v) \phi(x, v) dx dv = - \int_{(\mathbf{R}^3)^3} \mathfrak{J}_0[\rho_f, \rho_g, f, g](x, y, \bar{v}) \cdot \nabla_v \phi(x, \bar{v}) dx dy d\bar{v},$$

which is the formulation in the sense of distributions of the identity (21).  $\square$

**5.3. Momentum Currents: Proof of (22).** Applying Lemma 5.1 to the test function  $\phi(x, v)v_i$ , we find that

$$\begin{aligned} \int_{(\mathbf{R}^3)^2} \text{St}[f, g](x, v) \phi(x, v) v_i dx dv &= \int_{(\mathbf{R}^3)^4} (\phi(x, v'(v, w, n_{x,y})) v'_i(v, w, n_{x,y}) - \phi(x, v) v_i) \\ &\quad \times f(x, v) g(y, w) B[\rho_f, \rho_g](x, y, v - w) \mathbf{1}_{(v-w)|n_{x,y} < 0} dx dv dy dw. \end{aligned}$$

Then we split the difference

$$\begin{aligned} &\phi(x, v'(v, w, n_{x,y})) v'_i(v, w, n_{x,y}) - \phi(x, v) v_i \\ &= (\phi(x, v'(v, w, n_{x,y})) - \phi(x, v)) v'_i(v, w, n_{x,y}) \\ &\quad + (v'_i(v, w, n_{x,y}) - v_i) \phi(x, v) \\ &= (\phi(x, v'(v, w, n_{x,y})) - \phi(x, v)) v_i \\ &\quad - (\phi(x, v'(v, w, n_{x,y})) - \phi(x, v)) (v|n_{x,y})(n_{x,y})_i \\ &\quad + (\phi(x, v'(v, w, n_{x,y})) - \phi(x, v)) (w|n_{x,y})(n_{x,y})_i \\ &\quad - \phi(x, v) (v - w|n_{x,y})(n_{x,y})_i \\ &= S_1 - S_2 + S_3 - S_4. \end{aligned}$$

Thus

$$\begin{aligned} &\int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](x, y, v - w) \mathbf{1}_{(v-w)|n_{x,y} < 0} f(x, v) f(y, w) S_1 dx dv dy dw \\ &= - \int_{(\mathbf{R}^3)^3} \mathfrak{J}_0[\rho_f, \rho_f, v_i f, f](x, y, v) \cdot \nabla_v \phi(x, v) dx dy dv, \end{aligned}$$

while

$$\begin{aligned} & \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](x, y, v-w) \mathbf{1}_{(v-w|n_{x,y}) < 0} f(x, v) f(y, w) S_2 dx dv dy dw \\ &= - \int_{(\mathbf{R}^3)^3} \mathfrak{J}_0[\rho_f, \rho_f, (v|n_{x,y})(n_{x,y})_i f, f](x, y, v) \cdot \nabla_v \phi(x, v) dx dy dv, \end{aligned}$$

and

$$\begin{aligned} & \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](x, y, v-w) \mathbf{1}_{(v-w|n_{x,y}) < 0} f(x, v) f(y, w) S_3 dx dv dy dw \\ &= - \int_{(\mathbf{R}^3)^3} \mathfrak{J}_0[\rho_f, \rho_f, f, (w|n_{x,y})(n_{x,y})_i f](x, y, v) \cdot \nabla_v \phi(x, v) dx dy dv. \end{aligned}$$

We are left with the term

$$\begin{aligned} & \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](x, y, v-w) \mathbf{1}_{(v-w|n_{x,y}) < 0} f(x, v) f(y, w) S_4 dx dv dy dw \\ &= \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](x, y, v-w) (v-w|n_{y,x})_+ (n_{y,x})_i \phi(x, v) f(x, v) f(y, w) dx dv dy dw \\ &= \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](y, x, w-v) (v-w|n_{y,x})_+ (n_{y,x})_i \phi(x, v) f(x, v) f(y, w) dx dv dy dw, \end{aligned}$$

where the last equality follows from assumption (i) on the collision kernel  $B$ . This last integral is transformed with the change of variables

$$(x, y, v, w) \mapsto (y, x, w, v)$$

so that

$$\begin{aligned} & \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](y, x, w-v) (v-w|n_{y,x})_+ (n_{y,x})_i \phi(x, v) f(x, v) f(y, w) dx dv dy dw \\ &= \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](x, y, v-w) (w-v|n_{x,y})_+ (n_{x,y})_i \phi(y, w) f(x, v) f(y, w) dx dv dy dw \\ &= - \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](x, y, v-w) (v-w|n_{y,x})_+ (n_{y,x})_i \phi(y, w) f(x, v) f(y, w) dx dv dy dw. \end{aligned}$$

Thus

$$\begin{aligned} & \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](x, y, v-w) \mathbf{1}_{(v-w|n_{x,y}) < 0} f(x, v) f(y, w) S_4 dx dv dy dw \\ &= \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](x, y, v-w) (v-w|n_{y,x})_+ (n_{y,x})_i \phi(x, v) f(x, v) f(y, w) dx dv dy dw \\ &= - \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](x, y, v-w) (v-w|n_{y,x})_+ (n_{y,x})_i \phi(y, w) f(x, v) f(y, w) dx dv dy dw, \end{aligned}$$

so that, eventually,

$$\begin{aligned} & \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](x, y, v-w) \mathbf{1}_{(v-w|n_{x,y}) < 0} f(x, v) f(y, w) S_4 dx dv dy dw \\ &= \frac{1}{2} \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](x, y, v-w) (v-w|n_{y,x})_+ (n_{y,x})_i (\phi(x, v) - \phi(y, w)) \\ & \quad \times f(x, v) f(y, w) dx dv dy dw. \end{aligned}$$

Set

$$\begin{pmatrix} x_\theta \\ y_\theta \end{pmatrix} := \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_\theta \\ w_\theta \end{pmatrix} := \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}.$$

We shall express the difference  $\phi(x, v) - \phi(y, w)$  as follows:

$$\phi(x, v) - \phi(y, w) = - \int_0^{\pi/2} \frac{d}{d\theta} \phi(x_\theta, v_\theta) d\theta.$$

