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Pour l'obtention du grade de
DOCTEUR DE L'ÉCOLE POLYTECHNIQUE

Specialité: Mathématiques Appliquées

Présenté par

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Path dependent partial differential equation: theory and applications

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Résumé

Dans les travaux précédents, Ekren, Keller, Touzi & Zhang [35] et Ekren, Touzi & Zhang [37, 38], les auteurs ont introduit une notion de solutions de viscosité des EDPs dépendantes de la trajectoire (EDP-P) et ils ont montré des résultats d'unicité et d'existence par un argument appelé 'trajectoire gelée'. Les solutions de viscosité des EDP-Ps généralisent les solutions de viscosité des EDPs, en particulier, elles peuvent être utilisées pour caractériser les fonctions valeur des problèmes de contrôle stochastique non-markovien. Dans cette thèse, nous présentons le développement récent de la nouvelle théorie. Au cas des EDP-Ps semi-linéaires, nous améliorons l'argument pour le résultat de la comparaison et nous proposons une méthode de Perron pour prouver l'existence de solution de viscosité. En outre, comme dans le travail de Barles et Souganidis [4] dans le contexte d'EDP, nous montrons qu'une famille de schémas numériques satisfaisant les conditions de monotonie fournit les solutions numériques convergeant vers les solutions de viscosité des EDP-P's. De plus, nous essayons de développer la notion des EDP-Ps elliptiques et nous arrivons à montrer les résultats d'unicité et d'existence en suivant les arguments dans [38]. Cette thèse contient aussi certaines applications intéressantes des EDP-Ps. L'une des application concerne les grandes déviations des diffusions non-markoviennes. Comme Fleming a utilisé la stabilité des solutions de viscosité des EDPs pour montrer le principe de grandes déviations dans le cas markovien (voir [51]), nous utilisons les équations différentielles stochastiques rétrogrades et les EDP-Ps pour généraliser son résultat au cas non-markovien. En plus, on applique ce résultat de grandes déviations à l'étude du comportement asymptotique de la surface de volatilité implicite en mathématiques financières. Enfin, nous présentons un algorithme dual pour des problèmes de contrôle stochastique. Lorsque les simulations de Monte Carlo des problèmes de contrôle stochastique fournissent des estimations biaisées inférieurement, l'algorithme dual donne des bornes supérieures des fonctions valeur. L'idée de 'trajectoire gelée' est utilisée pour donner des représentations duales des problèmes de contrôle stochastique non-markovien.

Abstract

In the previous works, Ekren, Keller, Touzi & Zhang [35] and Ekren, Touzi & Zhang [37, 38], the new notion of viscosity solutions to path dependent PDEs is introduced, and a wellposedness theory is proved by a ‘path-frozen’ argument. This new notion generalizes that of viscosity solutions to PDEs developed intensively in the years of 80’s and 90’s, and can be used to characterize the value function of non-Markovian stochastic control problem. In this thesis, we report the recent development of the new theory. We improve the argument for the comparison result, and provide a PDE-style Perron’s method for proving the existence of viscosity solutions to semi-linear path dependent PDEs. As in the seminar work of Barles and Souganidis [4] in the context of PDEs, we show that a family of numerical schemes satisfying the so-called *monotonicity condition* provides numerical solutions converging to viscosity solutions of fully nonlinear path dependent PDEs. Further, we develop a notion of elliptic path dependent PDEs, and provide a wellposedness theory by following the lines of arguments in [38]. This thesis also includes some interesting applications of path dependent PDEs. One of them is on the large deviations of non-Markovian diffusion. As Fleming used the stability of viscosity solutions of PDEs to establish the large deviation principle in Markovian case (see [51]), we use the theory of backward stochastic differential equations and that of path dependent PDEs to generalize his result for non-Markovian diffusions. Moreover, the large deviation result is applied to investigate the short maturity asymptotics of the implied volatility surface in financial mathematics. Finally, a study of dual algorithm for stochastic control problems is presented. As Monte-Carlo simulations for the stochastic control problems provide low-biased estimate, a dual algorithm offer upper bounds of the true values. The idea of ‘path-frozen’ is exploited to give a dual representation of non-Markovian stochastic control problems.

Chapitre I

Introduction

1 Motivation

1.1 Des modèles markoviens aux modèles non-markoviens

L'équation d'évolution est l'un des outils les plus utilisés dans la modélisation mathématique. Soit $d \in \mathbb{N}$ et $u : (t, x) \in [0, T] \times \mathbb{R}^d \mapsto u(t, x)$. Considérons une équation d'évolution du seconde ordre de la forme :

$$\partial_t u(t, x) + G(t, x, u(t, x), Du(t, x), D^2u(t, x)) = 0, \quad (\text{I.1.1})$$

où G est une fonction génératrice, et D, D^2 sont les opérateurs différentiels du premier et du deuxième ordre en la variable x , respectivement. En particulier, si $G : (t, x, y, z, \gamma) \mapsto G(t, x, y, z)$ est croissante en γ , nous disons que l'équation (I.1.1) est une équation parabolique. Les équations paraboliques sont liées aux processus stochastiques et aux problèmes de contrôle stochastique.

Exemple I.1.1. *Un exemple simple peut être la relation de Feynman-Kac. Soit W un mouvement brownien de dimension d , $h : \mathbb{R}^d \rightarrow \mathbb{R}$ une fonction Borel-mesurable et bornée, définissons*

$$v(t, x) := \mathbb{E}[h(W_T) | W_t = x]. \quad (\text{I.1.2})$$

En supposant $v \in C^{1,2}$ (en effet, ceci peut être facilement vérifié), nous pouvons appliquer la formule d'Itô et conclure que v est une solution classique de l'équation de la chaleur avec la condition terminale h , i.e.

$$\partial_t v + \frac{1}{2} D^2 v = 0, \quad v(T, x) = h(x). \quad (\text{I.1.3})$$

Inversement, soit u une solution de (I.1.3) vérifiant une condition de croissance adéquate à l'infini. Le résultat classique de régularité montre que $u \in C^{1,2}$. Encore une fois, par la formule d'Itô, nous observons que $u(t, W_t)$ est une martingale, donc $u = v$.

L'équation de Hamilton-Jacobi-Bellman dans le cadre du contrôle stochastique donne une généralisation non-linéaire de l'exemple précédent.

Exemple I.1.2. Soit K un sous-ensemble compact de \mathbb{R}^m ($m \in \mathbb{N}$), et \mathcal{K} l'ensemble de tous les processus adaptés à valeurs dans K . Considérons une diffusion markovienne contrôlée X^κ de la forme :

$$X_t^\kappa = X_0 + \int_0^t b(s, X_s^\kappa, \kappa_s) ds + \int_0^t \sigma(s, X_s^\kappa, \kappa_s) dW_s, \quad \kappa \in \mathcal{K}. \quad (\text{I.1.4})$$

Nous considérons le problème d'optimisation :

$$u_0 = \sup_{\kappa \in \mathcal{K}} \mathbb{E} \left[\int_0^T e^{\int_0^t c(s, X_s^\kappa, \kappa_s) ds} f(t, X_t^\kappa, \kappa_t) dt + e^{\int_0^T c(s, X_s^\kappa, \kappa_s) ds} h(X_T^\kappa) \right]. \quad (\text{I.1.5})$$

Afin de résoudre ce problème, nous introduisons la version dynamique du problème d'optimisation :

$$u(t, x) = \sup_{\kappa \in \mathcal{K}} \mathbb{E} \left[\int_0^T e^{\int_0^t c(s, X_s^\kappa, \kappa_s) ds} f(t, X_t^\kappa, \kappa_t) dt + e^{\int_0^T c(s, X_s^\kappa, \kappa_s) ds} h(X_T^\kappa) \mid X_t^\kappa = x \right]$$

Le principe de programmation dynamique dit essentiellement que pour tout les temps d'arrêt $\tau \in [0, T]$, nous avons

$$u(t, X_t^\kappa) = \sup_{\kappa \in \mathcal{K}} \mathbb{E} \left[\int_t^\tau e^{\int_t^s c(r, X_r^\kappa, \kappa_r) dr} f(s, X_s^\kappa, \kappa_s) ds + e^{\int_t^\tau c(s, X_s^\kappa, \kappa_s) ds} u(\tau, X_\tau^\kappa) \right].$$

Supposons que la fonction valeur $u \in C^{1,2}$. En utilisant la formule d'Itô, nous pouvons vérifier que u est une solution classique de l'équation de Hamilton-Jacobi-Bellman :

$$\partial_t u + \sup_{k \in K} \left\{ b(t, x, k) \cdot Du + \frac{1}{2} \text{Tr} \left((\sigma \sigma^T)(t, x, k) D^2 u \right) + c(t, x, k) u + f(t, x, k) \right\} = 0, \quad u_T = h.$$

Il y a de nombreux avantages de caractériser les fonctions valeur comme solutions de l'équation de HJB. Par exemple, dans le problème initial (I.1.5), il s'agit d'une optimisation dans un espace de dimension infinie, de plus avec une contrainte d'adaptabilité. Par contre, dans l'équation de HJB, il s'agit de la maximisation sur un ensemble de \mathbb{R}^m . Cela offre de la simplicité en particulier pour des approches numériques.

Il est à noter que dans les deux exemples précédents, nous considérons des modèles markoviens. Mon travail de recherche dans le cadre de cette thèse soulève la question suivante : que se passe-t-il dans des problèmes de contrôle stochastiques non-markoviens (en d'autres mots, ceux dépendant de trajectoire) ? Ils arrivent en effet très souvent dans des applications. L'exemple suivant fournit certaines situations où la dépendance de trajectoire peut être réduite au cas markovien par l'augmentation de l'espace d'état.

Exemple I.1.3. *Notons à nouveau par W un mouvement brownien de dimension d . Soit \bar{W} , W^* la moyenne courante et le maximum courant, respectivement, i.e.*

$$\bar{W}_t = \frac{1}{t} \int_0^t W_s ds, \quad W_t^* = \max_{s \leq t} W_s.$$

Notons que \bar{W} et W^ ne sont pas markoviens. Toutefois, en prenant en compte le mouvement brownien lui-même, nous constatons que les processus vecteur (W, \bar{W}) et (W, W^*) sont markoviens.*

1. *Considérons $\bar{u}(t, x, y) = \mathbb{E}[h(\bar{W}_T) | W_t = x, \bar{W}_t = y]$. Ensuite, en supposant que \bar{u} est assez régulière, nous pouvons vérifier que \bar{u} est une solution classique de l'équation suivante :*

$$\partial_t \bar{u} + \frac{1}{2} D_x^2 \bar{u} + \frac{x-y}{t} D_y \bar{u} = 0, \quad \bar{u}(T, x, y) = h(y).$$

2. *Considérons $u^*(t, x, y) = \mathbb{E}[h(W_T^*) | W_t = x, W_t^* = y]$ pour tout $x \leq y$. Cette espérance conditionnelle peut être formellement liée à l'EDP :*

$$\partial_t u + \frac{1}{2} D_x^2 u = 0 \text{ pour } x \leq y, \quad D_y u(t, x, x) = 0 \text{ pour tout } x \in \mathbb{R}^d, \quad u(T, x, y) = h(y).$$

Dans le deuxième exemple, même si nous pouvons encore écrire l'EDP correspondante, l'équation a une condition au bord plus complexe (précisément, ici c'est une condition de Neumann). Des conditions au bord complexes peuvent causer des difficultés dans l'analyse des EDP's, sans oublier qu'il n'y a pas de méthode unifiée pour traiter toutes sortes de conditions au bord. En outre, il y a des exemples de modèles non-markoviens dans lesquels l'augmentation de l'espace ne fonctionne pas. Ceci est illustré par l'exemple suivant.

Exemple I.1.4. *1. Soit μ une mesure σ -finie et singulière par rapport à la mesure de Lebesgue. Définissons $U_t := \int_0^t W_s d\mu_s$. Le processus vecteur (W, U) est markovien, et on peut encore définir $\tilde{u}(t, x, y) := \mathbb{E}[h(U_T) | W_t = x, U_t = y]$. Cependant, nous ne pouvons plus trouver l'EDP correspondante.*

2. *Un autre exemple est le contrôle avec retard. Au lieu de (I.1.4), nous considérons une diffusion contrôlée de la forme :*

$$X_t^\kappa = X_0 + \int_0^t b(s, X_{s-\delta}^\kappa, \kappa_s) ds + \int_0^t \sigma(s, X_{s-\delta}^\kappa, \kappa_s) dW_s, \quad \kappa \in \mathcal{K},$$

où $\delta > 0$ est un paramètre représentant le retard. Notons que le processus X n'est plus markovien. Nous considérons le même problème d'optimisation que dans (I.1.5). Dans ce cas, il est impossible de construire un processus markovien par une augmentation de dimension finie de l'espace d'état. Si on prend une augmentation de dimension infinie de l'espace d'état, le calcul différentiel devient compliqué.

Les exemples ci-dessus justifient la nécessité de développer une généralisation non-markovienne de l'équation d'évolution.

1.2 De la généralisation de la formule d'Itô aux EDP's dépendantes de trajectoire

Comme nous avons vu dans la section précédente, le lien entre les fonctions valeur et les solutions d'équations d'évolution est souvent établi à l'aide de la formule d'Itô. Ainsi, une compréhension de la formule d'Itô pour une fonction de la trajectoire pourrait être un bon point de départ de la théorie des EDP's dépendantes de trajectoire (EDP-P).

Dans l'article original de Dupire [34], l'auteur a étudié les dérivées horizontales et verticales pour des fonctions de trajectoire. Soit \mathcal{D} l'ensemble de toutes les trajectoires càdlàg sur $[0, T]$ à valeurs dans \mathbb{R}^d et partant de l'origine; soit $u : \mathbb{R}^+ \times \mathcal{D} \rightarrow \mathbb{R}$ une fonction non-anticipative, i.e. $u(t, \omega) = u(t, \omega_t \wedge \cdot)$ pour $\omega \in \mathcal{D}$. Soit

$$\begin{aligned} \partial_t u(t, \omega) &:= \lim_{\epsilon \rightarrow 0} \frac{u(t+\epsilon, \omega_{t+\epsilon} \wedge \cdot) - u(t, \omega)}{\epsilon}, \\ \partial_{\omega^i} u(t, \omega) &:= \lim_{\epsilon \rightarrow 0} \frac{u(t, \omega + \epsilon \mathbf{1}_{[t, T]} e_i)}{\epsilon}, \quad \text{et} \quad \partial_\omega u(t, \omega) := \left(\partial_{\omega^i} u(t, \omega) \right)_{1 \leq i \leq d}, \end{aligned} \tag{I.1.6}$$

si ces limites existent. Notons que si la fonction u dépend de la trajectoire uniquement à travers de la valeur courante, i.e. il y a une fonction $\hat{u} : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}$ telle que $u(t, \omega) = \hat{u}(t, \omega_t)$, alors les dérivées définies ci-dessus se réduisent aux dérivées partielles habituelles définies sur l'espace réel. Nous disons que la fonction $u : \mathbb{R}^+ \times \mathcal{D} \rightarrow \mathbb{R}$ est dans la classe $C^{1,2}$, si u est continue, et toutes les dérivées $\partial_t u, \partial_\omega u, \partial_{\omega\omega}^2 u$ existent et sont continues. Dans [17, 34], les auteurs ont montré que, sous certaines conditions générales, une fonction $u \in C^{1,2}$ satisfait la formule d'Itô

généralisée :

$$u(t, \omega) - u_0 = \int_0^t \partial_t u(s, \omega) ds + \frac{1}{2} \int_0^t \partial_{\omega\omega}^2 u(s, \omega) d\langle \omega \rangle_s + \int_0^t \partial_\omega u(s, \omega) d\omega_s, \quad \mathbb{P}\text{-a.s.} \quad (\text{I.1.7})$$

pour toutes les mesures \mathbb{P} sous lesquelles le processus canonique dans \mathcal{D} est une semimartingale continue. Dans l'exemple suivant, nous allons voir que la formule d'Itô généralisée joue le rôle de pont entre les modèles probabilistes et les équations d'évolution *dépendantes de trajectoire*.

Exemple I.1.5. *Nous considérons un modèle-jouet similaire à celui de l'exemple I.1.1. Soit Ω l'ensemble de toutes les trajectoires continues à partir de l'origine. Considérons une fonction mesurable et bornée $h : \Omega \rightarrow \mathbb{R}$. Définissons la fonction de l'espérance conditionnelle :*

$$u(t, \omega) := \mathbb{E}[h(W_{T \wedge \cdot}) | \mathcal{F}_t](\omega),$$

où \mathcal{F}_t est la filtration engendrée par le mouvement brownien W . En supposant que $u \in C^{1,2}$ dans le sens de Dupire, nous pouvons appliquer la formule d'Itô généralisée (I.1.7). Comme $u(t, W)$ est une martingale, nous obtenons

$$\partial_t u + \frac{1}{2} \partial_{\omega\omega}^2 u = 0,$$

une equation de la chaleur dépendante de la trajectoire.

Cet exemple simple non seulement généralise le modèle dans l'exemple I.1.1, mais aussi couvre ceux dans l'exemple I.1.3, ainsi que le premier modèle dans l'exemple I.1.4. Donc, une théorie des EDP-P's peut unifier l'analyse des EDP's avec différents types de conditions au bord, et traiter les problèmes que les EDP's habituelles ne peuvent pas décrire.