Since

$$\frac{d}{d\theta} \phi(x_\theta, v_\theta) = y_\theta \cdot \nabla_x \phi(x_\theta, v_\theta) + w_\theta \cdot \nabla_v \phi(x_\theta, v_\theta),$$

we find that

$$\begin{aligned} & \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](x, y, v-w) \mathbf{1}_{(v-w|n_{x,y}) < 0} f(x, v) f(y, w) S_4 dx dv dy dw \\ &= -\frac{1}{2} \int_{(\mathbf{R}^3)^4} \int_0^{\pi/2} B[\rho_f, \rho_f](x, y, v-w) (v-w|n_{y,x})_+ (n_{y,x})_i y_\theta \cdot \nabla_x \phi(x_\theta, v_\theta) \\ & \quad \times f(x, v) f(y, w) d\theta dx dv dy dw \\ & \quad -\frac{1}{2} \int_{(\mathbf{R}^3)^4} \int_0^{\pi/2} B[\rho_f, \rho_f](x, y, v-w) (v-w|n_{y,x})_+ (n_{y,x})_i w_\theta \cdot \nabla_v \phi(x_\theta, v_\theta) \\ & \quad \times f(x, v) f(y, w) d\theta dx dv dy dw. \end{aligned}$$

Since the Jacobian determinants

$$\det \frac{\partial(x_\theta, y_\theta)}{\partial(x, y)} = \det \frac{\partial(v_\theta, w_\theta)}{\partial(v, w)} = 1,$$

one can transform both integrals in the right-hand side of the identity above into

$$\begin{aligned} & - \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](x, y, v-w) \mathbf{1}_{(v-w|n_{x,y}) < 0} f(x, v) f(y, w) S_4 dx dv dy dw \\ &= \frac{1}{2} \int_{(\mathbf{R}^3)^4} \nabla_x \phi(x, v) \cdot y \int_0^{\pi/2} B[\rho_f, \rho_f](x_\theta, y_\theta, v_\theta - w_\theta) (v_\theta - w_\theta | n_{y_\theta, x_\theta})_+ \\ & \quad \times (n_{y_\theta, x_\theta})_i f(x_\theta, v_\theta) f(y_\theta, w_\theta) d\theta dx dv dy dw \\ & \quad + \frac{1}{2} \int_{(\mathbf{R}^3)^4} \nabla_v \phi(x, v) \cdot w \int_0^{\pi/2} B[\rho_f, \rho_f](x_\theta, y_\theta, v_\theta - w_\theta) (v_\theta - w_\theta | n_{y_\theta, x_\theta})_+ \\ & \quad \times (n_{y_\theta, x_\theta})_i f(x_\theta, v_\theta) f(y_\theta, w_\theta) d\theta dx dv dy dw. \end{aligned}$$

Summarizing, we have found that

$$\begin{aligned} \int_{(\mathbf{R}^3)^2} \text{St}[f, f](x, v) \phi(x, v) v_k dx dv &= - \int_{(\mathbf{R}^3)^2} \mathcal{I}_i[f](x, v) \cdot \nabla_x \phi(x, v) dx dv \\ & \quad - \int_{(\mathbf{R}^3)^2} \mathcal{J}_i[f](x, v) \cdot \nabla_v \phi(x, v) dx dv, \end{aligned}$$

where

$$\begin{aligned} \mathcal{I}_i[f](x, v) &:= -\frac{1}{2} \int_{(\mathbf{R}^3)^4} \int_0^{\pi/2} B[\rho_f, \rho_f](x_\theta, y_\theta, v_\theta - w_\theta) (v_\theta - w_\theta | n_{y_\theta, x_\theta})_+ \\ & \quad \times f(x_\theta, v_\theta) f(y_\theta, w_\theta) (n_{y_\theta, x_\theta})_i y d\theta dy dw, \end{aligned}$$

while

$$\begin{aligned} \mathcal{J}_i[f](x, v) &:= \int_{\mathbf{R}^3} \mathfrak{J}_0[\rho_f, \rho_f, (v_i - (v|n_{x,y})(n_{x,y})_i)f, f](x, y, v) dy \\ &\quad + \int_{\mathbf{R}^3} \mathfrak{J}_0[\rho_f, \rho_f, f, (w|n_{x,y})(n_{x,y})_i f](x, y, v) dy \\ &- \frac{1}{2} \int_{(\mathbf{R}^3)^4} \int_0^{\pi/2} B[\rho_f, \rho_f](x_{-\theta}, y_{-\theta}, v_{-\theta} - w_{-\theta})(v_{-\theta} - w_{-\theta} | n_{y_{-\theta}, x_{-\theta}})_+ \\ &\quad \times f(x_{-\theta}, v_{-\theta}) f(y_{-\theta}, w_{-\theta}) (n_{y_{-\theta}, x_{-\theta}})_i w d\theta dy dw. \end{aligned}$$

This is the weak formulation (in the sense of distributions) of (22).  $\square$

**5.4. Energy Currents: Proof of (23).** Applying Lemma 5.1 to the test function  $\phi(x, v)|v|^2$ , we find that

$$\begin{aligned} &\int_{(\mathbf{R}^3)^2} \text{St}[f, g](x, v) \phi(x, v) |v|^2 dx dv \\ &= \int_{(\mathbf{R}^3)^4} (\phi(x, v'(v, w, n_{x,y})) |v'(v, w, n_{x,y})|^2 - \phi(x, v) |v|^2) f(x, v) g(y, w) \\ &\quad \times B[\rho_f, \rho_g](x, y, v - w) \mathbf{1}_{(v-w|n_{x,y}) < 0} dx dv dy dw \end{aligned}$$

Observe that

$$v'(v, w, n) = (v - (v|n)n) + (w|n)n \in (\mathbf{R}n)^\perp + \mathbf{R}n.$$

Hence

$$|v'(v, w, n)|^2 = |v - (v|n)n|^2 + (w|n)^2 = |v|^2 - (v|n)^2 + (w|n)^2.$$

Then we split the difference

$$\begin{aligned} &\phi(x, v'(v, w, n_{x,y})) |v'(v, w, n_{x,y})|^2 - \phi(x, v) |v|^2 \\ &= (\phi(x, v'(v, w, n_{x,y})) - \phi(x, v)) |v'(v, w, n_{x,y})|^2 \\ &\quad + (|v'(v, w, n_{x,y})|^2 - |v|^2) \phi(x, v) \\ &= (\phi(x, v'(v, w, n_{x,y})) - \phi(x, v)) |v|^2 \\ &\quad - (\phi(x, v'(v, w, n_{x,y})) - \phi(x, v)) |(v|n_{x,y})|^2 \\ &\quad + (\phi(x, v'(v, w, n_{x,y})) - \phi(x, v)) |(w|n_{x,y})|^2 \\ &\quad - \phi(x, v) ((v|n_{x,y})^2 - (w|n_{x,y})^2) \\ &= S_1 - S_2 + S_3 - S_4. \end{aligned}$$

Proceeding as in the case of the momentum current, we find that

$$\begin{aligned} &\int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](x, y, v - w) \mathbf{1}_{(v-w|n_{x,y}) < 0} f(x, v) f(y, w) \\ &\quad \times (S_1 - S_2 + S_3) dx dv dy dw \\ &= - \int_{(\mathbf{R}^3)^3} \mathfrak{J}_0[\rho_f, \rho_f, (|v|^2 - (v|n_{x,y})^2)f, f](x, y, v) \cdot \nabla_v \phi(x, v) dx dy dv \\ &\quad - \int_{(\mathbf{R}^3)^3} \mathfrak{J}_0[\rho_f, \rho_f, f, (w|n_{x,y})^2 f](x, y, v) \cdot \nabla_v \phi(x, v) dx dy dv. \end{aligned}$$

Then we are left with the term

$$\begin{aligned}
& - \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](x, y, v-w) \mathbf{1}_{(v-w|n_{x,y}) < 0} f(x, v) f(y, w) S_4 dx dv dy dw \\
&= \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](x, y, v-w) \mathbf{1}_{(v-w|n_{x,y}) < 0} ((v|n_{x,y})^2 - (w|n_{x,y})^2) \phi(x, v) \\
&\quad \times f(x, v) f(y, w) dx dv dy dw \\
&= \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](x, y, v-w) (v-w|n_{y,x})_+ (v+w|n_{y,x}) \phi(x, v) \\
&\quad \times f(x, v) f(y, w) dx dv dy dw \\
&= - \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](y, x, w-v) (w-v|n_{x,y})_+ (w+v|n_{x,y}) \phi(x, v) \\
&\quad \times f(x, v) f(y, w) dx dv dy dw.
\end{aligned}$$

by assumption (i) on the collision kernel  $B$ , observing that  $n_{y,x} = -n_{x,y}$ . With the change of variables

$$(x, v, y, w) \mapsto (y, w, x, v)$$

this last integral is transformed into

$$\begin{aligned}
& - \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](x, y, v-w) \mathbf{1}_{(v-w|n_{x,y}) < 0} f(x, v) f(y, w) S_4 dx dv dy dw \\
&= - \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](x, y, v-w) (v-w|n_{y,x})_+ (v+w|n_{y,x}) \phi(y, w) \\
&\quad \times f(x, v) f(y, w) dx dv dy dw.
\end{aligned}$$

Therefore

$$\begin{aligned}
& - \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](x, y, v-w) \mathbf{1}_{(v-w|n_{x,y}) < 0} f(x, v) f(y, w) S_4 dx dv dy dw \\
&= \frac{1}{2} \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](x, y, v-w) (v-w|n_{y,x})_+ (v+w|n_{y,x}) (\phi(x, v) - \phi(y, w)) \\
&\quad \times f(x, v) f(y, w) dx dv dy dw.
\end{aligned}$$

With the same notation as in the preceding section, i.e.

$$\begin{pmatrix} x_\theta \\ y_\theta \end{pmatrix} := \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_\theta \\ w_\theta \end{pmatrix} := \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix},$$

the difference  $\phi(x, v) - \phi(y, w)$  is represented as

$$\phi(y, w) - \phi(x, v) = \int_0^{\pi/2} (y_\theta \cdot \nabla_x \phi + w_\theta \cdot \nabla_v \phi)(x_\theta, v_\theta) d\theta.$$



Substituting this expression in the identity above leads to

$$\begin{aligned}
& - \int_{(\mathbf{R}^3)^4} B[\rho_f, \rho_f](x, y, v-w) \mathbf{1}_{(v-w|n_{x,y}) < 0} f(x, v) f(y, w) S_4 dx dv dy dw dn \\
& = \frac{1}{2} \int_{(\mathbf{R}^3)^4} \int_0^{\pi/2} B[\rho_f, \rho_f](x, y, v-w) (y_\theta \cdot \nabla_x \phi + w_\theta \cdot \nabla_v \phi)(x_\theta, v_\theta) \\
& \quad \times (v-w|n_{y,x})_+ (v+w|n_{y,x}) f(x, v) f(y, w) d\theta dx dv dy dw \\
& = \frac{1}{2} \int_{(\mathbf{R}^3)^4} \int_0^{\pi/2} B[\rho_f, \rho_f](x_{-\theta}, y_{-\theta}, v_{-\theta}-w_{-\theta}) (y \cdot \nabla_x \phi + w \cdot \nabla_v \phi)(x, v) \\
& \quad \times (v_{-\theta}-w_{-\theta}|n_{y_{-\theta}, x_{-\theta}})_+ (v_{-\theta}+w_{-\theta}|n_{y_{-\theta}, x_{-\theta}}) f(x_{-\theta}, v_{-\theta}) f(y_{-\theta}, w_{-\theta}) d\theta dx dv dy dw.
\end{aligned}$$

Summarizing, we have proved that

$$\begin{aligned}
\int_{(\mathbf{R}^3)^2} \text{St}[f, f](x, v) \phi(x, v) |v|^2 dx dv & = - \int_{(\mathbf{R}^3)^2} \mathcal{I}_4[f](x, v) \cdot \nabla_x \phi(x, v) dx dv \\
& \quad - \int_{(\mathbf{R}^3)^2} \mathcal{J}_4[f](x, v) \cdot \nabla_v \phi(x, v) dx dv,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{I}_4[f](x, v) & := -\frac{1}{2} \int_{(\mathbf{R}^3)^2} \int_0^{\pi/2} B[\rho_f, \rho_f](x_{-\theta}, y_{-\theta}, v_{-\theta}-w_{-\theta}) \\
& \quad \times (v_{-\theta}-w_{-\theta}|n_{y_{-\theta}, x_{-\theta}})_+ (v_{-\theta}+w_{-\theta}|n_{y_{-\theta}, x_{-\theta}}) f(x_{-\theta}, v_{-\theta}) f(y_{-\theta}, w_{-\theta}) y d\theta dy dw.
\end{aligned}$$

while

$$\begin{aligned}
\mathcal{J}_4[f](x, v) & := - \int_{\mathbf{R}^3} \mathfrak{J}_0[\rho_f, \rho_f, (|v|^2 - (v|n_{x,y})^2) f, f](x, y, v) dy \\
& \quad - \int_{\mathbf{R}^3} \mathfrak{J}_0[\rho_f, \rho_f, f, (w|n_{x,y})^2 f](x, y, v) dy \\
& \quad - \frac{1}{2} \int_{(\mathbf{R}^3)^2} \int_0^{\pi/2} B[\rho_f, \rho_f](x_{-\theta}, y_{-\theta}, v_{-\theta}-w_{-\theta}) (v_{-\theta}-w_{-\theta}|n_{y_{-\theta}, x_{-\theta}})_+ \\
& \quad \times (v_{-\theta}+w_{-\theta}|n_{y_{-\theta}, x_{-\theta}}) f(x_{-\theta}, v_{-\theta}) f(y_{-\theta}, w_{-\theta}) w d\theta dy dw.
\end{aligned}$$