1.3 Des solutions classiques aux solutions de viscosité

Dans les sections précédentes, nous supposons toujours que la régularité des fonctions valeur est suffisante pour appliquer la formule d'Itô et obtenir les EDP's correspondantes ou EDP-P's. Cependant, en réalité les solutions ne vérifient pas en général une telle régularité.

Prenons l'équation de la chaleur de l'exemple I.1.1 et l'équation de la chaleur dépendante de la trajectoire de l'exemple I.1.5. Comme nous avons mentionné dans l'exemple I.1.1, l'équation de la chaleur a toujours des solutions régulières. Qu'est-ce qui se passe dans le cas dépendant de trajectoire ?

Example I.1.6. *Considérons la fonction valeur*

$$u(t, \omega) := \mathbb{E}\left[W_{\frac{T}{2} \wedge t} \middle| \mathcal{F}_t\right](\omega) = \omega_{\frac{T}{2} \wedge t}.$$

Selon l'exemple I.1.5, la fonction u est un candidat naturel pour la solution de l'équation de la chaleur dépendante de la trajectoire avec la condition terminale $u(T, \omega) = \omega_{\frac{T}{2}}$. Selon le calcul de (I.1.6), la dérivée verticale peut être facilement calculée : $\partial_\omega u(t, \omega) = 1_{\{s \leq \frac{T}{2}\}}$, elle n'est pas continue. Par conséquent, la fonction u n'est pas de classe $C^{1,2}$ au sens de Dupire.

Si nous appelons les solutions de classe $C^{1,2}$ (au sens de Dupire) les solutions classiques, alors, en général, l'équation de la chaleur dépendante de la trajectoire n'a probablement pas de solution classique. Il est nécessaire de développer une théorie de solutions faibles.

1.3.1 Solutions de Sobolev des EDP-P's

Il vaut bien de se rappeler de la théorie des équations différentielles stochastiques rétrogrades (EDSR).

Une sous-classe intéressante des équations non-linéaires paraboliques (I.1.1) est la classe d'équations semi-linéaires de la forme :

$$\partial_t u + \frac{1}{2} D^2 u = f(t, x, u, Du), \quad u(T, \cdot) = h. \quad (\text{I.1.8})$$

En prenant la même fonction génératrice f et l'état terminal h , nous pouvons générer une EDSR :

$$dY_t = f(t, W_t, Y_t, Z_t) dt + Z_t dW_t, \quad Y_T = h(W_T). \quad (\text{I.1.9})$$

Une solution de l'EDSR ci-dessus est une paire de processus adaptés (Y, Z) dans l'espace L^2 , i.e.

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} Y_t^2\right] < \infty, \quad \mathbb{E}\left[\int_0^T Z_s^2 ds\right] < \infty.$$

Il y a un lien fort entre l'EDP semilinéaire (I.1.8) et l'EDSR (I.1.9). En supposant que (I.1.8) a une solution classique u , puis en utilisant la formule d'Itô, on peut vérifier que $Y_T = u(t, W_t)$, $Z_t = Du(t, W_t)$ est une solution de l'EDSR (I.1.9). En effet, comme il est souligné dans Barles & Lesigne [3], et aussi dans Bally & Matoussi [11], les solutions des EDSR's correspondent aux solutions de Sobolev dans l'analyse des EDP's classiques. Toutes les deux affai-

blissent l'exigence de régularité des solutions. Un des avantages des solutions de Sobolev est que l'on peut prouver le résultat de l'unicité et l'existence (existence et unicité) par un argument de point fixe, après avoir trouvé un espace de Sobolev approprié, et c'est ainsi que Pardoux et Peng ont prouvé le résultat de l'unicité et l'existence pour l'EDSR (I.1.9) dans [89].

Un progrès important introduit par l'EDSR est qu'elles peuvent naturellement être définies dans le cadre dépendant de trajectoire. Considérons une fonction génératrice $F : (t, \omega, y, z) \mapsto \mathbb{R}$ et un état terminal $h : \Omega \rightarrow \mathbb{R}$. Légèrement différent de (I.1.9), nous considérons une EDSR :

$$dY_t = F(t, W, Y_t, Z_t)dt + Z_t dW_t, \quad Y_T = h(W). \quad (\text{I.1.10})$$

La solution de (I.1.10) peut être considérée comme une solution de Sobolev de l'EDP-P semi-linéaire suivante :

$$\partial_t u + \frac{1}{2} \partial_{\omega\omega}^2 u = F(t, \omega, u, Du), \quad u(T, \cdot) = h. \quad (\text{I.1.11})$$

En effet, supposons que $Y_t(\omega) \in C^{1,2}$ dans le sens de Dupire, alors

$$dY_t = (\partial_t Y_t + \frac{1}{2} \partial_{\omega\omega}^2 Y_t)dt + \partial_\omega dW_t,$$

en identifiant les terms de dt et de dW_t , on obtient formellement l'EDP-P (I.1.11). Il est à noter qu'il y a des analogues complètement non-linéaires d'EDSR. Nous nous référons à Cheridito, Soner, Touzi et Victoir [16] ainsi qu'à Soner, Touzi et Zhang [107] pour la théorie de 2EDSR. Nous renvoyons le lecteur également à Hu, Ji et Peng [68] pour une généralisation similaire, appelée G-EDSR.

1.3.2 Solutions de viscosité des EDP's non-markoviennes

La théorie des solutions de viscosité offre une approche alternative de solution faible pour les EDP's. Au lieu d'exiger que la solution soit régulière et satisfasse l'équation, nous introduisons les fonctions de test. Notons $Q := [0, T] \times \mathbb{R}^d$. Nous considérons l'EDP habituelle :

$$-\partial_t v(t, x) - g(t, x, v(t, x), Dv(t, x), D^2v(t, x)) = 0, \quad t < T, \quad x \in \mathbb{R}^d. \quad (\text{I.1.12})$$

Soit $\text{USC}(Q)$ (resp. $\text{LSC}(Q)$) l'ensemble des fonctions semi-continues supérieurement (resp. inférieurement) définies sur $Q \subset \mathbb{R}^d$. Pour $(t, x) \in Q$, $u \in \text{USC}(Q)$ et $v \in \text{LSC}(Q)$, on note :

$$\underline{A}u(t, x) := \left\{ \varphi \in C^{1,2}(Q) : (\varphi - u)(t, x) = \min_Q(\varphi - u) \right\}, \quad (\text{I.1.13})$$

$$\overline{A}v(t, x) := \left\{ \varphi \in C^{1,2}(Q) : (\varphi - v)(t, x) = \max_Q(\varphi - v) \right\}. \quad (\text{I.1.14})$$

Definition I.1.7. (i) $u \in \text{USC}(Q)$ est sous-solution de viscosité de l'équation (I.1.12) si :

$$\left\{ -\partial_t \varphi - g(\cdot, u, D\varphi, D^2\varphi) \right\}(t, x) \leq 0 \quad \text{pour tous } (t, x) \in Q, \varphi \in \underline{A}u(t, x).$$

(ii) $v \in \text{LSC}(Q)$ est sur-solution de viscosité de l'équation (I.1.12) si :

$$\left\{ -\partial_t \varphi - g(\cdot, u, D\varphi, D^2\varphi) \right\}(t, x) \geq 0 \quad \text{pour tous } (t, x) \in Q, \varphi \in \overline{A}v(t, x).$$

(iii) Une solution de viscosité de (I.1.12) est sous-solution de viscosité et sur-solution de viscosité de l'EDP (I.1.12).

La première différence entre une solution de Sobolev et une solution de viscosité est que la solution de Sobolev est un objet dans un espace de Sobolev, tandis que la solution de viscosité est une fonction définie à chaque point. Une caractéristique étonnante de solutions de viscosité est que la définition ne concerne que certaines propriétés locales des fonctions. Par conséquent, il est généralement facile de vérifier qu'une fonction donnée est une solution de viscosité de l'équation correspondante. Ceci est particulièrement utile dans la théorie de contrôle optimal.

Cette thèse est consacrée à l'élaboration d'une théorie des solutions de viscosité des EDP-P's. La difficulté principale dans la preuve du résultat de l'unicité et l'existence des solutions de viscosité est le principe de comparaison, i.e. soit u une sous-solution de viscosité et v une sur-solution de viscosité, nous nous attendons à ce que

$$u(T, \cdot) \leq v(T, \cdot) \quad \Rightarrow \quad u \leq v \text{ partout.}$$

Dans la théorie classique des EDP's, la preuve du résultat de comparaison se base sur le fait que l'espace est localement compact. Cependant, l'espace de trajectoire n'est pas localement compact, et ceci provoque de vraies difficultés pour notre projet.

Il est à noter qu'il y a eu des tentatives d'introduire des solutions de viscosité des EDP-P's dans la littérature existante. Par exemple, Lukoyanov introduit une théorie des solutions de viscosité des EDP-P's du premier ordre dans [82]. Dans ce cas particulier, il réduit l'espace

de trajectoire, et il examine seulement les trajectoires absolument continues avec des densités uniformément bornées. Par conséquent, il considère de nouveau un espace d'état compact. Toutefois, pour des EDP-P's du second ordre, cette astuce ne fonctionne plus.

Pour surmonter cette difficulté, il est raisonnable de modifier un peu la définition des solutions de viscosité. Dans le travail de Ekren, Touzi et Zhang [37], les auteurs ont introduit une nouvelle définition des solutions de viscosité pour les EDP-P's, en agrandissant la famille de fonctions de test. Un constat simple mais crucial est que, en introduisant plus de fonctions de test, nous avons moins de sous/sur-solutions de viscosité, et donc le résultat de comparaison devrait être plus facile à prouver.

Dans la section suivante, nous allons introduire la définition des solutions de viscosité des EDP-P's.

2 Définition des solutions de viscosité des EDP-P's

2.1 Notations

Soit $\Omega := \{\omega \in C^0([0, T], \mathbb{R}^d) : \omega_0 = 0\}$ l'espace canonique des trajectoires continues partant de l'origine, B le processus canonique défini par $B_t(\omega) := \omega_t$, $t \in [0, T]$, et $\mathbb{F} := \{\mathcal{F}_t, t \in [0, T]\}$ la filtration correspondante. On note aussi $\Theta := [0, T] \times \Omega$. Comme dans Dupire [34], nous introduisons la pseudo-distance

$$d((t, \omega), (t', \omega')) := |t - t'| + \|\omega_{\wedge t} - \omega'_{\wedge t'}\|_\infty \quad \text{pour tout } t, t' \in [0, T], \omega, \omega' \in \Omega. \quad (\text{I.2.1})$$

Tous les processus $u : [0, T] \times \Omega \rightarrow \mathbb{R}$ continus par rapport à d , sont \mathbb{F} -progressivement mesurable, en particulier, on a $u(t, \omega) = u(t, (\omega_s)_{s \leq t})$. Soit \mathcal{T} l'ensemble de tous les \mathbb{F} -temps d'arrêt, $\mathcal{T}^+ \subset \mathcal{T}$ la collection de tous les temps d'arrêt strictement positifs, et $\mathcal{T}^t \subset \mathcal{T}$ le sous-ensemble de \mathbb{F} -temps d'arrêt plus grands que t .

Pour $\omega, \omega' \in \Omega$ et $t \in [0, T]$, nous définissons

$$(\omega \otimes_t \omega')_s := \omega_s \mathbf{1}_{\{s < t\}} + (\omega_t + \omega'_{s-t}) \mathbf{1}_{\{s \geq t\}}.$$

Soit $\xi : \Omega \rightarrow \mathbb{R}$ une variable aléatoire \mathcal{F}_T -mesurable. Pour $(t, \omega) \in \Theta$, on définit

$$\xi^{t, \omega}(\omega') := \xi(\omega \otimes_t \omega') \quad \text{pour tous } \omega' \in \Omega.$$

De toute évidence, $\xi^{t, \omega}$ est \mathcal{F}_{T-t} -mesurable, et donc \mathcal{F}_T -mesurable. De même, étant donné un

processus X défini sur Ω , on note :

$$X_s^{t,\omega}(\omega') := X_{t+s}(\omega \otimes_t \omega'), \text{ pour } s \in [0, T - t].$$

Clairement, si X est \mathbb{F} -adapté, alors $X^{t,\omega}$ a la même propriété.

Soit \mathcal{P} une famille de mesures de probabilité sur Ω . Nous introduisons les opérateurs d'espérance sous-linéaire et sur-linéaire associés à \mathcal{P} :

$$\bar{\mathcal{E}}^{\mathcal{P}} := \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \quad \text{and} \quad \underline{\mathcal{E}}^{\mathcal{P}} := \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}.$$

2.2 Dérivabilité

Dans la section 1.2, nous avons déjà discuté des dérivées de Dupire et de la formule d'Itô généralisée. En fait, dans la théorie actuelle des solutions de viscosité des EDP-P's, la régularité $C^{1,2}$ des fonctions de test est seulement nécessaire afin d'appliquer la formule d'Itô. Par conséquent, nous pouvons définir les processus lisses directement :

Definition I.2.1 (Processus lisses). *Soit \mathcal{P} un ensemble de mesures de probabilité sur Ω tel que B est \mathbb{P} -semimartingale pour tous $\mathbb{P} \in \mathcal{P}$. Nous disons que $u \in C_{\mathcal{P}}^{1,2}(\Theta)$, si $u \in C^0(\Theta)$ et il existe des processus $\alpha, Z, \Gamma \in C^0(\Theta)$ à valeur dans \mathbb{R}, \mathbb{R}^d et \mathbb{S}_d , respectivement, tels que :*

$$du_t = \alpha_t dt + \frac{1}{2} \Gamma_t : d\langle B \rangle_t + Z_t dB_t, \quad \mathbb{P} - p.s. \text{ pour tous } \mathbb{P} \in \mathcal{P}.$$

Les processus α, Z et Γ sont appelés la dérivée en temps, le gradient spatial et la Hessienne spatiale, respectivement, et on note $\partial_t u := \alpha, \partial_{\omega} u_t := Z_t, \partial_{\omega\omega}^2 u_t := \Gamma_t$.

Nous observons que tous les processus $C^{1,2}$ au sens de Dupire sont dans $C_{\mathcal{P}}^{1,2}(\Theta)$. En particulier, notre notion de processus lisse est plus faible que celle de Dupire. Nous notons également que, lorsque \mathcal{P} est assez riche, nos dérivées de trajectoire sont uniques.

Remark I.2.2. La définition précédente ne nécessite pas que $\partial_{\omega\omega}^2 u_t$ soit la dérivée (en quelque sorte) de $\partial_{\omega} u_t$. Ceci est très bien illustré par l'exemple suivant. Soit $d = 2, u_t := \int_0^t B_s^1 dB_s^2$ qui est défini pour chaque trajectoire selon Karandikar [70].

- Evidemment $\partial_t u = 0$. Comme $du_t = B_t^1 dB_t^2$, par rapport aux mesures semimartingale, on en déduit également que $\partial_{\omega} u_t = (0, B_t^1)^{\top}$ et $\partial_{\omega\omega}^2 u_t = 0$. Donc $u \in C_{\mathcal{P}}^{1,2}(\Theta)$ pour tout sous-ensemble \mathcal{P} de la collection de toutes les mesures semimartingales pour B .
- Soit $\partial_{\omega}^{\mathbb{D}} u_t$ et $\partial_{\omega\omega}^{\mathbb{D}^2} u_t$ les dérivées verticales du premier et du deuxième ordre au sens de Dupire.

Le calcul direct montre que $\partial_\omega^D u_t = (0, B_t^1)^T = \partial_\omega u_t$. Cependant,

$$\partial_{\omega\omega}^{D^2} u_t = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

qui est non symétrique !

- Toutefois, nous devons souligner que dans cet exemple u n'appartient pas à $C^0(\Theta)$.

2.3 Définition des solutions de viscosité

Nous étudions l'EDP-P :

$$-\partial_t u(t, \omega) - G(t, \omega, u(t, \omega), \partial_\omega u(t, \omega), \partial_{\omega\omega}^2 u(t, \omega)) = 0, \quad t < T, \quad \omega \in \Omega, \quad (\text{I.2.2})$$

avec la condition au bord $u(T, \omega) = \xi(\omega)$. Ici, $G : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \rightarrow \mathbb{R}$ est lipschitzienne en les variables (y, z, γ) , et satisfait la condition d'ellipticité :

$$\gamma \in \mathbb{S}_d \mapsto G(t, \omega, y, z, \gamma) \quad \text{est non-décroissante.} \quad (\text{I.2.3})$$

Le processus inconnu $u(t, \omega)$ doit être \mathbb{F} -progressivement mesurable.

2.3.1 Définition via des fonctions de test

Nous introduisons les ensembles de processus de test :

$$\begin{aligned} \mathcal{A}^{\mathcal{P}} u_t(\omega) &:= \left\{ \varphi \in C_p^{1,2}(\Theta) : \exists \epsilon > 0, (\varphi - u^{t,\omega})_0 = \min_{\tau \in \mathcal{T}_{H_\epsilon}} \underline{\mathcal{E}}^{\mathcal{P}}[(\varphi - u^{t,\omega})_\tau] \right\}, \\ \overline{\mathcal{A}}^{\mathcal{P}} v_t(\omega) &:= \left\{ \varphi \in C_p^{1,2}(\Theta) : \exists \epsilon > 0, (\varphi - v^{t,\omega})_0 = \max_{\tau \in \mathcal{T}_{H_\epsilon}} \overline{\mathcal{E}}^{\mathcal{P}}[(\varphi - v^{t,\omega})_\tau] \right\}, \end{aligned} \quad (\text{I.2.4})$$

où \mathcal{T}_{H_ϵ} désigne tous les temps d'arrêt plus petits que

$$H_\epsilon := \epsilon \wedge \inf\{t \geq 0 : \|B_t\| \geq \epsilon\}. \quad (\text{I.2.5})$$

Dans la suite, nous allons appeler H_ϵ le temps d'arrêt de localisation (ou la localisation) du processus de test correspondant φ .

Definition I.2.3 (Solution de viscosité de l'EDP-P). *Soient $u, v \in \mathbb{L}^0(\mathbb{F})$.*

(i) u est une \mathcal{P} -sous-solution de viscosité de (I.2.2) si :

$$\left\{ -\partial_t \varphi - G(\cdot, u, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi) \right\}(t, \omega) \leq 0 \quad \text{pour tout } (t, \omega) \in \Theta, \varphi \in \underline{\mathcal{A}}^{\mathcal{P}} u_t(\omega).$$

(ii) v est une \mathcal{P} -sur-solution de viscosité de (I.2.2) si :

$$\left\{ -\partial_t \varphi - G(\cdot, v, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi) \right\}(t, \omega) \geq 0 \quad \text{pour tout } (t, \omega) \in \Theta, \varphi \in \overline{\mathcal{A}}^{\mathcal{P}} v_t(\omega).$$

(iii) Une \mathcal{P} -solution de viscosité de (I.2.2) est à la fois une \mathcal{P} -sous-solution de viscosité et une \mathcal{P} -sur-solution de viscosité.

Remark I.2.4. Dans le cas markovien, on peut aussi bien utiliser la dernière définition comme une alternative à la notion classique de la solution de viscosité. Par rapport à la notion classique revue dans la section 1.3.2, nous voyons que tout $\phi \in \underline{A}u(t, x)$ induit un processus $\varphi(t, \omega) := \phi(t, \omega_t)$ qui se trouve évidemment dans $\underline{\mathcal{A}}^{\mathcal{P}} u_t(\omega)$. Cependant, même dans le cas markovien $u_t(\omega) = u(t, \omega_t)$, un processus de test dans $\underline{\mathcal{A}}^{\mathcal{P}} u_t(\omega)$ n'induit pas nécessairement une fonction de test dans $\underline{A}u(t, \omega_t)$. Ainsi, notre notion de solution de viscosité implique plus de fonctions de test que la notion classique. Une sous-solution/sur-solution de viscosité au sens de la définition I.2.3 est limitée par une famille plus riche de fonctions de test. Par conséquent :

- selon notre définition, nous pouvons espérer profiter de la famille plus riche de fonctions de test afin d'obtenir une preuve d'unicité plus facile,
- selon notre définition, le problème d'existence est plus restreint que dans la théorie classique des solutions de viscosité.

Remark I.2.5. A cause de la localisation, la propriété de viscosité introduite dans la définition I.2.3 est une propriété locale. En effet, afin de vérifier la propriété de viscosité de u à (t, ω) , il suffit de connaître la valeur de $u^{t, \omega}$ sur $[0, H_\varepsilon]$ pour un $\varepsilon > 0$ arbitrairement petit. En particulier, puisque u et φ sont localement bornées, il n'y a pas de problème d'intégrabilité dans la définition de l'ensemble des fonctions de test $\underline{\mathcal{A}}^{\mathcal{P}}$ et $\overline{\mathcal{A}}^{\mathcal{P}}$.

Remark I.2.6. Dans les premiers papiers sur les EDP-P's, les localisations H_ε peuvent être temps d'arrêt arbitraires au cas des EDP-P's semi-linéaires, par contre, on utilise seulement les localisations comme dans (I.2.5) au cas des EDP-P's complètement non-linéaires. En effet, on a montré dans le papier plus récent [98] que c'est équivalent de considérer seulement les localisations comme dans (I.2.5) au cas des EDP-P's semi-linéaires.

On a aussi montré dans [100] que c'est équivalent d'utiliser les localisations constantes. Ça peut simplifier les démonstrations dans certains contextes.

2.3.2 Définition de Semijets

Comme dans la notion classique des solutions de viscosité dans l'espace de dimension finie, nous allons montrer que nous pouvons réduire notre Définition 1.2.3 aux paraboloides :

$$\phi_s^{q,p,\gamma}(\omega) := qs + p \cdot \omega_s + \frac{1}{2}\gamma : \omega_s \omega_s^T, \quad s \in [0, T - t], \quad \omega \in \Omega,$$

pour $(q, p, \gamma) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d$. Nous introduisons alors le sous-jet et le sur-jet correspondant :

$$\begin{aligned} \underline{\mathcal{J}}^p u_t(\omega) &:= \left\{ (q, p, \gamma) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d : \phi^{q,p,\gamma} \in \underline{\mathcal{A}}^p u_t(\omega) \right\}, \\ \overline{\mathcal{J}}^p v_t(\omega) &:= \left\{ (q, p, \gamma) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d : \phi^{q,p,\gamma} \in \overline{\mathcal{A}}^p v_t(\omega) \right\}. \end{aligned} \quad (\text{I.2.6})$$

La proposition suivante peut être facilement prouvée, et elle montre que nous pouvons donner une définition équivalente des solutions de viscosité des EDP-P's via les jets.

Proposition I.2.7. *Un processus $u \in C^0(\Theta)$ est une \mathcal{P} -sous-solution de viscosité de (1.2.2) si et seulement si :*

$$-q - G(t, \omega, u_t(\omega), p, \gamma) \leq 0 \quad \text{pour tous } (t, \omega) \in \Theta, (q, p, \gamma) \in \underline{\mathcal{J}}^p u_t(\omega). \quad (\text{I.2.7})$$

La proposition correspondante est aussi juste pour les sur-solutions de viscosité.

Remark I.2.8. *Ainsi la notion de solution de viscosité des EDP-P's ne dépend pas de la définition des fonctions test lisses. Celles-ci sont uniquement utiles pour la cohérence entre les solutions de viscosité et les solutions classiques.*

3 Contribution principale de la thèse

3.1 Résultat de comparaison pour une EDP-P semi-linéaire

Dans le travail de Ekren, Touzi et Zhang [38], les auteurs ont prouvé le résultat de l'unicité et l'existence des solutions de viscosité des EDP-P's non linéaires. Dans leur approche, ils utilisent un argument de 'trajectoire gelée' pour approcher les EDP-P's par des EDP's classiques sur de petits intervalles. Cependant, cette approximation ne fonctionne que sous certaines conditions techniques qui excluent certaines applications intéressantes, par exemple, les équations de Hamilton-Jacobi-Bellman de la forme générale ne sont pas totalement couvertes. Cela conduit à l'exploration de nouveaux arguments. Il est raisonnable de commencer par les EDP-P's semi-linéaires. Dans la littérature existante, Ekren, Keller, Touzi et Zhang [35] ont prouvé le résultat

de comparaison dans le cas semi-linéaire sous des conditions plus générales à travers une représentation probabiliste des solutions. Cependant, cette approche est difficile à généraliser au cas complètement non-linéaire, car il n’y a pas de telle représentation des solutions des équations complètement non-linéaires. Il est à noter que les deux arguments ci-dessus fournissent les résultats de comparaison et d’existence des solutions de viscosité simultanément. Rappelons qu’au contraire, dans la théorie classique des solutions de viscosité des EDP’s, nous avons séparé les arguments pour les deux problèmes. En particulier, cette séparation d’arguments permet d’éviter des conditions inutiles pour le résultat de comparaison.

Un progrès prometteur est fait dans l’étude des EDP-P’s semi-linéaires dans [102]. Comme dans la théorie des EDP’s classiques, nous séparons les arguments pour les résultats de comparaison et d’existence des solutions de viscosité. Le nouvel argument exploite une famille élargie de fonctions de test et simplifie la preuve du résultat de comparaison. Notre preuve contourne complètement le lemma délicat et profond de Crandall et Ishii (voir Lemme 3.2 [21]). En particulier, notre preuve du principe de comparaison pour l’équation de la chaleur dépendante de la trajectoire est élémentaire et ne nécessite pas de pénalisation (le résultat de comparaison standard pour EDP’s s’applique à un domaine borné ; l’extension à un domaine non-borné entraîne la nécessité d’une pénalisation utilisant les conditions de croissance). Le résultat de l’unicité et l’existence de l’équation de la chaleur dépendante de la trajectoire est une conséquence directe de l’équivalence entre les sous-solutions de viscosité et les sous-martingales.

Rappelons que la définition des solutions de viscosité des EDP-P’s dépend de la famille choisie de mesures de probabilité \mathcal{P} . Dans le cas semi-linéaire, nous utilisons :

$$\mathcal{P}_L := \left\{ \mathbb{P}_\lambda : \frac{d\mathbb{P}_\lambda}{d\mathbb{P}_0} = \exp \left(\int_0^T \lambda_t \cdot dB_t - \frac{1}{2} \int_0^T |\lambda_t|^2 dt \right), \lambda \in \mathbb{L}^0(\mathbb{F}), \|\lambda\|_\infty \leq L \right\},$$

où \mathbb{P}_0 est la mesure de Wiener et $\mathbb{L}^0(\mathbb{F})$ est l’ensemble de tous les processus \mathbb{F} -progressivement mesurables. Nos arguments sont inspirés de la preuve du résultat de comparaison de Caffarelli et Cabré [15]. Avec la définition de jets (dans la section 2.3.2), nous pouvons introduire l’ingrédient clé de nos arguments, la dérivation ponctuelle.

Definition I.3.1. *Fonction u est \mathcal{P} -ponctuellement $C^{1,2}$ en (t, ω) , si*

$$\text{cl}(\bar{\mathcal{J}}^{\mathcal{P}} u(t, \omega)) \cap \text{cl}(\underline{\mathcal{J}}^{\mathcal{P}} u(t, \omega)) \neq \emptyset.$$

Ce nom est justifié par le fait qu’une fonction dans la classe $C^{1,2}$ au sens de Dupire est ponctuellement dérivable en tout point (t, ω) . Nous démontrons un résultat de régularité important.

Proposition I.3.2. *Soit u une \mathbb{P}_0 -semimartingale avec décomposition : $du_t = Z_t \cdot dB_t + dA_t$,*

avec $\mathbb{E}^{\mathbb{P}_0} \left[\int_0^T |Z_t|^2 dt \right] < \infty$ continu et à variation finie, \mathbb{P}_0 -p.s. Alors il existe un ensemble borélien $\mathbb{T}^u \subset [0, T]$ et $\Omega_t^u \in \mathcal{F}_t$ pour chaque $t \in \mathbb{T}^u$ tel que pour tout $L > 0$,

$Leb(\mathbb{T}^u) = T$, $\mathbb{P}_0(\Omega_t^u) = 1$, et u est \mathcal{P}_L -ponctuellement $C^{1,2}$ en (t, ω) pour tous $t \in \mathbb{T}^u$, $\omega \in \Omega_t^u$.

Ce résultat peut être considéré comme un analogue du résultat de régularité d'Aleksandroff pour les fonctions convexes. Dans le cas semilinéaire, une propriété importante de notre notion de solutions de viscosité est que les sous-solutions de viscosité (resp. les sur-solutions) sont sous-martingales (resp. sur-martingales) à un processus absolument continu près. En particulier, les sous-solutions de viscosité et les sur-solutions sont ponctuellement $C^{1,2}$ $Leb \otimes \mathbb{P}$ -p.s. Le résultat de régularité conduit au résultat de comparaison final, i.e. sous une certaine condition générale, nous avons prouvé :

Theorem I.3.3. *Soient u, v une \mathcal{P} -sous-solution de viscosité continue et une \mathcal{P} -sur-solution de viscosité continue, respectivement, d'une EDP-P semi-linéaire. Si $u_T \leq v_T$ sur Ω , alors $u \leq v$ sur Θ .*

Nous nous référons au Chapitre III pour les détails.

3.2 Méthode de Perron pour les EDP-P's semilinéaires

Comme l'unicité de la solution de viscosité est simplement une conséquence du résultat de comparaison, il reste à prouver l'existence. Dans [35, 102], les auteurs ont montré que dans le cas semi-linéaire, les solutions des l'EDSRs correspondantes sont les solutions de viscosité. Au lieu de cela, nous nous proposons de prouver l'existence de solutions de viscosité par des arguments de type EDP, à savoir, par la méthode de Perron.

Dans le papier [98], nous montrons comment adapter la méthode de Perron au contexte des EDP-P's. Rappelons notre espace canonique $(\Omega, \mathbb{F}, \mathbb{P}_0)$, et notons $\Theta = [0, T] \times \Omega$. Une fonction $u : \Theta \rightarrow \mathbb{R}$ appartient à USC_b (resp. LSC_b), si u est bornée et satisfait

$$u(\theta) \geq \limsup_{d(\theta, \theta') \rightarrow 0} u(\theta') \quad (\text{resp. } \leq \liminf_{d(\theta, \theta') \rightarrow 0} u(\theta')).$$

Sous certaines conditions générales, nous avons prouvé :

Theorem I.3.4. *Supposons qu'il y a une sous-solution de viscosité $\underline{u} \in USC_b(\Theta)$ et une sur-solution $\bar{v} \in LSC_b(\Theta)$ d'une EDP-P semilinéaire, vérifiant la condition au bord $(\underline{u}_*)_T = \bar{v}_T^* = \xi$.*

Notons

$$\mathcal{D} := \left\{ \phi : \phi \in \text{USC}_b(\Theta) \text{ est sous-solution de viscosité de l'EDP-P } \underline{u} \leq \phi \leq \bar{v} \right\}.$$

Alors, $u(\theta) := \sup\{\phi(\theta) : \phi \in \mathcal{D}\}$ est une solution de viscosité continue de l'EDP-P semilinéaire, et vérifie la condition au bord $u_T = \xi$.

Bien que notre résultat, qui concerne des EDP-P's semilinéaires, ne peut pas être appliqué au cas complètement non linéaire directement, de nombreux arguments dans [98] pourraient être utiles pour d'autres recherches. En outre, la méthode de Perron est non seulement utile pour prouver l'existence de solutions de viscosité, mais a également des applications dans de divers contextes, par exemple, le résultat de l'unicité et l'existence de la solution de viscosité de l'enveloppe (voir [2]), l'unicité dans des problèmes de martingales [18], etc. Dans la démonstration de la méthode de Perron, nous suivons la même idée que dans la littérature classique sur les solutions de viscosité des EDP's, mais les arguments se révèlent différents et non triviaux.

Il est bien connu dans la littérature EDP que le résultat de comparaison des solutions de viscosité continues ne suffit pas pour l'existence de solutions. Dans la méthode de Perron, nous avons besoin d'un résultat de comparaison pour les solutions de viscosité semicontinues. Toutefois, l'argument dans [102] ne peut pas être adapté dans notre contexte, car on ne sait pas si les sous-martingales semicontinues supérieures sont presque partout ponctuellement dérivables (un résultat intermédiaire crucial dans [102]). Dans [98], nous appliquons une régularisation sur les solutions de viscosité semicontinues afin de se ramener aux approximations continues. Soit u une sous-solution de viscosité, et soit u^n sa version régularisée. Une régularisation raisonnable devrait satisfaire :

$$u^n \text{ est continue; } u^n \rightarrow u, \text{ quand } n \rightarrow \infty; \quad u^n \text{ est encore sous-solution de viscosité.}$$

La régularisation que nous proposons implique une distance rétrograde pour les trajectoires :

$$\overleftarrow{d}((t, \omega), (t', \omega')) := |t - t'| + \sup_{s \geq 0} |\omega_{(t-s) \vee 0} - \omega'_{(t'-s) \vee 0}|.$$

Cette distance est nouvelle dans la littérature, elle satisfait toutes les conditions ci-dessus et contribue à montrer le résultat de comparaison. Il est à noter que la régularisation est probablement inévitable dans l'étude du résultat de comparaison pour les EDP-P's complètement non linéaires. La régularisation que nous trouvons dans [98] pourrait contribuer à éclairer la recherche future.

En outre, comme dans tous les travaux sur les solutions de viscosité des EDP-P's, le résultat de l'arrêt optimal joue un rôle crucial pour surmonter l'absence de compacité locale de l'espace de trajectoires. Puisque nous traitons les solutions de viscosité semicontinues dans [98], nous avons besoin d'un résultat correspondant de l'arrêt optimal sous l'espérance non linéaire pour des obstacles semicontinus. Dans la littérature existante, Kobylanski et Quenez [74] contient le résultat souhaité, mais seulement dans le cas de l'espérance linéaire. Peng et Xu ont étudié dans [95] les EDSRs réfléchies avec des L^2 -obstacles, et ils ont montré un résultat intermédiaire crucial qui peut entraîner le résultat de l'arrêt optimal. Cependant, comme leur intérêt principal est les EDSRs réfléchies, il n'y a pas de théorème direct que nous pouvons appliquer. Dans [98], nous donnons une nouvelle preuve simple pour le problème de l'arrêt optimal, en utilisant la condition minimale de la décomposition de Skorokhod.

Le Chapitre IV est consacré au développement de la méthode de Perron.