This is the weak formulation (in the sense of distributions) of (23).  $\square$

## 6. ENTROPY INEQUALITIES FOR DELOCALIZED COLLISION INTEGRALS

**6.1. Proof of Theorem 3.4.** Start from the weak formulation of the collision integral in Lemma 5.2, with test function  $\phi := \ln f + 1$ :

$$\begin{aligned}
& \int_{(\mathbf{R}^3)^2} \text{St}[f, f](t, x, v) (\ln f(t, x, v) + 1) dx dv \\
& = \frac{1}{2} \int_{(\mathbf{R}^3)^4} f(t, x, v) f(t, y, w) \ln \frac{f(t, x, v'(v, w, n_{y,x})) f(t, y, w'(v, w, n_{y,x}))}{f(t, x, v) f(t, y, w)} \\
& \quad \times b(|x-y|) (v-w|n_{y,x})_+ dx dv dy dw.
\end{aligned}$$

Thus, in terms of the function  $\ell$ ,

$$\begin{aligned}
& \int_{(\mathbf{R}^3)^2} \text{St}[f, f](t, x, v) (\ln f(t, x, v) + 1) dx dv \\
= & -\frac{1}{2} \int_{(\mathbf{R}^3)^4} f(t, x, v) f(t, y, w) \ell \left( \frac{f(t, x, v'(v, w, n_{y,x})) f(t, y, w'(v, w, n_{y,x}))}{f(t, x, v) f(t, y, w)} \right) \\
& \quad \times b(|x - y|) (v - w|n_{y,x})_+ dx dv dy dw \\
& + \frac{1}{2} \int_{(\mathbf{R}^3)^4} (f(t, x, v'(v, w, n_{y,x})) f(t, y, w'(v, w, n_{y,x})) - f(t, x, v) f(t, y, w)) \\
& \quad \times b(|x - y|) (v - w|n_{y,x})_+ dx dv dy dw \\
& = S_1 + S_2.
\end{aligned}$$

Obviously  $S_1 \leq 0$ , while

$$\begin{aligned}
& \int_{(\mathbf{R}^3)^4} f(t, x, v'(v, w, n_{y,x})) f(t, y, w'(v, w, n_{y,x})) b(|x - y|) (v - w|n_{y,x})_+ dx dv dy dw \\
= & \int_{(\mathbf{R}^3)^4} f(t, x, v) f(t, y, w) b(|x - y|) (v'(v, w, n_{y,x}) - w'(v, w, n_{y,x})|n_{y,x})_+ dx dv dy dw \\
& = \int_{(\mathbf{R}^3)^4} f(t, x, v) f(t, y, w) b(|x - y|) (v - w|n_{x,y})_+ dx dv dy dw.
\end{aligned}$$

Hence

$$\begin{aligned}
S_2 = & \frac{1}{2} \int_{(\mathbf{R}^3)^4} f(t, x, v) f(t, y, w) ((v - w|n_{x,y})_+ - (v - w|n_{x,y})_-) b(|x - y|) dx dv dy dw \\
& = \frac{1}{2} \int_{(\mathbf{R}^3)^4} f(t, x, v) f(t, y, w) (v - w|n_{x,y}) b(|x - y|) dx dv dy dw.
\end{aligned}$$

The formula

$$\beta(r) := \int_r^\infty b(s) ds$$

implies that

$$\begin{aligned}
(30) \quad (v \cdot \nabla_x + w \cdot \nabla_y) \beta(|x - y|) &= (v - w \frac{x-y}{|x-y|}) \beta'(|x - y|) \\
&= -(v - w|n_{x,y}) b(|x - y|).
\end{aligned}$$

Hence

$$\begin{aligned}
& \int_{(\mathbf{R}^3)^4} f(t, x, v) f(t, y, w) (v - w|n_{x,y}) b(|x - y|) dx dv dy dw \\
= & \int_{(\mathbf{R}^3)^4} (v \cdot \nabla_x + w \cdot \nabla_y) (f(t, x, v) f(t, y, w)) \beta(|x - y|) dx dv dy dw \\
= & \int_{(\mathbf{R}^3)^4} (\text{St}[f, f](t, x, v) f(t, y, w) + f(t, x, v) \text{St}[f, f](t, y, w)) \beta(|x - y|) dx dv dy dw \\
& - \int_{(\mathbf{R}^3)^4} (\partial_t f(t, x, v) f(t, y, w) + f(t, x, v) \partial_t f(t, y, w)) \beta(|x - y|) dx dv dy dw.
\end{aligned}$$

Observing that the local mass conservation implies that

$$\int_{\mathbf{R}^3} \text{St}[f, f](t, x, v) dv = \int_{\mathbf{R}^3} \text{St}[f, f](t, y, w) dw = 0,$$

we conclude that

$$S_2 = -\frac{1}{2} \frac{d}{dt} \int_{(\mathbf{R}^3)^4} f(t, x, v) f(t, y, w) \beta(|x - y|) dx dv dy dw.$$

Summarizing, we have proved that

$$\frac{d}{dt} \int_{(\mathbf{R}^3)^2} f \ln f(t, x, v) dx dv = \int_{(\mathbf{R}^3)^2} \text{St}[f, f](t, x, v) (\ln f(t, x, v) + 1) dx dv = S_1 + S_2$$

and the conclusion follows from the expression of  $S_2$  obtained above, after defining  $\Lambda$  by the formula  $\Lambda := -S_1 \geq 0$ .  $\square$

**6.2. Proof of Theorem 3.5.** This proof involves somewhat lengthy computations, and will be split in six steps for the reader's convenience. Throughout this proof,  $\psi$  designates a test function satisfying the following conditions

$$\psi \in C^1(\mathbf{R}^3 \times \mathbf{R}^3) \cap L^\infty(\mathbf{R}^3 \times \mathbf{R}^3), \quad \psi(x, v) \geq 0 \text{ for all } x, v \in \mathbf{R}^3.$$