3.3 Schema monotone pour les EDP-P's

Outre le résultat de l'unicité et l'existence, un autre sujet intéressant est de calculer numériquement les solutions de viscosité des EDP-P's. Dans [4], Barles et Souganidis ont prouvé un théorème de convergence des schémas numériques monotones vers les solutions de viscosité des EDP's non linéaires. Notons \mathbb{T}_h un schéma numérique avec le pas de temps h , i.e. la solution numérique u^h satisfaisant

$$u^h(t, x) := \mathbb{T}_h^{t,x}[u^h(t+h, \cdot)].$$

Un schema dit monotone vérifie les propriétés suivantes :

(i) Cohérence : pour tout $(t, x) \in [0, T) \times \mathbb{R}^d$ et toute fonction lisse $\mathbb{F} \in C^{1,2}([0, T) \times \mathbb{R}^d)$,

$$\lim_{(t', x', h, c) \rightarrow (t, x, 0, 0)} \frac{(c + \varphi)(t', x') - \mathbb{T}_h^{t', x'}[(c + \varphi)(t' + h, \cdot)]}{h} = \mathbb{L}\varphi(t, x).$$

(ii) Monotonie : $\mathbb{T}_h^{t,x}[\varphi] \leq \mathbb{T}_h^{t,x}[\psi]$ lorsque $\varphi \leq \psi$.

(iii) Stabilité : u^h est bornée uniformément en h lorsque g est bornée.

(iv) Condition au bord : $\lim_{(t', x', h) \rightarrow (T, x, 0)} u^h(t', x') = g(x)$ pour tout $x \in \mathbb{R}^d$.

Supposant que le principe de comparaison est vrai pour les solutions de viscosité de l'EDP, Barles et Souganidis ont montré que

$$u := \lim_{h \rightarrow 0} u^h \text{ est l'unique solution de viscosité de l'EDP.}$$

Ils utilisent principalement la stabilité des solutions de viscosité des EDP's et la compacité

locale de l'espace d'état. Grace à leur résultat, il suffit de vérifier certaines propriétés locales du schéma numérique afin d'obtenir un résultat global de la convergence.

Il serait intéressant d'étendre le théorème de convergence dans [4] au contexte des EDP-P's. La difficulté principale à une extension directe de leurs arguments est que l'espace d'état n'est plus localement compact. Zhang et Zhuo [114] ont fourni récemment une formulation du théorème de la convergence des schémas monotones pour les EDP-P's. Considérons un schéma numérique dépendant de trajectoire :

$$u^h(t, \omega) := \mathbb{T}_h^{t, \omega}[u^h(t + h, \cdot)].$$

La condition de monotonie introduite par Zhang et Zhuo [114] est :

$$(ii') \quad \mathbb{T}_h^{t, \omega}[\varphi] \leq \mathbb{T}_h^{t, \omega}[\psi] \text{ lorsque } \bar{\mathcal{E}}^{\mathcal{P}}[(\varphi - \psi)^{t, \omega}] \leq 0,$$

où \mathcal{P} est la famille de mesures de probabilité dans la définition des solutions de viscosité des EDP-P's. Leur résultat principal est le même théorème de convergence que dans Barles et Souganidis [4] :

$$u := \lim_{h \rightarrow 0} u^h \text{ est l'unique solution de viscosité de l'EDP-P.}$$

Ils utilisent principalement la stabilité des solutions de viscosité des EDP-P's et ils surmontent la difficulté de l'absence de compacité locale par un argument d'arrêt optimal. Ils fournissent également un schéma numérique illustratif qui satisfait toutes les conditions de leur théorème de convergence. Cependant, ce schéma numérique illustratif n'est pas implémentable dans le cas général. En outre, la plupart des schémas numériques monotones au sens de Barles et Souganidis [4], par exemple, le schéma aux différences finies, ne satisfont pas leurs conditions, en particulier, la condition (ii').

Notre papier récent [100] fournit une nouvelle formulation du théorème de convergence des schémas numériques pour les EDP-P's. Au lieu de la condition (ii) ou (ii'), nous introduisons une nouvelle condition :

((ii) ") il existe une espérance non linéaire $\underline{\mathcal{E}}_h$ satisfaisant certaines conditions telles que, pour tout $\varphi, \psi \in \mathbb{L}^0(\mathcal{F}_{t+h})$, il y a

$$\mathbb{T}_h^{t, \omega}[\varphi] - \mathbb{T}_h^{t, \omega}[\psi] \geq \inf_{0 \leq \alpha \leq L} \underline{\mathcal{E}}_h[e^{\alpha h}(\varphi - \psi)^{t, \omega}] - h\rho(h).$$

Par conséquent, nous prouvons le théorème de convergence :

Theorem I.3.5. *Supposons que*

- la non-linéarité G de l'EDP-P (I.2.2) est elliptique (voir (I.2.3)),
- la non-linéarité G et l'état terminal ξ sont continus en tous les arguments, et $G(t, \omega, y, z, \gamma)$ est uniformément Lipschitz en y ,
- les solutions de viscosité de l'équation (I.2.2) satisfont le principe de comparaison, i.e. si u, v sont \mathcal{P} -sous-solution et sur-solution bornées et uniformément continues de (I.2.2), respectivement, et $u(T, \cdot) \leq v(T, \cdot)$, alors $u \leq v$ sur Θ .

Si le schéma numérique \mathbb{T} satisfait les conditions de monotonie, i.e. la condition (i), (ii'') et (iii), alors l'équation (I.2.2) admet une unique solution de viscosité bornée u , et

$$u^h \rightarrow u \text{ uniformément localement, lorsque } h \rightarrow 0.$$

Nos conditions sont légèrement plus fortes que les conditions classiques de Barles et Souganidis [4], si l'EDP-P dégénère en une EDP. Néanmoins, ces conditions sont vérifiées par tous les schémas numériques monotones connus dans le contexte du contrôle optimal, y compris le schéma aux différences finies dépendant de trajectoire, le schéma de Monte-Carlo de Fahim, Touzi et Warin [48], le schéma de semi-Lagrangien, le schéma de l'arbre trinomial de Guo, Zhang et Zhuo [59], le schéma du système de commutation de Kharroubi, Langrené et Pham [72], etc. Par conséquent, le résultat dans [100] étend tous ces schémas numériques au cas de dépendance de trajectoire. En particulier, il fournit des schémas numériques pour les EDSRs non-markoviennes du second ordre, et des jeux différentiels stochastiques, ce qui est nouveau dans la littérature, voir aussi Possamaï et Tan [97].

Le Chapitre V est consacré à l'article [100].

3.4 EDP-P's elliptiques

Rappelons le problème stochastique du contrôle optimal dans l'exemple I.1.2. Définissons le temps d'arrêt $\tau(\omega) := \inf\{t \geq 0 : \omega_t \notin O\}$, où O est un sous-ensemble ouvert borné de \mathbb{R}^d , et la fonction valeur $u(x) := \mathbb{E}[h(X_\tau) | X_0 = x]$ pour $x \in O$. Alors, la fonction u est une solution candidat de l'équation elliptique suivante :

$$\text{Tr}[D^2u] = 0,$$

avec la condition au bord $u = h$ sur ∂O . De toute évidence, nous pouvons généraliser cet exemple au cas de dépendance de trajectoire. Donc, il est intéressant de se pencher sur les EDP-P's elliptiques.

Notre point de départ est l'approche EDSR de Darling et Pardoux [23] qui peut être consi-

dérée comme une théorie des solutions de Sobolev pour les EDP-P's elliptiques. Comme dans le cas parabolique, la représentation de Feynman-Kac prévue dans [23] montre un lien entre les EDSRs markoviennes avec le temps terminal aléatoire et les EDP's semilinéaires elliptiques avec une condition au bord de type Dirichlet.

Suite aux travaux de Ekren, Touzi et Zhang [37, 38] sur les EDP-P's paraboliques, dans [99] nous essayons de développer la notion de solution de viscosité des EDP-P's elliptiques. Soit Q un sous-ensemble borné fermé et convexe de \mathbb{R}^d , \mathcal{Q} l'ensemble des trajectoires à valeur dans Q . Nous considérons les EDP-P's elliptiques de la forme :

$$G(\cdot, u, \partial_\omega u, \partial_{\omega\omega}^2 u)(\omega) = 0, \quad \omega \in \mathcal{Q}, \quad \text{et} \quad u = \xi \text{ pour } \omega \in \partial\mathcal{Q}. \quad (\text{I.3.1})$$

Une différence claire entre l'équation parabolique et l'équation elliptique est l'indépendance de la variable temporelle. Cependant, on ne sait pas comment décrire l'indépendance de temps dans le cas dépendant de trajectoire, parce que la variable de trajectoire contient des informations de temps. Nous réduisons l'espace de trajectoire au sous-espace $\Omega^e = \{\omega \in \Omega : \omega = \omega_{t\wedge \cdot} \text{ pour certains } t\}$ et désignons $\bar{t}(\omega) := \sup\{t : \omega_t \neq \omega_\infty\}$. Pour une fonction $u : \Omega^e \rightarrow \mathbb{R}$, nous introduisons $\tilde{u}(t, \omega) := u(\omega_{t\wedge \cdot})$. Rappelons la dérivée de Dupire :

$$\partial_t \tilde{u}(t, \omega) := \lim_{h \rightarrow 0} \frac{\tilde{u}(t+h, \omega_{t\wedge \cdot}) - \tilde{u}(t, \omega)}{h}.$$

L'indépendance de temps devrait impliquer $\partial_t \tilde{u} = 0$. Nous introduisons alors une nouvelle distance sur Ω^e :

$$d^e(\omega, \omega') := \inf_{\ell \in I} \sup_t |\omega_t - \omega'_{\ell(t)}|,$$

où $I = \{\ell : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \mid \ell \text{ est une bijection croissante et } \ell(0) = 0\}$. En particulier, une fonction continue par rapport à la distance d^e a la propriété d'invariance par rapport au temps :

$$u(\omega) = u(\omega_{\ell(\cdot)}) \quad \text{pour tout } \omega \text{ et toutes les bijections croissantes } \ell : \mathbb{R}^+ \rightarrow \mathbb{R}^+.$$

Intuitivement, une application u invariante par rapport au temps est invariante par le changement d'échelle en temps. Cette propriété implique la propriété désirée $\partial_t \tilde{u} = 0$. Il est à noter que si nous prenons une trajectoire en temps-espace $\bar{\omega}_t = (t, \omega_t)$, alors l'analogie de la distance d^e serait

$$\bar{d}^e(\bar{\omega}, \bar{\omega}') = \inf_{\ell \in I} \sup_t (|t - \ell(t)| + |\omega_t - \omega'_{\ell(t)}|),$$

qui est équivalente à la distance de Skorokhod.

Le résultat principal de [99] est :

Theorem I.3.6. *Sous certaines conditions techniques, l'EDP-P elliptique (I.3.1) a une unique solution de viscosité bornée et uniformément continue.*

Afin de prouver ce résultat de l'unicité et l'existence, nous suivons le raisonnement de [38], i.e. la technique de 'trajectoire gelée' déjà mentionnée dans la Section 3.1. Mais nous devons nous attaquer aux nouvelles difficultés dans le contexte elliptique. Une des principales difficultés est due à la condition de Dirichlet au bord. En particulier, comme le temps d'arrivée au bord $H_Q(\omega)$ n'est pas continu en ω , il est non-trivial de vérifier la continuité de la solution de viscosité construite via les EDP's aux trajectoires gelées.

Le Chapitre VI de la thèse est consacré aux EDP-P's elliptiques.

3.5 Grandes déviations pour les diffusions non-markoviennes

Après la construction précédente de la théorie des EDP-P's, nous considérons à présent une application dans au problème des grandes déviations pour les diffusions non-markoviennes. La théorie des grandes déviations considère le taux de convergence d'une suite de probabilités $(\mathbb{P}[A_n])_{n \geq 1}$ qui tend vers zéro, où $(A_n)_{n \geq 1}$ est une suite des événements rares. Après une renormalisation convenable, la limite est appelée *la fonction de taux*, elle est généralement représentée en termes d'un problème de contrôle. Le travail pionnier des Freidlin et Wentzell [55] considère les événements rares induits par des diffusions markoviennes. Les techniques sont basées sur le théorème de Girsanov pour le changement de mesures équivalentes, et la dualité convexe classique. Une contribution importante par Fleming [51] est d'utiliser la propriété de stabilité des solutions de viscosité afin d'obtenir une approche significativement simplifiée. Nous nous référons à Feng et Kurtz [49] pour une application systématique de cette méthodologie avec des extensions pertinentes.

L'objectif principal de notre travail [84] est d'étendre l'approche des solutions de viscosité à certains problèmes de grandes déviations avec des événements rares induits par les diffusions non-markoviennes

$$X_t^\epsilon = X_0 + \int_0^t b_s(X^\epsilon) ds + \int_0^t \sqrt{\epsilon} \sigma_s(X^\epsilon) dW_s, \quad t \geq 0, \quad (\text{I.3.2})$$

où W est un mouvement brownien, et b, σ sont des fonctions non-anticipatives des trajectoires de X^ϵ satisfaisant les conditions d'existence et d'unicité de la solution de l'équation différentielle stochastique (EDS). Notons que le principe de grande déviation pour des diffusions

non-markoviennes de type (I.3.2) n'est pas nouveau. Par exemple, Gao & Liu [56] ont étudié un tel problème par la méthode de Fredlin-Wentzell, en utilisant différentes normes dans des espaces de dimension infinie. Bien que les techniques dans leur papier soient assez profondes et sophistiquées, la méthodologie est plus ou moins "classique". Notre objectif principal est d'étendre l'approche EDP de Fleming [51] au cadre de dépendance de trajectoire, avec des outils différents. Ceux-ci comprennent la théorie des EDSRs, le contrôle stochastique et la théorie des solutions de viscosité des EDP-P's. Plus précisément, la théorie des EDSRs, fondée par Pardoux & Peng [89], peut être effectivement utilisée comme un substitut des EDP's dans le cadre markovien. En effet, la transformation logarithmique de la probabilité résout une EDP semilinéaire dans le cas markovien. Toutefois, en raison de la nature "fonctionnelle" des coefficients dans (I.3.2), à la fois l'EDSR et l'EDP impliquées deviendront non-markoviennes.

Nous considérons deux types de comportement asymptotique dans [84], celui de la transformation de Laplace et celui du temps de sortie. Sous certaines conditions générales, nous avons prouvé :

Theorem I.3.7 (Transformée de Laplace). *Soit ξ une variable aléatoire bornée, uniformément continue, et \mathcal{F}_T -mesurable. Alors nous avons que*

$$L_0^\epsilon := -\epsilon \ln \mathbb{E} \left[e^{-\frac{1}{\epsilon} \xi(X^\epsilon)} \right]$$

converge vers L_0 lorsque $\epsilon \rightarrow 0$, où

$$L_0 := \inf_{\alpha \in \mathbb{L}_d^2} \ell_0^\alpha, \quad \ell_0^\alpha := \xi(x^\alpha) + \frac{1}{2} \int_0^T |\alpha_t|^2 dt, \tag{I.3.3}$$

et x^α est défini par l'équation différentielle ordinaire contrôlée :

$$\omega_t^\alpha = \int_0^t \alpha_s ds, \quad x_t^\alpha = X_0 + \int_0^t b_s(x^\alpha) ds + \int_0^t \sigma_s(x^\alpha) d\omega_s^\alpha, \quad t \in [0, T].$$

Theorem I.3.8 (Temps de sortie). *Soit O un sous-ensemble ouvert borné de \mathbb{R}^n au bord dans C^3 , et définissons*

$$\mathcal{O} := \left\{ \omega \in \Omega_n : \omega_t \in O \text{ pour tout } t \leq T \right\}. \tag{I.3.4}$$

Alors nous avons que

$$Q_0^\epsilon := -\epsilon \ln \mathbb{P}^\epsilon[H < T], \quad \text{où } H := \inf\{t > 0 : X_t \notin O\}, \tag{I.3.5}$$

converge vers

$$Q_0 := \inf \left\{ q_0^\alpha : \alpha \in \mathbb{L}_d^2, x_{T \wedge \cdot}^\alpha \notin \mathcal{O} \right\}, \quad \text{où } q_0^\alpha := \frac{1}{2} \int_0^T |\alpha_s|^2 ds,$$

et x^α est défini comme dans le Théorème [I.3.7](#).

Plusieurs points techniques sont à noter. Tout d’abord, comme l’EDSR impliquée dans notre problème est non-linéaire en le terme de gradient (quadratique, pour être précis), nous avons besoin donc de l’extension par Kobylanski [\[73\]](#) aux EDSRs dans ce contexte. Deuxièmement, afin d’obtenir la fonction de taux, nous exploitons la représentation de contrôle stochastique de la transformation logarithmique, et nous faisons l’analyse asymptotique en utilisant les propriétés BMO des solutions des EDSRs. Enfin, nous utilisons la notion de solutions de viscosité de l’équation de Hamilton-Jacobi dépendante de la trajectoire introduite par Lukoyanov [\[82\]](#) afin de caractériser la fonction de taux (une version dynamique de L_0 dans [\(I.3.3\)](#)) en tant que solution de viscosité unique d’une équation eikonale dépendante de la trajectoire.

Un autre objectif principal, en fait, la motivation initiale, de ce travail est une application en mathématiques financières. Il est connu que l’un des problèmes importants dans l’évaluation et la couverture des options exotiques est de caractériser le comportement asymptotique en temps court de la surface de la volatilité implicite, étant donné le prix des options européennes pour toutes les échéances et les prix d’exercice. La nécessité d’étudier le comportement asymptotique est dû au fait que seul un ensemble discret des échéances et des prix d’exercice sont disponibles. On a contourné cette difficulté, en utilisant le comportement asymptotique afin d’étendre la surface de volatilité aux régimes non-observables, nous nous référons à Henry-Labordère [\[?\]](#). Les résultats disponibles dans la littérature ont été limités au cas markovien, et nos résultats dans un sens ouvrent la porte au cas général non-markovien, autrement dit, au cas dépendant de trajectoire.

Nous nous référons au Chapitre [VII](#) pour l’exposé détaillé de ces résultats.

3.6 Algorithme dual des problèmes de contrôle stochastique

La dernière partie de la thèse est consacré à l’algorithme dual des problèmes de contrôle stochastique. Nous nous référons au Chapitre [VIII](#) pour le travail de [\[66\]](#).

Résoudre les problèmes de contrôle stochastique est important en mathématiques appliquées. Les méthodes classiques des EDP’s sont efficaces pour résoudre de tels problèmes au cas de dimensions réduites, mais deviennent numériquement insolubles lorsque la dimension du problème augmente. Les méthodes probabilistes, comme la simulation de Monte-Carlo, sont

moins sensibles à la dimension du problème.

Il a été démontré dans Pardoux & Peng [89], Cheridito, Soner, Touzi & Victoir [16] et Soner, Touzi & Zhang [107] que les EDSRs du premier et du deuxième ordre peuvent fournir des représentations stochastiques pour les EDP's paraboliques semi-linéaires ou complètement non-linéaires. Les EDSRs fournissent une généralisation non-linéaire de la formule classique de Feynman-Kac .

L'approche numérique du problème de contrôle stochastique via EDSR associée a été d'abord proposée indépendamment dans Bouchard & Touzi [13] et Zhang [113]. En outre, des généralisations ont été obtenues dans Fahim, Touzi & Warin [48], Guyon & Henry-Labordère [60] et Tan [110]. Dans l'algorithme de [60], il est nécessaire d'évaluer des espérances conditionnelles de grande dimension, qui sont en général calculées à l'aide des techniques de régression paramétriques. En résolvant des EDSRs, on obtient une estimation sous-optimale du contrôle stochastique. En faisant une simulation de Monte-Carlo supplémentaire, indépendante avec ce contrôle sous-optimal, on obtient une estimation biaisée : une borne inférieure de la valeur du problème de contrôle stochastique sous-jacent. En pratique, il est difficile de choisir la bonne base pour l'étape de régression, en particulier dans le cas de grande dimension. Donc, l'appréciation de la précision de la borne inférieure est confortée par l'approche duale qui entraîne une borne supérieure correspondante.

Une telle expression duale a été obtenue par Rogers [103], basé sur le travail plus ancien par Davis et Burstein [24]. Alors que le travail de Rogers traite le cas du temps discret, il est applicable à une classe générale de processus markoviens. Les travaux de Davis et Burstein [24] reliant le contrôle déterministe et le contrôle stochastique en utilisant des techniques de décomposition de flot (voir aussi Diehl, Friz, Gassiat [32] pour une approche via le chemin rugueux à ce problème) sont limités au cas où le contrôle n'est que dans le terme de drift.

Dans notre papier [66], nous considérons également le contrôle des processus de diffusion, mais nous permettons au contrôle d'agir à la fois sur le terme de drift et sur le terme de volatilité. Pour un certain $M > 0$, prenons A un sous-ensemble compact $O_M := \{x \in \mathbb{R}^k : |x| \leq M\}$ pour un $k \in \mathbb{N}$, et notons $\mathcal{A} := \left\{ \alpha : \Theta \rightarrow \mathbb{R}^K : \alpha \text{ est } \mathbb{F}\text{-adapté, et prend ses valeurs dans } A \right\}$. Considérons la fonction valeur d'un problème de contrôle stochastique :

$$u_0 = \sup_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_0^T e^{-\int_0^t r(s, \alpha_s, X_s^\alpha) ds} f(t, \alpha_t, X_t^\alpha) dt + e^{-\int_0^T r(s, \alpha_s, X_s^\alpha) ds} g(X_T^\alpha) \right],$$

où X^α est une diffusion contrôlée de dimension d définie par

$$X^\alpha := \int_0^\cdot \mu(t, \alpha_t, X_t^\alpha) dt + \int_0^\cdot \sigma(t, \alpha_t, X_t^\alpha) dW_t.$$

Soit N^h un h -net fini de A , i.e. pour tout $x, y \in N^h \subset A$, nous avons $|x - y| \leq h$. On note $\mathcal{D}_h := \left\{ a : [0, T] \rightarrow \mathbb{R}^k : a \text{ est constante sur } [t_i^h, t_i + 1^h) \text{ pour } i, \text{ et prend des valeurs dans } N^h \right\}$. Notre résultat dual dit que $u_0 = \lim_{h \rightarrow 0} v^h$, où

$$v^h := \inf_{\varphi \in \mathcal{U}} \mathbb{E} \left[\sup_{a \in \mathcal{D}_h} \left\{ e^{-\int_0^T r(s, a_s, X_s^a) ds} g(X_T^a) + \int_0^T e^{-\int_0^t r(s, a_s, X_s^a) ds} f(t, a_t, X_t^a) dt - \int_0^T e^{-\int_0^t r(s, a_s, X_s^a) ds} \varphi_t(X^a)^\top \sigma(t, a_t, X_t^a) dW_t \right\} \right].$$

L'idée de base de l'algorithme dual est de remplacer le contrôle stochastique par une famille de problèmes de contrôle déterministe (voir la maximisation en espérance dans la définition de v^h). Contrairement à [24], [32] nous ne traitons pas les problèmes de contrôle déterministe en temps continu. Au lieu de cela, nous utilisons un résultat de discrétisation pour l'équation d'HJB par Krylov [76], nous récupérons la solution du problème de contrôle stochastique comme la limite de problèmes de contrôle déterministe sur un ensemble fini de contrôles discrétisés. Inspiré par la technique "trajectoire gelée" introduite dans Ekren, Touzi et Zhang [38], nous généralisons également le résultat dual au cas dépendant de trajectoire. Sous certaines conditions générales, nous avons prouvé :

Theorem I.3.9. *Considérons le problème de contrôle stochastique :*

$$u_0 = \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}^0} \left[g(X_{T \wedge \cdot}^\alpha) \right],$$

où X^α est une diffusion de dimension d définie par $X^\alpha := \int_0^\cdot \mu(t, \alpha_t) dt + \int_0^\cdot \sigma(t, \alpha_t) dB_t$. Alors, nous avons

$$u_0 = \lim_{h \rightarrow 0} v^h, \text{ où } v^h := \inf_{\varphi \in \mathcal{U}} \mathbb{E}^{\mathbb{P}^0} \left[\sup_{a \in \mathcal{D}_h} \left\{ g(X_{T \wedge \cdot}^a) - \int_0^T \varphi_t^\top(X^a) \sigma(t, a_t) dB_t \right\} \right].$$

Chapitre II

English Introduction

1 Motivation

1.1 From Markovian models to non-Markovian ones

Evolution equation is one of the most used tools in mathematical modeling. Let $d \in \mathbb{N}$ and $u : (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d \mapsto u(t, x)$. Consider a second order evolution equation of the form :

$$\partial_t u(t, x) = G(t, x, u(t, x), Du(t, x), D^2u(t, x)), \quad (\text{II.1.1})$$

where G is a so-called generator function, and D, D^2 are the first and second order differential operator in x -variable, respectively. In particular, if $G : (t, x, y, z, \gamma) \mapsto G(t, x, y, z)$ is non-increasing in γ , we say Equation (II.1.1) is a parabolic equation. Parabolic equations are closely related to the theory of stochastic process and that of stochastic control.

Example II.1.1. *A toy model can be the well known Feynman-Kac relation. Let W be a d -dimension Brownian motion, $h : \mathbb{R}^d \rightarrow \mathbb{R}$ be a bounded Borel-measurable function, and define*

$$v(t, x) := \mathbb{E}[h(W_T) | W_t = x]. \quad (\text{II.1.2})$$

By assuming $v \in C^{1,2}$ (indeed, it can be easily verified), we may apply the Itô formula and conclude that v is a classical solution of the heat equation with the terminal condition h , i.e.

$$\partial_t v + \frac{1}{2} D^2 v = 0, \quad v(T, x) = h(x). \quad (\text{II.1.3})$$

Conversely, let u be a solution to (II.1.3). The classical regularity result shows that $u \in C^{1,2}$. Again, by the Itô formula, we observe that $u(t, W_t)$ is a martingale, and thus $u = v$.

A nonlinear generalization of the toy example can be the Hamilton-Jacobi-Bellman equation.

Example II.1.2. Let K be a compact subset of \mathbb{R}^m ($m \in \mathbb{N}$), and \mathcal{K} be the set of all adapted processes taking values in K . Consider a controlled Markovian diffusion X^κ of the form :

$$X_t^\kappa = X_0 + \int_0^t b(s, X_s^\kappa, \kappa_s) ds + \int_0^t \sigma(s, X_s^\kappa, \kappa_s) dW_s, \quad \kappa \in \mathcal{K}. \quad (\text{II.1.4})$$

We are concerned with the optimization :

$$u_0 = \sup_{\kappa \in \mathcal{K}} \mathbb{E} \left[\int_0^T e^{\int_0^t c(s, X_s^\kappa, \kappa_s) ds} f(t, X_t^\kappa, \kappa_t) dt + e^{\int_0^T c(s, X_s^\kappa, \kappa_s) ds} h(X_T^\kappa) \right]. \quad (\text{II.1.5})$$

In order to solve this problem, we introduce the dynamic version of the optimization :

$$u(t, x) = \sup_{\kappa \in \mathcal{K}} \mathbb{E} \left[\int_0^T e^{\int_0^t c(s, X_s^\kappa, \kappa_s) ds} f(t, X_t^\kappa, \kappa_t) dt + e^{\int_0^T c(s, X_s^\kappa, \kappa_s) ds} h(X_T^\kappa) \mid X_t^\kappa = x \right]$$

The dynamic programming principle says that for any stopping time $\tau \in [0, T]$ we have

$$u(t, X_t^\kappa) = \sup_{\kappa \in \mathcal{K}} \mathbb{E} \left[\int_t^\tau e^{\int_t^s c(r, X_r^\kappa, \kappa_r) dr} f(s, X_s^\kappa, \kappa_s) ds + e^{\int_t^\tau c(s, X_s^\kappa, \kappa_s) ds} u(\tau, X_\tau^\kappa) \right].$$

We next assume that the value function $u \in C^{1,2}$. With the help of Itô's formula, we may verify that u is a classical solution of the Hamilton-Jacobi-Bellman equation :

$$\partial_t u + \sup_{k \in K} \left\{ b(t, x, k) \cdot Du + \frac{1}{2} \text{Tr} \left((\sigma \sigma^\top)(t, x, k) D^2 u \right) + c(t, x, k) u + f(t, x, k) \right\} = 0, \quad u_T = h.$$

There are many advantages of characterizing the value functions as a solution of the HJB equation. For example, in the initial problem (II.1.5), we are facing an optimization in an infinite dimensional space, not to mention the constraint of adaptedness. Instead, in the HJB equation, we only see a maximization over a compact set in \mathbb{R}^m . It provides simplicity in particular for numerical approaches.

It is noteworthy that in both previous examples we consider Markovian models. How about non-Markovian ones (in other words, path-dependent ones)? They are indeed more than usual in applications. The following example provides some situations where path dependence can be reduced to the Markovian setting by augmentation of the state space.

Example II.1.3. Denote again by W a d -dimension Brownian motion. Let \bar{W}, W^* be the running average and the running maximum, respectively, i.e.

$$\bar{W}_t = \int_0^t W_s ds, \quad W_t^* = \max_{s \leq t} W_s.$$

Note that neither \bar{W} or W^* is Markovian. However, by taking into account the Brownian motion itself, we find that both the vector processes (W, \bar{W}) and (W, W^*) are Markovian.

1. Consider $\bar{u}(t, x, y) = \mathbb{E}[h(\bar{W}_T)|W_t = x, \bar{W}_t = y]$. Then, by assuming \bar{u} is smooth enough, we may verify that \bar{u} is a classical solution to the following equation :

$$\partial_t u + \frac{1}{2} D_x^2 u + x D_y u = 0, \quad u(T, x, y) = h(y).$$

2. Consider $u^*(t, x, y) = \mathbb{E}[h(W_T^*)|W_t = x, W_t^* = y]$ for all $x \leq y$. The expectation value can be formally related to the PDE :

$$\partial_t u + \frac{1}{2} D_x^2 u = 0 \text{ on } \{x \leq y\}, \quad D_y u(t, \cdot, \cdot) = 0 \text{ on } \{x = y\}, \quad u(T, x, y) = h(y).$$

In the second example, although we may still write down the corresponding PDE, the equation has a more complex boundary condition (precisely, here it is of a Neumann condition). The involvement of complicated boundary conditions can cause difficulty in the analysis of PDEs. Not to mention that there is no unified method to treat all different kinds of boundary conditions. Further, there are examples of non-Markovian models in which the augmentation of space does not help. This is illustrated by the following example.

Example II.1.4. 1. Let μ be a σ -finite measure singular to the Lebesgue measure. Define $U_t := \int_0^t W_s d\mu_s$. Then, the vector process (W, U) is Markovian, and it still makes sense to define $\tilde{u}(t, x, y) := \mathbb{E}[h(U_T)|W_t = x, U_t = y]$. However, we can no longer find the corresponding PDE.

2. Another example is the control with delay. Instead of (II.1.4), we consider a controlled diffusion of the form :

$$X_t^\kappa = X_0 + \int_0^t b(s, X_{s-\delta}^\kappa, \kappa_s) ds + \int_0^t \sigma(s, X_{s-\delta}^\kappa, \kappa_s) dW_s, \quad \kappa \in \mathcal{K}.$$

Note that the process X is no longer Markovian. We are concerned with the same optimization problem as in (II.1.5). In this case, it is not possible to construct a Markovian process by a finite-dimension augmentation of space.

The above examples justify the need of developing a generalization for evolution equations to the non-Markovian setting.

1.2 From generalized Itô formula to path dependent PDEs

As we have seen in the previous section, the relation between value functions and solutions of evolution equations is often established with the help of the Itô formula. So, an understanding of path-dependent Itô formula could be a good start of the theory of path dependent PDEs.

In Dupire's original work [34], he studied the so-called horizontal and vertical derivatives for functions of paths. Let \mathcal{D} be the set of all càdàg paths on $[0, T]$ taking values in \mathbb{R}^d and starting from the origin, and $u : \mathbb{R}^+ \times \mathcal{D} \rightarrow \mathbb{R}$ be a non-anticipative function, i.e. $u(t, \omega) = u(t, \omega_{t\wedge \cdot})$ for $\omega \in \mathcal{D}$. Define

$$\begin{aligned} \partial_t u(t, \omega) &:= \lim_{\epsilon \rightarrow 0} \frac{u(t+\epsilon, \omega_{t\wedge \cdot}) - u(t, \omega)}{\epsilon}, \\ \partial_{\omega^i} u(t, \omega) &:= \lim_{\epsilon \rightarrow 0} \frac{u(t, \omega + \epsilon \mathbf{1}_{[t, T]} e_i)}{\epsilon}, \quad \text{and} \quad \partial_{\omega} u(t, \omega) := \left(\partial_{\omega^i} u(t, \omega) \right)_{1 \leq i \leq d}. \end{aligned} \quad (\text{II.1.6})$$

Note that if the function u is not path dependent, i.e. there is a function $\hat{u} : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that $u(t, \omega) = \hat{u}(t, \omega_t)$, then the derivatives defined above reduce to the normal partial derivatives defined for real spaces. We next say that a function $u : \mathbb{R}^+ \times \mathcal{D} \rightarrow \mathbb{R}$ is in the class $C^{1,2}$, if u is continuous, and all the derivatives $\partial_t u, \partial_{\omega} u, \partial_{\omega\omega}^2 u$ exist and are continuous. In [17, 34], the authors showed that under some general conditions, a $C^{1,2}$ -function u satisfies a generalized Itô formula :

$$u(t, \omega) - u_0 = \int_0^t \partial_t u(s, \omega) ds + \frac{1}{2} \int_0^t \partial_{\omega\omega}^2 u(s, \omega) d\langle \omega \rangle_s + \int_0^t \partial_{\omega} u(s, \omega) d\omega_s, \quad \mathbb{P}\text{-a.s.} \quad (\text{II.1.7})$$

for all measures \mathbb{P} under which the canonical process in \mathcal{D} is a semimartingale. From the following example, we will see that the generalized Itô formula plays the role of bridge between probabilistic models and *path dependent* evolution equations.

Example II.1.5. *We consider a toy model similar to the one in Example II.1.1. Let Ω be the set of all continuous paths starting from the origin. Consider a bounded measurable function $h : \Omega \rightarrow \mathbb{R}$. Define the function of the conditional expectation :*

$$u(t, \omega) := \mathbb{E} \left[h(W_{T\wedge \cdot}) \middle| \mathcal{F}_t \right] (\omega),$$

where \mathcal{F}_t is the filtration generated by Brownian motion W . By assuming that $u \in C^{1,2}$ in the sense of Dupire, we may apply the generalized Itô formula (II.1.7). Since $u(t, W)$ is clearly a martingale, we obtain

$$\partial_t u + \frac{1}{2} \partial_{\omega\omega}^2 u = 0,$$

a path dependent heat equation.

This simple example not only generalizes the model in Example II.1.1, but also covers the ones in Example II.1.3 as well as the first model in Example II.1.4. So, a theory of path dependent PDE can unify the analysis of PDEs with different kinds of boundary conditions, and carry the information that normal PDEs cannot interpret.