Step 1. Start from the weak formulation of the collision integral in Lemma 5.2, with test function  $\phi := \psi \ln f$ : observing that  $v'(v, w, n) = v'(v, w, -n)$  and  $w'(v, w, n) = w'(v, w, -n)$ ,

$$\begin{aligned} & \int_{(\mathbf{R}^3)^2} \text{St}[f, f](t, x, v) \ln f(t, x, v) \psi(x, v) dx dv \\ &= \int_{(\mathbf{R}^3)^4} \frac{(\psi \ln f)(t, x, v'(v, w, n_{y, x})) + (\psi \ln f)(t, y, w'(v, w, n_{y, x})) - (\psi \ln f)(t, x, v) - (\psi \ln f)(t, y, w)}{2} \\ & \quad \times f(t, x, v) f(t, y, w) b(|x - y|) (v - w |n_{y, x}|)_+ dx dv dy dw. \end{aligned}$$

Decompose

$$S := \frac{(\psi \ln f)(t, x, v'(v, w, n_{y, x})) + (\psi \ln f)(t, y, w'(v, w, n_{y, x})) - (\psi \ln f)(t, x, v) - (\psi \ln f)(t, y, w)}{2}$$

as

$$\begin{aligned} S &= \frac{1}{2} \psi(x, v'(v, w, n_{y, x})) \ln \frac{f(t, x, v'(v, w, n_{y, x})) f(t, y, w'(v, w, n_{y, x}))}{f(t, x, v) f(t, y, w)} \\ & \quad + \frac{1}{2} (\psi(y, w'(v, w, n_{y, x})) - \psi(x, v'(v, w, n_{y, x}))) \ln f(t, y, w'(v, w, n_{y, x})) \\ & \quad - \frac{1}{2} (\psi(x, v) - \psi(x, v'(v, w, n_{y, x}))) \ln f(t, x, v) \\ & \quad - \frac{1}{2} (\psi(y, w) - \psi(x, v'(v, w, n_{y, x}))) \ln f(t, y, w) \\ & = S_1 + S_2 - S_3 - S_4. \end{aligned}$$

The term  $S_4$  is decomposed further into

$$\begin{aligned} S_4 &= \frac{1}{2} (\psi(y, w) - \psi(x, v'(v, w, n_{y, x}))) \ln f(t, y, w) \\ &= \frac{1}{2} (\psi(y, w) - \psi(x, v)) \ln f(t, y, w) \\ & \quad + \frac{1}{2} (\psi(x, v) - \psi(x, v'(v, w, n_{y, x}))) \ln f(t, y, w) \\ &= S_{41} + S_{42}. \end{aligned}$$

Step 2. Similarly, with  $\ell(z) := z - 1 - \ln z \geq 0$  for  $z > 0$ ,

$$\begin{aligned} S_1 &= -\frac{1}{2}\psi(x, v'(v, w, n_{y,x}))\ell\left(\frac{f(t, x, v'(v, w, n_{y,x}))f(t, y, w'(v, w, n_{y,x}))}{f(t, x, v)f(t, y, w)}\right) \\ &\quad + \frac{1}{2}\psi(x, v'(v, w, n_{y,x}))\left(\frac{f(t, x, v'(v, w, n_{y,x}))f(t, y, w'(v, w, n_{y,x}))}{f(t, x, v)f(t, y, w)} - 1\right) \\ &= -S_{11} + S_{12}. \end{aligned}$$

Thus

$$\begin{aligned} \int_{(\mathbf{R}^3)^2} \text{St}[f, f] \ln f(t, x, v) dx dv &= \int_{(\mathbf{R}^3)^4} (-S_{11} + S_{12} + S_2 - S_3 - S_{41} - S_{42}) \\ &\quad \times f(t, x, v)f(t, y, w)b(|x - y|)(v - w|n_{y,x})_+ dx dv dy dw. \end{aligned}$$

Set

$$\Lambda[\psi](t) := \int_{(\mathbf{R}^3)^4} S_{11} f(t, x, v)f(t, y, w)b(|x - y|)(v - w|n_{y,x})_+ dx dv dy dw;$$

clearly

$$\psi \geq 0 \implies \Lambda[\psi] \geq 0.$$

Step 3. Next

$$\begin{aligned} \Gamma_{12} &:= \int_{(\mathbf{R}^3)^4} S_{12} f(t, x, v)f(t, y, w)b(|x - y|)(v - w|n_{y,x})_+ dx dv dy dw \\ &= \frac{1}{2} \int_{(\mathbf{R}^3)^4} \psi(x, v'(v, w, n_{y,x})) (f(t, x, v'(v, w, n_{y,x}))f(t, y, w'(v, w, n_{y,x})) \\ &\quad - f(t, x, v)f(t, y, w)) b(|x - y|)(v - w|n_{y,x})_+ dx dv dy dw \\ &= \frac{1}{2} \int_{(\mathbf{R}^3)^4} (\psi(t, x, v)(v - w|n_{x,y})_+ - \psi(x, v'(v, w, n_{y,x})) (v - w|n_{y,x})_+) \\ &\quad \times f(t, x, v)f(t, y, w)b(|x - y|) dx dv dy dw \end{aligned}$$

where the change of variables  $(v, w) \mapsto (v'(v, w, n_{y,x}), w'(v, w, n_{y,x}))$  leads to the last equality, upon observing that

$$(v - w|n_{y,x}) = (v'(v, w, n_{y,x}) - w'(v, w, n_{y,x})|n_{x,y}).$$

Since

$$\psi(t, x, v)(v - w|n_{x,y})_+ = \psi(t, x, v)(v - w|n_{x,y}) + \psi(t, x, v)(v - w|n_{y,x})_+,$$

the term  $\Gamma_{12}$  is recast as

$$\begin{aligned} \Gamma_{12} &= \frac{1}{2} \int_{(\mathbf{R}^3)^4} \psi(x, v)b(|x - y|)(v - w|n_{x,y})f(t, x, v)f(t, y, w) dx dv dy dw \\ &\quad + \frac{1}{2} \int_{(\mathbf{R}^3)^4} (\psi(x, v) - \psi(x, v'(v, w, n_{y,x})))(v - w|n_{y,x})_+ \\ &\quad \times f(t, x, v)f(t, y, w)b(|x - y|) dx dv dy dw \\ &= \Gamma_{121} + \Gamma_{122}. \end{aligned}$$

Step 4. The term  $\Gamma_{121}$  is handled exactly as the potential term in the global entropy inequality: using the identity (30) leads to

$$\begin{aligned}
\Gamma_{121} &= -\frac{1}{2} \int_{(\mathbf{R}^3)^4} \psi(x, v) f(t, x, v) f(t, y, w) (v \cdot \nabla_x + w \cdot \nabla_y) \beta(|x - y|) dx dv dy dw \\
&= \frac{1}{2} \int_{(\mathbf{R}^3)^4} (v \cdot \nabla_x + w \cdot \nabla_y) [\psi(x, v) f(t, x, v) f(t, y, w)] \beta(|x - y|) dx dv dy dw \\
&= \frac{1}{2} \int_{(\mathbf{R}^3)^4} v \cdot \nabla_x \psi(x, v) f(t, x, v) f(t, y, w) \beta(|x - y|) dx dv dy dw \\
&\quad + \frac{1}{2} \int_{(\mathbf{R}^3)^4} \psi(x, v) \text{St}[f, f](t, x, v) f(t, y, w) \beta(|x - y|) dx dv dy dw \\
&\quad - \frac{1}{2} \frac{d}{dt} \int_{(\mathbf{R}^3)^4} \psi(x, v) f(t, x, v) f(t, y, w) \beta(|x - y|) dx dv dy dw.
\end{aligned}$$

In the last equality above, we have used the equality

$$\begin{aligned}
(v \cdot \nabla_x + w \cdot \nabla_y) [\psi(x, v) f(t, x, v) f(t, y, w)] &= f(t, x, v) f(t, y, w) v \cdot \nabla_x \psi(x, v) \\
&\quad + \psi(x, v) f(t, y, w) (\text{St}[f, f](t, x, v) - \partial_t f(t, x, v)) \\
&\quad + \psi(x, v) f(t, x, v) (\text{St}[f, f](t, y, w) - \partial_t f(t, y, w)) \\
&= f(t, x, v) f(t, y, w) v \cdot \nabla_x \psi(x, v) \\
&\quad + \psi(x, v) (f(t, y, w) \text{St}[f, f](t, x, v) + f(t, x, v) \text{St}[f, f](t, y, w)) \\
&\quad - \psi(x, v) \partial_t (f(t, x, v) f(t, y, w)),
\end{aligned}$$

and the local conservation of mass (the first identity in Proposition 3.1)

$$\int_{\mathbf{R}^3} \text{St}[f, f](t, y, w) dw = 0$$

to dispose of the term  $\psi(x, v) f(t, x, v) \text{St}[f, f](t, y, w)$  in the integral.

Representing  $\text{St}[f, f](t, x, v)$  in terms of the Landau mass current  $\mathcal{J}_0[f] = \mathcal{J}_0[f, f]$  (see (21), and the definition of  $\mathcal{J}_0[f, g]$  in the statement of Theorem 3.5),

$$\begin{aligned}
\Gamma_{121} &= \frac{1}{2} \int_{(\mathbf{R}^3)^2} v \cdot \nabla_x \psi(x, v) f(t, x, v) \rho_f(t, \cdot) \star \beta(|\cdot|)(x) dx dv \\
&\quad - \frac{1}{2} \int_{(\mathbf{R}^3)^2} \nabla_v \psi(x, v) \cdot \mathcal{J}_0[f](t, x, v) \rho_f(t, \cdot) \star \beta(|\cdot|)(x) dx dv \\
&\quad - \frac{1}{2} \frac{d}{dt} \int_{(\mathbf{R}^3)^2} \psi(x, v) f(t, x, v) \rho_f(t, \cdot) \star \beta(|\cdot|)(x) dx dv.
\end{aligned}$$

Step 5. On the other hand

$$\begin{aligned}
\Gamma_{122} &= \frac{1}{2} \int_{(\mathbf{R}^3)^4} (\psi(x, v) - \psi(x, v'(v, w, n_{y,x}))) (v - w |n_{y,x})_+ \\
&\quad \times f(t, x, v) f(t, y, w) b(|x - y|) dx dv dy dw \\
&= -\frac{1}{2} \int_{(\mathbf{R}^3)^2} \psi(x, v) \text{St}[f, f](t, x, v) dx dv \\
&= \frac{1}{2} \int_{(\mathbf{R}^3)^2} \nabla_v \psi(x, v) \cdot \mathcal{J}_0[f](t, x, v) dx dv.
\end{aligned}$$

Next observe that

$$S_3 + S_{42} = \frac{1}{2}(\psi(x, v) - \psi(x, v'(v, w, n_{y,x}))) \ln(f(t, x, v)f(t, y, w)),$$

so that

$$\begin{aligned} \Gamma_{34} &:= \int_{(\mathbf{R}^3)^4} (S_3 + S_{42}) f(t, x, v) f(t, y, w) b(|x - y|) (v - w|n_{y,x})_+ dx dv dy dw \\ &= \frac{1}{2} \int_{(\mathbf{R}^3)^4} (\psi(x, v) - \psi(x, v'(v, w, n_{y,x}))) f(t, x, v) f(t, y, w) \ln(f(t, x, v)f(t, y, w)) \\ &\quad \times b(|x - y|) (v - w|n_{y,x})_+ dx dv dy dw \\ &= -\frac{1}{2} \int_{(\mathbf{R}^3)^2} \nabla_v \psi(x, v) \cdot (\mathcal{J}_0[f \ln f, f](t, x, v) + \mathcal{J}_0[f, f \ln f](t, x, v)) dx dv. \end{aligned}$$

Step 6. While the terms  $\Gamma_{12}$  and  $\Gamma_{34}$  only contribute divergences in  $v$  and time derivatives, the two terms considered below contribute divergences in  $x$ .