1.3 From classical solutions to viscosity solutions

In the previous sections, we always assume the regularity of value functions so as to apply the Itô formula and get the corresponding PDEs or path dependent PDEs. However, in reality solutions do not necessarily have enough regularity.

Let us consider the heat equation in Example II.1.1 and the path dependent heat equation in Example II.1.5. As we mentioned in Example II.1.1, the heat equation always have regular solutions. How about the path dependent case?

Example II.1.6. Consider the value function

$$u(t, \omega) := \mathbb{E}\left[W_{\frac{T}{2} \wedge t} \middle| \mathcal{F}_t\right](\omega) = \omega_{\frac{T}{2} \wedge t}.$$

As in Example II.1.5, the function u is a natural candidate as a solution of the path dependent heat equation with the terminal condition $u(T, \omega) = \omega_{\frac{T}{2}}$. According to the calculus in (II.1.6), the vertical derivate can be easily computed as $\partial_\omega u(t, \omega) = 1_{\{t \leq \frac{T}{2}\}}$, and is clearly not continuous. Therefore, function u is not in class $C^{1,2}$ in Dupire's sense.

If we call $C^{1,2}$ (in Dupire's sense) solutions are classical, then in general the path dependent heat equation probably has no classical solution. It is necessary to develop a theory of weak solutions.

1.3.1 Sobolev solutions of path dependent PDEs

It is worth recalling the theory of backward stochastic differential equations (BSDE). An interesting subclass of fully nonlinear parabolic equation (II.1.1) is the class of semilinear equations of the form

$$\partial_t u + \frac{1}{2} D^2 u = f(t, x, u, Du), \quad u(T, \cdot) = h. \tag{II.1.8}$$

By taking the same generator function f and terminal condition h , we can generate a BSDE :

$$dY_t = f(t, W_t, Y_t, Z_t)dt + Z_t dW_t, \quad Y_T = h(W_T). \quad (\text{II.1.9})$$

A solution of the above BSDE is a pair of adapted process (Y, Z) in the L^2 space, i.e.

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} Y_t^2\right] < \infty, \quad \mathbb{E}\left[\int_0^T Z_s^2 ds\right] < \infty.$$

There is a strong connection between the semilinear PDE (II.1.8) and the BSDE (II.1.9). By assuming that (II.1.8) has a classical solution u , then by using the Itô formula one may verify that $Y_t = u(t, W_t)$, $Z_t = Du(t, W_t)$ is a solution to the BSDE (II.1.9). Indeed, as it is pointed out in Barles & Lesigne [3] and also in Bally & Matoussi [11], solutions of BSDEs resemble Sobolev solutions in the classical PDE analysis. Both of them weaken the differentiability requirement for solutions. One of the advantages of Sobolev solutions is that one may prove the wellposedness (existence and uniqueness) by a fix point argument, after finding an appropriate Sobolev space, and that is how Pardoux and Peng proved the wellposedness for BSDEs in [89].

An important progress introduced by BSDEs is that they are naturally defined in a path dependent setting. Consider a generator function $F : (t, \omega, y, z) \mapsto \mathbb{R}$ and a terminal condition $h : \Omega \rightarrow \mathbb{R}$. Slightly different to (II.1.9), we are concerned with a BSDE

$$dY_t = F(t, W, Y_t, Z_t)dt + Z_t dW_t, \quad Y_T = h(W). \quad (\text{II.1.10})$$

A solution of (II.1.10) can be considered as a Sobolev solution of the following semilinear path dependent PDE :

$$\partial_t u + \frac{1}{2} \partial_{\omega\omega}^2 u = F(t, \omega, u, Du), \quad u(T, \cdot) = h.$$

It is noteworthy that there are fully nonlinear analogs of BSDEs. We refer to Cheridito, Soner, Touzi and Victoir [16] as well as Soner, Touzi and Zhang [107] for the theory of 2BSDE. We also refer to Hu, Ji and Peng [68] for a similar generalization developed independently, called G-BSDE.

1.3.2 Viscosity solutions of path dependent PDEs

There is another weak solution approach in the classical PDE theory, that is, the viscosity solution. Instead of demanding the solution to be regular and satisfy the equation, we introduce

the notion of test functions. Denote $Q := [0, T) \times \mathbb{R}^d$. We consider the standard PDE :

$$-\partial_t v(t, x) - g(t, x, v(t, x), Dv(t, x), D^2v(t, x)) = 0, \quad t < T, \quad x \in \mathbb{R}^d. \quad (\text{II.1.11})$$

For $(t, x) \in Q$, $u \in \text{USC}(Q)$, and $v \in \text{LSC}(Q)$, we denote :

$$\underline{A}u(t, x) := \left\{ \varphi \in C^{1,2}(Q) : (\varphi - u)(t, x) = \min_Q(\varphi - u) \right\}, \quad (\text{II.1.12})$$

$$\overline{A}v(t, x) := \left\{ \varphi \in C^{1,2}(Q) : (\varphi - v)(t, x) = \max_Q(\varphi - v) \right\}. \quad (\text{II.1.13})$$

Definition II.1.7. (i) $u \in \text{USC}(Q)$ is a viscosity subsolution of equation (II.1.11) if :

$$\left\{ -\partial_t \varphi - g(\cdot, u, D\varphi, D^2\varphi) \right\}(t, x) \leq 0 \quad \text{for all } (t, x) \in Q, \quad \varphi \in \underline{A}u(t, x).$$

(ii) $v \in \text{LSC}(Q)$ is a viscosity supersolution of equation (II.1.11) if :

$$\left\{ -\partial_t \varphi - g(\cdot, u, D\varphi, D^2\varphi) \right\}(t, x) \geq 0 \quad \text{for all } (t, x) \in Q, \quad \varphi \in \overline{A}v(t, x).$$

(iii) A viscosity solution of (II.1.11) is a viscosity subsolution and supersolution of (II.1.11).

The first different feature between a Sobolev solution and a viscosity solution is that a Sobolev solution is an object in a Sobolev space, while a viscosity solution is a pointwise defined function. An amazing feature of viscosity solutions is that the definition is only concerned with some local properties of the functions. Consequently, it is usually easy to verify that a function is a viscosity solution to the corresponding equation. This is in particular useful in the optimal control theory.

This thesis is devoted to developing a theory of viscosity solutions to path dependent PDEs. The main difficulty in proving the wellposedness of viscosity solutions is the so-called comparison principal, i.e. let u be a viscosity subsolution and v be a viscosity supersolution, and then we expect

$$u(T, \cdot) \leq v(T, \cdot) \quad \Rightarrow \quad u \leq v \text{ everywhere.}$$

In the classical PDE theory, the argument to prove the comparison result relies a lot on the fact that the space is locally compact. However, the path space is not locally compact, and it causes real difficulties for our project.

It is noteworthy that there is effort to introduce viscosity solutions to path dependent PDEs in the existing literature. For example, Lukoyanov introduced a theory of viscosity solutions to

first order path dependent PDEs in [82]. In this special case, he reduced the path space, and only considered the absolutely continuous paths with uniformly bounded densities. Consequently, he is again concerned with a compact state space. However, for general second order path dependent PDEs, this trick does no longer work.

To overcome this trouble, it is reasonable to modify a little the definition of viscosity solutions. In the work of Ekren, Touzi and Zhang [37], they introduced a new definition of viscosity solutions for path dependent PDEs, by enlarging the family of test functions. A simple but crucial observation is that by introducing more test functions, we would have fewer viscosity sub-/super-solutions, and thus the comparison result should be easier to prove.

In the next section, we will introduce the definition of viscosity solutions to path dependent PDEs.

2 Definition of viscosity solutions to path dependent PDEs

2.1 Notations

Let $\Omega := \{\omega \in C^0([0, T], \mathbb{R}^d) : \omega_0 = 0\}$ be the canonical space of continuous paths starting from the origin, B the canonical process defined by $B_t(\omega) := \omega_t$, $t \in [0, T]$, and $\mathbb{F} := \{\mathcal{F}_t, t \in [0, T]\}$ the corresponding filtration. Following Dupire [34], we introduce the pseudo-distance

$$d((t, \omega), (t', \omega')) := |t - t'| + \|\omega_{\wedge t} - \omega'_{\wedge t'}\|_{\infty} \quad \text{for all } t, t' \in [0, T], \omega, \omega' \in \Omega. \quad (\text{II.2.1})$$

Then, any process $u : [0, T] \times \Omega \rightarrow \mathbb{R}$, continuous with respect to d , is \mathbb{F} -progressively measurable, in particular $u(t, \omega) = u(t, (\omega_s)_{s \leq t})$. We denote by \mathcal{T} the set of all \mathbb{F} -stopping times, $\mathcal{T}^+ \subset \mathcal{T}$ the collection of all strictly positive stopping times, and $\mathcal{T}^t \subset \mathcal{T}$ the subset of the \mathbb{F} -stopping times larger than t .

For $\omega, \omega' \in \Omega$ and $t \in [0, T]$, we define

$$(\omega \otimes_t \omega')_s := \omega_s \mathbf{1}_{\{s < t\}} + (\omega_t + \omega'_{s-t}) \mathbf{1}_{\{s \geq t\}}.$$

Let $\xi : \Omega \rightarrow \mathbb{R}$ be \mathcal{F}_T -measurable random variable. For any $(t, \omega) \in \Theta$, define

$$\xi^{t, \omega}(\omega') := \xi(\omega \otimes_t \omega') \quad \text{for all } \omega' \in \Omega.$$

Clearly, $\xi^{t, \omega}$ is \mathcal{F}_{T-t} -measurable, and thus \mathcal{F}_T -measurable. Similarly, given a process X defined

on Ω , we denote :

$$X_s^{t,\omega}(\omega') := X_{t+s}(\omega \otimes_t \omega'), \text{ for } s \in [0, T - t].$$

Clearly, if X is \mathbb{F} -adapted, then so is $X^{t,\omega}$.

Let \mathcal{P} be a family of probability measures on Ω . We also introduce the sublinear and super-linear expectation operators associated to \mathcal{P} :

$$\bar{\mathcal{E}}^{\mathcal{P}} := \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \quad \text{and} \quad \underline{\mathcal{E}}^{\mathcal{P}} := \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}.$$

2.2 Differentiability

In Section 1.2, we already discussed the Dupire derivative and the corresponding Itô formula. Indeed, in the current theory of viscosity solution to path dependent PDE, the $C^{1,2}$ smoothness of the test functions is only needed in order to apply the Itô formula. Therefore, we may define the smooth processes directly :

Definition II.2.1 (Smooth processes). *Let \mathcal{P} be a set of probability measures on Ω with B a \mathbb{P} -semimartingale for all $\mathbb{P} \in \mathcal{P}$. We say that $u \in C_{\mathcal{P}}^{1,2}(\Theta)$ if $u \in C^0(\bar{\Theta})$ and there exist processes $\alpha, Z, \Gamma \in C^0(\Theta)$ valued in \mathbb{R}, \mathbb{R}^d and \mathbb{S}_d , respectively, such that :*

$$du_t = \alpha_t dt + \frac{1}{2} \Gamma_t : d\langle B \rangle_t + Z_t dB_t, \quad \mathbb{P} - a.s. \text{ for all } \mathbb{P} \in \mathcal{P}.$$

The processes α, Z and Γ are called the time derivative, spacial gradient and spacial Hessian, respectively, and we denote $\partial_t u := \alpha, \partial_{\omega} u_t := Z_t, \partial_{\omega\omega}^2 u_t := \Gamma_t$.

We observe that any $C^{1,2}$ process in the Dupire sense is in $C_{\mathcal{P}}^{1,2}(\Theta)$. In particular, our notion of smooth processes is weaker than the corresponding one in Dupire [34]. We also note that, when \mathcal{P} is rich enough, our path derivatives are unique.

Remark II.2.2. The previous definition does not require that $\partial_{\omega\omega}^2 u_t$ be the derivative (in some sense) of $\partial_{\omega} u_t$. This is very well illustrated by the following example. Let $d = 2$, and $u_t := \int_0^t B_s^1 dB_s^2$ which is defined pathwise due to the results of Karandikar [70].

- Clearly $\partial_t u = 0$. Since $du_t = B_t^1 dB_t^2$, under any semimartingale measure, we also deduce that $\partial_{\omega} u_t = (0, B_t^1)^{\top}$, and $\partial_{\omega\omega}^2 u_t = 0_{\mathbb{S}_2}$. Hence $u \in C_{\mathcal{P}}^{1,2}(\Theta)$ for any subset \mathcal{P} of the collection of all semimartingale measures for B .

- Let $\partial_{\omega}^D u_t$ and $\partial_{\omega\omega}^{D^2} u_t$ denote the vertical first and second derivatives in the Dupire sense. Direct

calculation reveals that $\partial_\omega^D u_t = (0, B_t^1)^T = \partial_\omega u_t$. However,

$$\partial_{\omega\omega}^{D^2} u_t = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

which is not symmetric!

- However, we need to point out that in this example u does not belong to $C^0(\bar{\Theta})$.

2.3 Definition of Viscosity Solutions

We study the path dependent PDE :

$$-\partial_t u(t, \omega) - G(t, \omega, u(t, \omega), \partial_\omega u(t, \omega), \partial_{\omega\omega}^2 u(t, \omega)) = 0, \quad t < T, \quad \omega \in \Omega. \quad (\text{II.2.2})$$

with boundary condition $u(T, \omega) = \xi(\omega)$. Here, $G : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \rightarrow \mathbb{R}$ is Lipschitz-continuous in the variables (y, z, γ) , and satisfies the ellipticity condition :

$$\gamma \in \mathbb{S}_d \mapsto G(t, \omega, y, z, \gamma) \quad \text{is non-decreasing.} \quad (\text{II.2.3})$$

The unknown process $u(t, \omega)$ is required to be \mathbb{F} -progressively measurable.

2.3.1 Definition via test functions

We introduce the sets of test processes :

$$\begin{aligned} \underline{\mathcal{A}}^{\mathcal{P}} u_t(\omega) &:= \left\{ \varphi \in C_{\mathcal{P}}^{1,2}(\Theta) : (\varphi - u^{t,\omega})_0 = \min_{\tau \in \mathcal{T}_{H_\epsilon}} \underline{\mathcal{E}}^{\mathcal{P}} [(\varphi - u^{t,\omega})_\tau] \text{ for some } \epsilon > 0 \right\}, \\ \overline{\mathcal{A}}^{\mathcal{P}} v_t(\omega) &:= \left\{ \varphi \in C_{\mathcal{P}}^{1,2}(\Theta) : (\varphi - v^{t,\omega})_0 = \max_{\tau \in \mathcal{T}_{H_\epsilon}} \overline{\mathcal{E}}^{\mathcal{P}} [(\varphi - v^{t,\omega})_\tau] \text{ for some } \epsilon > 0 \right\}, \end{aligned} \quad (\text{II.2.4})$$

where \mathcal{T}_{H_ϵ} denotes all stopping times smaller than $H_\epsilon := \epsilon \wedge \inf\{t \geq 0 : \|B_t\| \geq \epsilon\}$. Later, we will call H_ϵ the localizing stopping time (or the localization) of the corresponding test process φ .

Definition II.2.3 (Viscosity solution of path-dependent PDE). *Let $u, v \in \mathbb{L}^0(\mathbb{F})$.*

(i) u is a \mathcal{P} -viscosity subsolution of (II.2.2) if :

$$\left\{ -\partial_t \varphi - G(\cdot, u, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi) \right\}(t, \omega) \leq 0 \quad \text{for all } (t, \omega) \in \Theta, \quad \varphi \in \underline{\mathcal{A}}^{\mathcal{P}} u_t(\omega).$$

(ii) v is a \mathcal{P} -viscosity supersolution of (II.2.2) if :

$$\left\{ -\partial_t \varphi - G(\cdot, v, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi) \right\}(t, \omega) \geq 0 \quad \text{for all } (t, \omega) \in \Theta, \varphi \in \overline{\mathcal{A}}^{\mathcal{P}} v_t(\omega).$$

(iii) A \mathcal{P} -viscosity solution of (II.2.2) is both a \mathcal{P} -subsolution and a \mathcal{P} -supersolution.

Remark II.2.4. In the Markovian case, we may as well use the last definition as an alternative to the standard notion of viscosity solutions. Compared to the standard notion reviewed in Section 1.3.2, we see that any $\phi \in \underline{A}u(t, x)$ induces a process $\varphi(t, \omega) := \phi(t, \omega_t)$ which obviously lies in $\underline{\mathcal{A}}^{\mathcal{P}} u_t(\omega)$. However, even in the Markovian case $u_t(\omega) = u(t, \omega_t)$, a test process in $\underline{\mathcal{A}}^{\mathcal{P}} u_t(\omega)$ does not necessarily induce a test function in $\underline{A}u(t, \omega_t)$. Thus, our notion of viscosity solution involves more test functions than the standard notion. A viscosity subsolution/supersolution in sense of Definition II.2.3 is restricted by a richer family of test functions. Consequently :

- under our definition, we may hope to take advantage of the richer family of test functions in order to obtain an easier uniqueness proof,
- under our definition, the existence problem is more restricted than under the standard theory of viscosity solutions.

Remark II.2.5. Due to the localization, the viscosity property introduced in Definition II.2.3 is a local property. Indeed, in order to check the viscosity property of u at (t, ω) , it suffices to know the value of $u^{t, \omega}$ on $[0, \mathbb{H}_\varepsilon]$ for an arbitrarily small $\varepsilon > 0$. In particular, since u and φ are locally bounded, there is no integrability issue in the definition of the set of test functions $\underline{\mathcal{A}}^{\mathcal{P}}$ and $\overline{\mathcal{A}}^{\mathcal{P}}$.

2.3.2 Semijets definition

Similar to the standard notion of viscosity solutions in finite-dimensional spaces, we will now prove that we may reduce our Definition II.2.3 to paraboloids :

$$\phi_s^{q,p,\gamma}(\omega) := qs + p \cdot \omega_s + \frac{1}{2} \gamma : \omega_s \omega_s^T, \quad s \in [0, T - t], \quad \omega \in \Omega,$$

for some $(q, p, \gamma) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d$. We then introduce the corresponding subjet and superjet :

$$\begin{aligned} \underline{\mathcal{J}}^{\mathcal{P}} u_t(\omega) &:= \left\{ (q, p, \gamma) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d : \phi^{q,p,\gamma} \in \underline{\mathcal{A}}^{\mathcal{P}} u_t(\omega) \right\}, \\ \overline{\mathcal{J}}^{\mathcal{P}} v_t(\omega) &:= \left\{ (q, p, \gamma) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d : \phi^{q,p,\gamma} \in \overline{\mathcal{A}}^{\mathcal{P}} v_t(\omega) \right\}. \end{aligned} \tag{II.2.5}$$

The following proposition can be easily proved, and show that we may give an equivalent definition of viscosity solutions to path dependent PDEs via the jets.

Proposition II.2.6. *A process $u \in C^0(\bar{\Theta})$ is a \mathcal{P} -viscosity subsolution of (II.2.2) if and only if :*

$$-q - G(t, \omega, u_t(\omega), p, \gamma) \leq 0 \quad \text{for all } (t, \omega) \in \Theta, (q, p, \gamma) \in \underline{\mathcal{J}}^p u_t(\omega). \quad (\text{II.2.6})$$

The corresponding statement holds for supersolutions.

3 Main contribution of the thesis

3.1 Comparison result for semilinear path dependent PDEs

In the work of Ekren, Touzi and Zhang [38], the authors proved the wellposedness of viscosity solutions to fully nonlinear path dependent PDEs. In their approach, they use a path-frozen argument so as to approximate path dependent PDEs by PDEs in small intervals. However, this approximation only works under certain technical conditions, which exclude some interesting applications, for example, the Hamilton-Jacobi-Bellman equations in the general form are not totally covered. This leads to the exploration for new arguments. It is reasonable to start from semilinear path dependent equations. In the existing literature, Ekren, Keller, Touzi and Zhang [35] proved the comparison result in the semilinear case under more general conditions through a probabilistic representation of solutions. However, this approach can hardly be generalized to the fully nonlinear case, because there is no such representation for general fully nonlinear equations. It is worth mentioning that in both arguments above, we obtain the comparison result and the existence of viscosity solutions simultaneously. Conversely, in the classical theory of viscosity solutions to PDEs, we have separated arguments for the two problems. In particular, this separation of results can release some unnecessary conditions for the comparison result.

A promising progress occurs in the study of semilinear path dependent PDEs in [102]. As in the PDE theory, we separate the arguments for the comparison result and the existence of viscosity solutions. The new argument exploits the enlarged family of test functions, and simplifies the proof of the comparison result. Our proof by-passes completely the delicate and deep Crandall and Ishii Lemma (see Lemma 3.2 in [21]). In particular, our proof of comparison result for the path-dependent heat equation is elementary, and does not require any penalization to address (the standard comparison result for second order PDEs applies to a bounded domain, the extension to an unbounded domain involves a penalization using the growth conditions). The wellposedness of the path-dependent heat equation is a direct consequence of the equivalence between the viscosity subsolution and the submartingale properties.

Recall that the definition of viscosity solutions to path dependent PDEs depends on the chosen family of probability measures \mathcal{P} . In the semilinear case, we set \mathcal{P} to be

$$\mathcal{P}_L := \left\{ \mathbb{P}_\lambda : \frac{d\mathbb{P}_\lambda}{d\mathbb{P}_0} = \exp \left(\int_0^T \lambda_t \cdot dB_t - \frac{1}{2} \int_0^T |\lambda_t|^2 dt \right), \lambda \in \mathbb{L}^0(\mathbb{F}), \|\lambda\|_\infty \leq L \right\},$$

where \mathbb{P}_0 is the Wiener measure and $\mathbb{L}^0(\mathbb{F})$ is the set of all \mathbb{F} -progressively measurable processes. Our arguments are inspired from the work of Caffarelli and Cabré [15]. With the definition of jets (in Section 2.3.2), we may introduce the key ingredient of our arguments, the punctual differentiation, i.e.

function u is \mathcal{P} -punctually $C^{1,2}$ at (t, ω) , if $\text{cl}(\bar{\mathcal{J}}^{\mathcal{P}} u(t, \omega)) \cap \text{cl}(\underline{\mathcal{J}}^{\mathcal{P}} u(t, \omega)) \neq \emptyset$.

It justifies the name that a function in class $C^{1,2}$ in Dupire's sense is punctually differentiable at every point (t, ω) . Further, we prove an important smoothness result.

Proposition II.3.1. *Assume u is a \mathbb{P}_0 -semimartingale with decomposition $du_t = Z_t \cdot dB_t + dA_t$, such that $\mathbb{E}^{\mathbb{P}_0} \left[\int_0^T Z_t^2 dt \right] < \infty$ and $A \in \mathbb{L}^0(\mathbb{F})$ is continuous and has finite variation, \mathbb{P}_0 -a.s. Then there exist a Borel set $\mathbb{T}^u \subset [0, T]$ and $\Omega_t^u \in \mathcal{F}_t$ for each $t \in \mathbb{T}^u$ such that, for any $L > 0$,*

$$\text{Leb}(\mathbb{T}^u) = T, \quad \mathbb{P}_0(\Omega_t^u) = 1, \quad \text{and } u \text{ is } \mathcal{P}_L\text{-punctually } C^{1,2} \text{ at } (t, \omega) \text{ for all } t \in \mathbb{T}^u, \omega \in \Omega_t^u.$$

This result can be viewed as the analogue of the Aleksandroff regularity result for convex functions. In the present semilinear case, an important property of our notion of viscosity solutions is that viscosity subsolutions (resp. supersolutions) are submartingales (resp. supermartingales) up to the addition of some absolutely continuous process. In particular, viscosity subsolutions and supersolutions are punctually differentiable $\text{Leb} \otimes \mathbb{P}$ -a.e. The regularity result leads to the final comparison result, i.e., under some general condition, we proved :

Theorem II.3.2. *Let u, v be continuous \mathcal{P} -viscosity subsolution and supersolution, respectively, of a semilinear path dependent PDE. If $u_T \leq v_T$ on Ω , then $u \leq v$ on Θ .*

We refer to Chapter III for the details.

3.2 Perron's method for semilinear path dependent PDEs

While the uniqueness of viscosity solution is simply implied by the comparison result, it remains to prove the existence. In [35, 102], the authors proved that in the semilinear case, the solutions of corresponding backward SDEs are viscosity solutions. Instead, we are interested

in proving the existence of viscosity solutions by PDE-type arguments, namely by Perron's method.

In the recent paper [98], we show how to adapt the Perron method to the context of path dependent PDEs. Recall our canonical setting $(\Omega, \mathbb{F}, \mathbb{P}_0)$, and denote $\Theta = [0, T] \times \Omega$. A function $u : \Theta \rightarrow \mathbb{R}$ belongs to USC_b (resp. LSC_b), if u is bounded and satisfies

$$u(\theta) \geq \limsup_{d(\theta, \theta') \rightarrow 0} u(\theta') \quad (\text{resp. } \leq \liminf_{d(\theta, \theta') \rightarrow 0} u(\theta')).$$

Under some general conditions, we proved :

Theorem II.3.3. *Assume that there is a viscosity subsolution $\underline{u} \in \text{USC}_b(\Theta)$ and a supersolution $\bar{v} \in \text{LSC}_b(\Theta)$ of a semilinear path dependent PDE, satisfying the boundary condition $(\underline{u}_*)_T = \bar{v}_T^* = \xi$. Denote*

$$\mathcal{D} := \left\{ \phi : \phi \in \text{USC}_b(\Theta) \text{ is a viscosity subsolution of the path dependent PDE and } \underline{u} \leq \phi \leq \bar{v} \right\}.$$

Then $u(\theta) := \sup\{\phi(\theta) : \phi \in \mathcal{D}\}$ is a continuous viscosity solution of the semilinear path dependent PDE, and satisfies the boundary condition $u_T = \xi$.

Although our result, which concerns the semilinear path dependent PDEs, cannot be applied to the fully nonlinear case directly, many arguments in [98] could be useful for further research. Also, the Perron method is not only useful in proving the existence of viscosity solutions, but also has applications in various contexts, for example, the wellposedness of envelope viscosity solution (see [2]), the uniqueness of martingale problems [18], etc. In the proof of Perron's method, we follow the same idea as the classical literature on viscosity solutions of PDEs, but the arguments turn out to be different and nontrivial.

It is well understood in PDE literature that the comparison result for continuous viscosity solutions is not sufficient for the existence of solutions. In Perron's method, we need a comparison result for semicontinuous viscosity solutions. However, the argument in [102] cannot be adapted into our context, because it is not clear whether upper semicontinuous submartingales are almost everywhere punctually differentiable (a crucial intermediate result in [102]). In [98], we apply a regularization on semicontinuous viscosity solutions so as to mollify them to be continuous. Let u be a viscosity subsolution, and u^n be its regularized version. A reasonable regularization should satisfy :

$$u^n \text{ is continuous; } \quad u^n \rightarrow u, \text{ as } n \rightarrow \infty; \quad u^n \text{ is still a viscosity subsolution.}$$

The regularization we propose involves a backward distance for paths, is new in literature, satisfies all the above conditions and helps to prove the comparison result. It is worth mentioning that a regularization is probably inevitable in the study of the comparison result for fully nonlinear path dependent PDEs. The regularization we find in [98] might shed light on the future research.

Further, as in all work on the viscosity solutions of path dependent PDEs, the optimal stopping result plays a crucial role to overcome the non-local-compactness of the path space. Since we treat semicontinuous viscosity solutions in [98], we need the corresponding result of optimal stopping under nonlinear expectation for semicontinuous obstacles. In the existing literature, Kobylanski and Quenez [74] contains the desired result but only in the case of linear expectation. Peng and Xu studied in [95] reflected backward SDEs with L^2 obstacles, and they proved a crucial intermediate result which can lead to the optimal stopping result. However, since their main interest is reflected backward SDEs, there is no direct theorem that we may apply. In [98], we give a new simple proof for the optimal stopping problem, by using the minimum condition of the Skorokhod decomposition.

Chapter IV is devoted to the development of the Perron method.

3.3 Monotone scheme for path dependent PDEs

Besides the wellposedness, another interesting topic is to compute viscosity solutions to path dependent PDEs numerically. In their seminal work [4], Barles and Souganidis proved a convergence theorem for monotone numerical schemes for viscosity solutions of fully nonlinear PDEs. Denote by \mathbb{T}_h a numerical scheme with time-step h , namely the numerical solution u^h satisfying

$$u^h(t, x) := \mathbb{T}_h^{t,x}[u^h(t+h, \cdot)].$$

A so-called monotone scheme satisfies the following properties :

(i) Consistency : for any $(t, x) \in [0, T) \times \mathbb{R}^d$ and any smooth function $\varphi \in C^{1,2}([0, T) \times \mathbb{R}^d)$,

$$\lim_{(t', x', h, c) \rightarrow (t, x, 0, 0)} \frac{(c + \varphi)(t', x') - \mathbb{T}_h^{t', x'}[(c + \varphi)(t' + h, \cdot)]}{h} = \mathbb{L}\varphi(t, x).$$

(ii) Monotonicity : $\mathbb{T}_h^{t,x}[\varphi] \leq \mathbb{T}_h^{t,x}[\psi]$ whenever $\varphi \leq \psi$.

(iii) Stability : u^h is bounded uniformly in h whenever g is bounded.

(iv) Boundary condition : $\lim_{(t', x', h) \rightarrow (T, x, 0)} u^h(t', x') = g(x)$ for any $x \in \mathbb{R}^d$.

Assuming that the comparison principle holds true for viscosity solutions of a PDE, Barles and

Souganidis showed that

$$u := \lim_{h \rightarrow 0} u^h \text{ is the unique viscosity solution to the PDE.}$$

They mainly use the stability of viscosity solutions of PDEs and the local compactness of the state space. Due to their result, one only needs to check some local properties of a numerical scheme in order to get a global convergence result.

It would be interesting to extend the convergence theorem in [4] to the context of path dependent PDE. The main obstacle for a direct extension of their arguments is that the state space is no longer locally compact. Zhang and Zhuo [114] provided recently a formulation of the convergence theorem of monotone schemes for path dependent PDEs. Roughly speaking, by denoting by

$$u^h(t, \omega) := \mathbb{T}_h^{t, \omega}[u^h(t+h, \cdot)]$$

a path dependent numerical schema they modified the monotonicity condition as :

$$(ii') \quad \mathbb{T}_h^{t, \omega}[\varphi] \leq \mathbb{T}_h^{t, \omega}[\psi] \text{ whenever } \bar{\mathcal{E}}^{\mathcal{P}}[(\varphi - \psi)^{t, \omega}] \leq 0,$$

where \mathcal{P} is the family of probability measures in the definition of viscosity solutions to path dependent PDEs. Their main result is the same convergence theorem as in Barles and Souganidis [4] :

$$u := \lim_{h \rightarrow 0} u^h \text{ is the unique viscosity solution to the path dependent PDE.}$$

They mainly use the stability of viscosity solutions to path dependent PDEs, and overcome the difficulty of non-local compactness by an optimal stopping argument as in the wellposedness theory. They also provide an illustrative numerical scheme which satisfies all the conditions of their convergence theorem. However, this illustrative numerical scheme is not applicable in general cases. Moreover, most of the monotone numerical schemes in the sense of Barles and Souganidis [4], for example the finite difference scheme, do not satisfy their conditions, in particular, the condition (ii').

Our recent work [100] provides a new formulation of the convergence theorem for numerical schemes of path dependent PDE. Instead of condition (ii) or (ii'), we introduce a new condition :

(ii'') there exists a nonlinear expectation $\underline{\mathcal{E}}_h$ satisfying certain conditions such that for any $\varphi, \psi \in \mathbb{L}^0(\mathcal{F}_{t+h})$, it holds that

$$\mathbb{T}_h^{t, \omega}[\varphi] - \mathbb{T}_h^{t, \omega}[\psi] \geq \inf_{0 \leq \alpha \leq L} \underline{\mathcal{E}}_h[e^{\alpha h}(\varphi - \psi)^{t, \omega}] - h\rho(h).$$

Then we prove the convergence theorem :

Theorem II.3.4. *Assume that*

- *the nonlinearity G of path dependent PDE (II.2.2) is elliptic (see (II.2.3)),*
- *the nonlinearity G and the terminal condition ξ are continuous in all arguments, and $G(t, \omega, y, z, \gamma)$ is uniformly Lipschitz in y ,*
- *the comparison principle of viscosity solutions of (II.2.2) holds, i.e. if u, v are bounded and uniformly continuous \mathcal{P} -viscosity subsolution and supersolution of (II.2.2), respectively, and $u(T, \cdot) \leq v(T, \cdot)$, then $u \leq v$ on $[0, T] \times \Omega$.*

If the numerical scheme \mathbb{T} satisfies the monotonicity conditions, namely condition (i), (ii'') and (iii), then PPDE (II.2.2) admits a unique bounded viscosity solution u , and

$$u^h \rightarrow u \text{ locally uniformly, as } h \rightarrow 0.$$

Our conditions are slightly stronger than the classical conditions of Barles and Souganidis [4], as path dependent PDEs degenerate to be PDEs. Nevertheless, to the best of our knowledge these conditions are satisfied by all classical monotone numerical schemes in the context of optimal control, including the path dependent finite difference scheme, the Monte-Carlo scheme of Fahim, Touzi and Warin [48], the semi-Lagrangian scheme, the trinomial tree scheme of Guo, Zhang and Zhuo [59], the switching system scheme of Kharroubi, Langrené and Pham [72], etc. Therefore, the result in [100] extends all these numerical schemes to the path-dependent case. In particular, it provides numerical schemes for non-Markovian second order BSDEs, and stochastic differential games, which is new in the literature, see also Possamaï and Tan [97].

We will show the result of [100] in Chapter V.

3.4 Elliptic path dependent PDEs

Recall the stochastic optimal control problem in Example II.1.2. Define a stopping time $\tau(\omega) := \inf\{t \geq 0 : \omega_t \notin O\}$, where O is a bounded open subset of \mathbb{R}^d , and consider the value function $u(x) := \mathbb{E}[h(X_\tau) | X_0 = x]$ for $x \in O$. Then the function u is a candidate solution of the following elliptic equation :

$$\text{Tr}[D^2u] = 0,$$

with the boundary condition $u = h$ on ∂O . Clearly, we may generalize this example to the path dependent case. So, it is interesting to look into elliptic path dependent PDEs.

Our starting point is the BSDE approach of Darling and Pardoux [23] which can be viewed as a theory of Sobolev solutions for semilinear path dependent PDEs. Similar to the parabolic case, the Feynman-Kac representation provided in [23] shows a close connection between Markovian BSDEs with random terminal times and elliptic semilinear PDEs with Dirichlet boundary conditions.