First consider

$$\begin{aligned} \Gamma_{41} &:= \int_{(\mathbf{R}^3)^4} S_{41} f(t, x, v) f(t, y, w) b(|x - y|) (v - w|n_{y,x})_+ dx dv dy dw \\ &= \frac{1}{2} \int_{(\mathbf{R}^3)^4} (\psi(y, w) - \psi(x, v)) f(t, x, v) f \ln f(t, y, w) b(|x - y|) (v - w|n_{y,x})_+ dx dv dy dw, \end{aligned}$$

and express

$$\psi(y, w) - \psi(x, v) = \int_0^{\pi/2} \frac{d}{d\theta} \psi(x_\theta, v_\theta) d\theta,$$

where

$$\begin{pmatrix} x_\theta \\ y_\theta \end{pmatrix} := \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_\theta \\ w_\theta \end{pmatrix} := \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}.$$

Since

$$\frac{d}{d\theta} \psi(x_\theta, v_\theta) = y_\theta \cdot \nabla_x \psi(x_\theta, v_\theta) + w_\theta \cdot \nabla_v \psi(x_\theta, v_\theta),$$

one has

$$\begin{aligned} \Gamma_{41} &= \frac{1}{2} \int_{(\mathbf{R}^3)^4} (y_\theta \cdot \nabla_x \psi(x_\theta, v_\theta) + w_\theta \cdot \nabla_v \psi(x_\theta, v_\theta)) f(t, x, v) f \ln f(t, y, w) \\ &\quad \times b(|x - y|) (v - w|n_{y,x})_+ dx dv dy dw \\ &= \frac{1}{2} \int_{(\mathbf{R}^3)^4} (y \cdot \nabla_x \psi + w \cdot \nabla_v \psi)(x, v) \int_0^{\pi/2} f(t, x_\theta, v_\theta) f \ln f(t, y_\theta, w_\theta) \\ &\quad \times b(|x_\theta - y_\theta|) (v_\theta - w_\theta |n_{y_\theta, x_\theta})_+ d\theta dx dv dy dw \end{aligned}$$

because

$$\det \frac{\partial(x_\theta, y_\theta)}{\partial(x, y)} = \det \frac{\partial(v_\theta, w_\theta)}{\partial(v, w)} = 1.$$

Step 7. Similarly, set

$$\begin{aligned} \Gamma_2 &:= \int_{(\mathbf{R}^3)^4} S_2 f(t, x, v) f(t, y, w) b(|x - y|) (v - w|n_{y,x})_+ dx dv dy dw \\ &= \frac{1}{2} \int_{(\mathbf{R}^3)^4} (\psi(y, w) - \psi(x, v)) f(t, x, v'(v, w, n_{y,x})) \ln f(t, y, w) \\ &\quad \times f(t, y, w'(v, w, n_{y,x})) b(|x - y|) (v - w|n_{y,x})_+ dx dv dy dw. \end{aligned}$$

By the same change of variables as in  $\Gamma_2$ , one finds that

$$\begin{aligned} \Gamma_2 = \frac{1}{2} \int_{(\mathbf{R}^3)^4} & (y \cdot \nabla_x \psi + w \cdot \nabla_v \psi)(x, v) \int_0^{\pi/2} f(t, x_{-\theta}, v'(v, w, n_{y,x})_{-\theta}) \\ & \times f(t, y_{-\theta}, w'(v, w, n_{y,x})_{-\theta}) \ln f(t, y_{-\theta}, w_{-\theta}) b(|x_{-\theta} - y_{-\theta}|) \\ & \times (v_{-\theta} - w_{-\theta} |n_{x_{-\theta}, y_{-\theta}})_+ d\theta dx dv dy dw. \end{aligned}$$

Conclusion. Gathering all these terms together, we arrive at the following identity:

$$\begin{aligned} & \int_{(\mathbf{R}^3)^2} \text{St}[f, f](t, x, v) \ln f(t, x, v) \psi(x, v) dx dv = -\Lambda[\psi](t) \\ & + \int_{(\mathbf{R}^3)^2} (\mathcal{K}[f](t, x, v) \cdot \nabla_x \psi(x, v) + \mathcal{L}[f](t, x, v) \cdot \nabla_v \psi(x, v)) dx dv \\ & \quad - \frac{1}{2} \frac{d}{dt} \int_{(\mathbf{R}^3)^2} \psi(x, v) f(t, x, v) \rho_f(t, \cdot) \star \beta(|\cdot|)(x) dx dv, \end{aligned}$$

where

$$\begin{aligned} \mathcal{L}[f](t, x, v) & := -\frac{1}{2} \mathcal{J}_0[f, f](t, x, v) (1 - \rho_f \star \beta(|\cdot|)(t, x)) \\ & \quad - \frac{1}{2} (\mathcal{J}_0[f \ln f, f](t, x, v) + \mathcal{J}_0[f, f \ln f](t, x, v)) \\ & + \frac{1}{2} \int_{(\mathbf{R}^3)^2} \int_0^{\pi/2} (f(t, x_{-\theta}, v'(v, w, n_{y,x})_{-\theta}) f(t, y_{-\theta}, w'(v, w, n_{y,x})_{-\theta}) \\ & \quad - f(t, x_{-\theta}, v_{-\theta}) f(t, y_{-\theta}, w_{-\theta})) \ln f(t, y_{-\theta}, w_{-\theta}) b(|x_{-\theta} - y_{-\theta}|) \\ & \quad \times (v_{-\theta} - w_{-\theta} |n_{x_{-\theta}, y_{-\theta}})_+ w d\theta dy dw, \end{aligned}$$

where the notation  $\rho_f(t, \cdot) \star \beta(|\cdot|)$  has been defined in (26), while

$$\begin{aligned} \mathcal{K}[f](t, x, v) & := \frac{1}{2} \int_{(\mathbf{R}^3)^2} \int_0^{\pi/2} (f(t, x_{-\theta}, v'(v, w, n_{y,x})_{-\theta}) f(t, y_{-\theta}, w'(v, w, n_{y,x})_{-\theta}) \\ & \quad - f(t, x_{-\theta}, v_{-\theta}) f(t, y_{-\theta}, w_{-\theta})) \ln f(t, y_{-\theta}, w_{-\theta}) b(|x_{-\theta} - y_{-\theta}|) \\ & \quad \times (v_{-\theta} - w_{-\theta} |n_{x_{-\theta}, y_{-\theta}})_+ y d\theta dy dw. \end{aligned}$$

## 7. CONCLUSION

Summarizing, we have seen that delocalized collision integrals in the style of the Enskog, Povzner or soft sphere integrals can be expressed in terms of momentum and energy currents which have nontrivial components in the space of positions. For that reason, such delocalized collision integrals contribute to the local conservation laws of momentum and energy which appear in the Euler system of gas dynamics in the case of dense gases. This effect has been analyzed at the level of asymptotic analysis in [18] by means of the Hilbert expansion method, along with the resulting virial expansion of the pressure. One can also find in chapter 16 of [12] (see in particular §16.32) a variant of the discussions in the present paper limited to an asymptotic expansion in terms of the molecular radius. What is proposed in the present article is a non-asymptotic discussion which parallels that of §16.32 in [12], along with a discussion of a local variant of the Boltzmann H Theorem in Theorem 3.5. All our discussions are based on Landau's formulation of the Boltzmann collision integral in terms of a mass current in the space of molecular velocities, to be found in §41 of [20], and discussed in [29] in full detail.

## REFERENCES

- [1] L. Arkeryd: On the Enskog Equation with Large Initial Data. *SIAM J. Math. Anal.* **21** (1990), 631–646
- [2] L. Arkeryd, C. Cercignani: Global Existence in  $L^1$  for the Enskog Equation and Convergence of the Solutions to Solutions of the Boltzmann Equation. *J. Stat. Phys.* **59** (1990), 845–867
- [3] C. Bardos, F. Golse: Différents aspects de la notion d’entropie au niveau de l’équation de Boltzmann et de Navier-Stokes. *C. R. Acad. Sci. Paris Sér. I* **299** (1984), 225–228
- [4] H. van Beijeren: Equilibrium distribution of hard-sphere systems and revised Enskog theory. *Phys. Rev. Lett.* **51** (1983), 1503–1505
- [5] H. van Beijeren, M.H. Ernst: The modified Enskog equation. *Physica* **68**(1973), 437–456
- [6] H. van Beijeren, M.H. Ernst: The modified Enskog equation for mixtures. *Physica* **70** (1973), 225–242
- [7] N. Bellomo, M. Lachowicz: On the Asymptotic Equivalence between the Enskog and the Boltzmann Equations. *J. Stat. Phys.* **51** (1988), 233–247
- [8] N. Bellomo, M. Lachowicz: On the Asymptotic Theory of the Boltzmann and Enskog Equations: A Rigorous H-Theorem for the Enskog Equation. In “Mathematical aspects of fluid and plasma dynamics”, G. Toscani, V. Boffi, S. Rionero eds., *Lect. Notes Math.* 1460, 15–30, Springer-Verlag, Berlin, Heidelberg 1991
- [9] R.E. Caflisch: The fluid dynamic limit of the nonlinear Boltzmann equation. *Comm. Pure Appl. Math.* **33** (1980), 651–666
- [10] C. Cercignani: The Grad Limit for a System of Soft Spheres. *Comm. Pure Appl. Math.* **36** (1983), 479–494
- [11] C. Cercignani, R. Illner, M. Pulvirenti: “The Mathematical Theory of Dilute Gases”. Springer-Verlag, New York Inc. 1994
- [12] S. Chapman, T.G. Cowling: “The Mathematical Theory of Non-Uniform Gases”. 2nd ed. Cambridge Univ. Press. 1952
- [13] C. Dafermos: “Hyperbolic Conservation Laws in Continuum Physics”. 3rd ed. Springer-Verlag, Berlin, Heidelberg 2010
- [14] C. De Lellis, L. Szekelyhidi: On Admissibility Criteria for Weak Solutions of the Euler Equations. *Arch. Rational Mech. Anal.* **195** (2010), 225–260
- [15] M. Esteban, B. Perthame: On the modified Enskog equation for elastic and inelastic collisions. Models with spin. *Ann. Inst. H. Poincaré sect. C Anal. non lin.* **8** (1991), 289–308
- [16] K.O. Friedrichs, P.D. Lax: Systems of Conservation Equations with a Convex Extension. *Proc. Nat. Acad. Sci. USA* **68** (1971), 1686–1688
- [17] S.K. Godunov: An interesting class of quasi-linear systems. [Russian]. *Dokl. Akad. Nauk SSSR* **139** (1961), 521–523
- [18] M. Lachowicz: On the Hydrodynamic Limit of the Enskog Equation. *Publ. RIMS Kyoto Univ.* **34** (1998), 191–210
- [19] L.D. Landau, E.M. Lifshitz: “Statistical Physics I”. *Course of Theoretical Physics Vol. 5.* 3rd ed. Pergamon Press Ltd 1980
- [20] L.D. Landau, E.M. Lifshitz: “Physical Kinetics”. *Course of Theoretical Physics Vol. 10.* Pergamon Press Ltd 1981
- [21] O.E. Lanford III: Time Evolution of Large Classical Systems. *Dynamical systems, theory and applications (Rencontres, Battelle Res. Inst., Seattle, Wash., 1974).* *Lecture Notes in Phys.*, Vol. 38. Springer, Berlin, pp. 1–111
- [22] P. Maynar, M. I. Garcia de Soria, J. J. Brey: The Enskog equation for confined elastic hard spheres *J. Stat. Phys.* **170** (2018), 999–1018
- [23] J. Polewczak: Global Existence in  $L^1$  for the Modified Nonlinear Enskog Equation in  $\mathbb{R}^3$ . *J. Stat. Phys.* **56** (1989), 159–173
- [24] A. Ya. Povzner: The Boltzmann Equation in the Kinetic Theory of Gases. [Russian] *Mat. Sb., N. Ser.* **100** (1962), 65–86. *Engl. Transl.: Amer. Math. Soc., Transl., Ser. 2*, **47** (1965), 193–216
- [25] P. Resibois: H-Theorem for the (Modified) Nonlinear Enskog Equation. *J. Stat. Phys.*, **19** (1978), 593–609
- [26] Y. Sone: “Molecular Gas Dynamics. Theory, Tehcniques and Applications”. Birkhäuser, Boston MA, 2007



- [27] S. Takata: On the thermal relaxation of a dense gas described by the modified Enskog equation in a closed system in contact with a heat bath. *Kinetic and Related Models* **17** (2024), 331-346
- [28] G. Toscani, N. Bellomo: The Enskog-Boltzmann equation in the whole space  $\mathbf{R}^3$ : some global existence, uniqueness and stability results. *Computers and Math. with Applications* (13) (1987), 851–859
- [29] C. Villani: Conservative Form of Boltzmann’s Collision Operator: Landau Revisited. *Math. Modelling Numer. Anal. M2AN* **33** (1999), 209–227

UNIVERSITÉ GRENOBLE ALPES, LJK, BÂT. IMAG, 150 PLACE DU TORRENT, 38400 SAINT MARTIN D’HÈRES, FRANCE

*Email address:* frederique.charles@univ-grenoble-alpes.fr

SORBONNE UNIVERSITÉ, LJLL, BOÎTE COURRIER 187, 75252 PARIS CEDEX 05 FRANCE

*Email address:* zhe.chen@sorbonne-universite.fr

ÉCOLE POLYTECHNIQUE & IP PARIS, CMLS, 91128 PALAISEAU CEDEX, FRANCE

*Email address:* francois.golse@polytechnique.edu