Following the work of Ekren, Touzi and Zhang [37, 38] on parabolic path dependent PDEs, in [99] we try to develop the notion of viscosity solution to elliptic path dependent PDE. Let Q be a bounded, closed and convex subset of \mathbb{R}^d , and \mathcal{Q} be the set of paths taking values in Q . We consider the elliptic path dependent PDEs of the form :

$$G(\cdot, u, \partial_\omega u, \partial_{\omega\omega}^2 u)(\omega) = 0, \quad \omega \in \mathcal{Q}, \quad \text{and} \quad u = \xi \text{ for } \omega \in \partial\mathcal{Q}. \quad (\text{II.3.1})$$

One clear difference between the parabolic equation and the elliptic one is the time-independence of the latter. However, it is unclear how to describe the time-independence in the path dependent case, because the path variable contains information of time. We reduce the path space to the subspace $\Omega^e = \{\omega \in \Omega : \omega = \omega_{t\wedge\cdot} \text{ for some } t\}$ and denote $\bar{t}(\omega) := \sup\{t : \omega_t \neq \omega_\infty\}$. For a function $u : \Omega^e \rightarrow \mathbb{R}$, we introduce $\tilde{u}(t, \omega) := u(\omega_{t\wedge\cdot})$. Recall the Dupire time-derivative :

$$\partial_t \tilde{u}(t, \omega) := \lim_{h \rightarrow 0} \frac{\tilde{u}(t+h, \omega_{t\wedge\cdot}) - \tilde{u}(t, \omega)}{h}.$$

To our understanding, the time independence should imply $\partial_t \tilde{u} = 0$. For this purpose, we introduce a distance on Ω^e :

$$d^e(\omega, \omega') := \inf_{\ell \in I} \sup_t |\omega_t - \omega'_{\ell(t)}|,$$

where $I = \{\ell : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \mid \ell \text{ is an increasing bijection and } \ell(0) = 0\}$. In particular, a function continuous with respect to the distance d^e has the so-called time invariant property :

$$u(\omega) = u(\omega_{\ell(\cdot)}) \quad \text{for all } \omega \text{ and all increasing bijection } \ell : \mathbb{R}^+ \rightarrow \mathbb{R}^+.$$

Loosely speaking, a time invariant map u is unchanged by any time scaling of path. This property implies the desired property $\partial_t \tilde{u} = 0$. It is noteworthy that if we take the time-space path $\bar{\omega}_t = (t, \omega_t)$, then the distance d^e reads

$$d^e(\bar{\omega}, \bar{\omega}') = \inf_{\ell \in I} \sup_t (|t - \ell(t)| + |\omega_t - \omega'_{\ell(t)}|),$$

equivalent to the Skorokhod metric.

The main result of [99] is that under some technical conditions

Theorem II.3.5. *The elliptic path dependent PDE (II.3.1) has a unique viscosity solution.*

In order to prove this wellposedness theorem, we follow the line of argument in [38], namely the path-frozen technique already mentioned in Section 3.1. But we have to address some new difficulties in the present elliptic context. One of the main difficulties is due to the boundary of Dirichlet problem. In particular, since the hitting time of the boundary $H_Q(\omega)$ is not continuous in ω , it is non-trivial to verify the continuity of the viscosity solution constructed through path-frozen PDEs.

Chapter VI of the thesis is devoted to elliptic path dependent PDEs.

3.5 Large deviations for non-Markovian diffusions

After the previous construction of the theory of path dependent PDEs, we also include some of its application in this thesis. The recent work [84] considers the large deviations for non-Markovian diffusions. The theory of large deviations is concerned with the rate of convergence of a vanishing sequence of probabilities $(\mathbb{P}[A_n])_{n \geq 1}$, where $(A_n)_{n \geq 1}$ is a sequence of *rare events*. After convenient scaling and normalization, the limit is called *rate function*, and is typically represented in terms of a control problem. The pioneering work of Freidlin and Wentzell [55] considers rare events induced by Markov diffusions. The techniques are based on the Girsanov theorem for equivalent change of measure, and classical convex duality. An important contribution by Fleming [51] is to use the powerful stability property of viscosity solutions in order to obtain a significantly simplified approach. We refer to Feng and Kurtz [49] for a systematic application of this methodology with relevant extensions.

The main objective of our recent work [84] is to extend the viscosity solutions approach to some problems of large deviations with rare events induced by non-Markov diffusions

$$X_t^\epsilon = X_0 + \int_0^t b_s(X^\epsilon) ds + \int_0^t \sqrt{\epsilon} \sigma_s(X^\epsilon) dW_s, \quad t \geq 0, \quad (\text{II.3.2})$$

where W is a Brownian motion, and b, σ are non-anticipative functions of the paths of X^ϵ satisfying convenient conditions for existence and uniqueness of the solution of the last stochastic differential equation (SDE). We should note that the Large Deviation Principle (LDP) for non-Markovian diffusions of type (II.3.2) is not new. For example, Gao & Liu [56] studied such a problem via the sample path LDP method by Freidlin-Wentzell, using various norms in infinite

dimensional spaces. While the techniques there are quite deep and sophisticated, the methodology is more or less “classical.” Our main focus in this work is to extend the PDE approach of Fleming [51] in the present path-dependent framework, with a different set of tools. These include the theories of backward SDEs, stochastic control, and the viscosity solution for path-dependent PDEs (PPDEs). Specifically, the theory of backward SDEs, pioneered by Pardoux & Peng [89], can be effectively used as a substitute to the partial differential equations in the Markovian setting. Indeed, the log-transformation of the vanishing probability solves a semilinear PDE in the Markovian case. However, due to the “functional” nature of the coefficients in (II.3.2), both backward SDE and PDE involved will become non-Markovian.

We are concerned with two types of asymptotics in [84], the one about the Laplace transform and the one about the exiting time. Under some general conditions, we proved :

Theorem II.3.6 (Laplace transform). *Let ξ be a bounded uniformly continuous \mathcal{F}_T -measurable r.v. Then we have that*

$$L_0^\epsilon := -\epsilon \ln \mathbb{E} \left[e^{-\frac{1}{\epsilon} \xi(X^\epsilon)} \right].$$

converges to L_0 as $\epsilon \rightarrow 0$, where

$$L_0 := \inf_{\alpha \in \mathbb{L}_d^2} \ell_0^\alpha, \quad \ell_0^\alpha := \xi(x^\alpha) + \frac{1}{2} \int_0^T |\alpha_t|^2 dt, \quad (\text{II.3.3})$$

and x^α is defined by the controlled ordinary differential equation :

$$\omega_t^\alpha = \int_0^t \alpha_s ds, \quad x_t^\alpha = X_0 + \int_0^t b_s(x^\alpha) ds + \int_0^t \sigma_s(x^\alpha) d\omega_s^\alpha, \quad t \in [0, T].$$

Theorem II.3.7 (Exiting time). *Let O be a bounded open set in \mathbb{R}^n with C^3 boundary, and define*

$$\mathcal{O} := \left\{ \omega \in \Omega_n : \omega_t \in O \text{ for all } t \leq T \right\}. \quad (\text{II.3.4})$$

Then we have that

$$Q_0^\epsilon := -\epsilon \ln \mathbb{P}^\epsilon[H < T], \quad \text{where } H := \inf\{t > 0 : X_t \notin O\}, \quad (\text{II.3.5})$$

converges to

$$Q_0 := \inf \left\{ q_0^\alpha : \alpha \in \mathbb{L}_d^2, x_{T \wedge \cdot}^\alpha \notin \mathcal{O} \right\}, \quad \text{where } q_0^\alpha := \frac{1}{2} \int_0^T |\alpha_s|^2 ds,$$

and x^α is defined as in Theorem II.3.6.

Several technical points are worth mentioning. First, since the PDE involved in our problem naturally has the nonlinearity in the gradient term (quadratic to be specific), we therefore need the extension by Kobylanski [73] on backward SDEs to this context. Second, in order to obtain the rate function, we exploit the stochastic control representation of the log-transformation, and proceed to the asymptotic analysis with crucial use of the BMO properties of the solution of the BSDE. Finally, we use the notion of viscosity solutions of path-dependent Hamilton-Jacobi equations introduced by Lukoyanov [82] in order to characterize the rate function (a dynamic version of L_0 in (II.3.3)) as the unique viscosity solution of a path dependent Eikonal equation.

Another main purpose, in fact the original motivation, of this work is an application in financial mathematics. It has been known that an important problem in the valuation and hedging of exotic options is to characterize the short time asymptotics of the implied volatility surface, given the prices of European options for all maturities and strikes. The need to resort to asymptotics is due to the fact that only a discrete set of maturities and strikes are available. This difficulty is bypassed by practitioners by using the asymptotics in order to extend the volatility surface to the un-observed regimes, for which we refer to Henry-Labordère [65]. The results available in this literature have been restricted to the Markovian case, and our results in a sense open the door to a general non-Markovian, path-dependent paradigm. We finally observe that the sequence of vanishing probabilities induced by non-Markov diffusions can be re-formulated in the Markov case by using the Gyöngy's [62] result which produces a Markov diffusion with the same marginals. However, the regularity of the coefficients of the resulting Markov diffusion $\sigma^X(t, x) := \mathbb{E}[\sigma_t | X_t = x]$ are in general not suitable for the application of the classical large deviation results.

We refer to Chapter VII for the details on the application on large deviation.

3.6 Dual algorithm for stochastic control problems

The last part of the thesis is to obtain a dual algorithm for stochastic control problems. We will report in Chapter VIII the work in [66].

Solving stochastic control problems, for example by approximating the Hamilton-Jacobi-Bellman equation, is an important problem in applied mathematics. Classical PDE methods are effective tools for solving such equations in low dimensional settings, but quickly become computationally intractable as the dimension of the problem increases : a phenomenon commonly referred to as "the curse of dimensionality". Probabilistic methods on the other hand such as Monte-Carlo simulation are less sensitive to the dimension of the problem.

It was demonstrated in Pardoux & Peng [89], Cheridito, Soner, Touzi & Victoir [16] and Soner, Touzi & Zhang [107] that first and second backward stochastic differential equations (in short BSDE) can provide stochastic representations that may be regarded as a non-linear generalisation of the classical Feynman-Kac formula for semi-linear and fully non-linear second order parabolic PDEs.

The numerical implementation of such a BSDE based scheme associated to a stochastic control problem was first proposed in Bouchard & Touzi [13], also independently in Zhang [113]. Further generalization was provided in Fahim, Touzi & Warin [48] and in Guyon & Henry-Labordère [60]. The algorithm in [60] requires evaluating high-dimensional conditional expectations, which are typically computed using parametric regression techniques. Solving the BSDE yields a sub-optimal estimation of the stochastic control. Performing an additional, independent (forward) Monte-Carlo simulation using this sub-optimal control, one obtains a biased estimation : a lower bound for the value of the underlying stochastic control problem. Choosing the right basis for the regression step is in practice a difficult task, particularly in high-dimensional settings. In fact, a similar situation arises for the familiar Longstaff-Schwarz algorithm, which also requires the computation of conditional expectations with parametric regressions and produces a low-biased estimate. As the algorithm in [60] provides a biased estimate, i.e. a lower bound it is of limited use in practice, unless it can be combined with a dual method that leads to a corresponding upper bound.

Such a dual expression was obtained by Rogers [103], building on earlier work by Davis and Burstein [24]. While the work of Rogers is in the discrete time setting, it applies to a general class of Markov processes. Previous work by Davis and Burstein [24] linking deterministic and stochastic control using flow decomposition techniques (see also Diehl, Friz, Gassiat [32] for a rough path approach to this problem) is restricted to the control of a diffusion in its drift term.

In [66] we are also concerned with the control of diffusion processes, but allow the control to act on both the drift and the volatility term in the diffusion equation. For some $M > 0$, let A be a compact subset of $O_M := \{x \in \mathbb{R}^k : |x| \leq M\}$ for some $k \in \mathbb{N}$, and denote $\mathcal{A} := \left\{ \alpha : \Theta \rightarrow \mathbb{R}^k : \alpha \text{ is } \mathbb{F}\text{-adapted, and takes values in } A \right\}$. Consider the value function of a stochastic control problem :

$$u_0 = \sup_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_0^T e^{-\int_0^t r(s, \alpha_s, X_s^\alpha) ds} f(t, \alpha_t, X_t^\alpha) dt + e^{-\int_0^T r(s, \alpha_s, X_s^\alpha) ds} g(X_T^\alpha) \right],$$

where X^α is a d -dimensional controlled diffusion defined by

$$X^\alpha := \int_0^\cdot \mu(t, \alpha_t, X_t^\alpha) dt + \int_0^\cdot \sigma(t, \alpha_t, X_t^\alpha) dW_t,$$

Let N^h be a finite h -net of A , i.e. for all $x, y \in N^h \subset A$, we have $|x - y| \leq h$, and denote $\mathcal{D}_h := \left\{ a : [0, T] \rightarrow \mathbb{R}^k : a \text{ is constant on } [t_i^h, t_{i+1}^h) \text{ for } i, \text{ and takes values in } N^h \right\}$. Our dual result reads $u_0 = \lim_{h \rightarrow 0} v^h$, where

$$v^h := \inf_{\varphi \in \mathcal{U}} \mathbb{E} \left[\sup_{a \in \mathcal{D}_h} \left\{ e^{-\int_0^T r(s, a_s, X_s^a) ds} g(X_T^a) + \int_0^T e^{-\int_0^t r(s, a_s, X_s^a) ds} f(t, a_t, X_t^a) dt - \int_0^T e^{-\int_0^t r(s, a_s, X_s^a) ds} \varphi_t(X^a)^T \sigma(t, a_t, X_t^a) dW_t \right\} \right].$$

The basic idea underlying the dual algorithm is to replace the stochastic control by a pathwise deterministic family of control problems (see the maximization in the expectation in the definition of v^h). The resulting "gain" of information is compensated by introducing a penalisation analogous to a Lagrange multiplier. In contrast to [24], [32] we do not consider continuous pathwise, i.e. deterministic, optimal control problems. Instead, we rely on a discretisation result for the HJB equation due to Krylov [76] and recover the solution of the stochastic control problem as the limit of deterministic control problems over a finite set \mathcal{D}_h of discretised controls. Inspired by the 'path-frozen' technique introduced in Ekren, Touzi and Zhang [38], we also generalize the dual result to the path dependent case, i.e. under some general conditions we proved :

Theorem II.3.8. *Consider the stochastic control problem :*

$$u_0 = \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}_0} \left[g(X_{T \wedge \cdot}^\alpha) \right],$$

where X^α is a d -dimensional diffusion defined by $X^\alpha := \int_0^\cdot \mu(t, \alpha_t) dt + \int_0^\cdot \sigma(t, \alpha_t) dB_t$. Then we have

$$u_0 = \lim_{h \rightarrow 0} v^h, \text{ where } v^h := \inf_{\varphi \in \mathcal{U}} \mathbb{E}^{\mathbb{P}_0} \left[\sup_{a \in \mathcal{D}_h} \left\{ g(X_{T \wedge \cdot}^a) - \int_0^T \varphi_t^T(X^a) \sigma(t, a_t) dB_t \right\} \right].$$

Chapitre III

Semilinear path dependent PDE : Comparison for continuous viscosity solutions

1 Notations

We briefly recall some notations already used in Chapter II and introduce some more for the study of semilinear path dependent PDEs.

Let $T > 0$ be a given finite maturity, $\Omega := \{\omega \in C([0, T]; \mathbb{R}^d) : \omega_0 = 0\}$ be the set of continuous paths starting from the origin, $\Theta := [0, T] \times \Omega$ and $\bar{\Theta} = [0, T] \times \Omega$. We denote B as the canonical process on Ω , $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ as the natural filtration, \mathcal{T} as the set of all \mathbb{F} -stopping times taking values in $[0, T]$. Further let \mathcal{T}^+ denote the subset of $\tau \in \mathcal{T}$ taking values in $(0, T]$, and for $\mathbb{H} \in \mathcal{T}$, let $\mathcal{T}_{\mathbb{H}}$ and $\mathcal{T}_{\mathbb{H}}^+$ be the subset of $\tau \in \mathcal{T}$ taking values in $[0, \mathbb{H}]$ and in $(0, \mathbb{H}]$, respectively. We also denote \mathbb{P}_0 as the Wiener measure on Ω , and define the augmented filtration by $\mathbb{F}^* := \{\mathcal{F}_t \vee \mathcal{N}; 0 \leq t \leq T\}$, where \mathcal{N} is the collection of all \mathbb{P}_0 -null sets.

Defining $\|\omega\| := \sup_{0 \leq s \leq T} |\omega_s|$ and $\|\omega\|_t := \sup_{0 \leq s \leq t} |\omega_s|$ for $t \in [0, T]$, we introduce the following pseudo-distance on Θ :

$$d(\theta, \theta') := |t - t'| + \|\omega_{t \wedge} - \omega'_{t' \wedge}\| \quad \text{for all } \theta = (t, \omega), \theta' = (t', \omega') \in \Theta.$$

We say a process valued in some metric space E is in $C^0(\Theta, E)$ whenever it is continuous with respect to d . Similarly, $\mathbb{L}^0(\mathcal{F}, E)$ and $\mathbb{L}^0(\mathbb{F}, E)$ denote the set of \mathcal{F} -measurable random variables and \mathbb{F} -progressively measurable processes, respectively. We remark that $C^0(\Theta, E) \subset \mathbb{L}^0(\mathbb{F}, E)$, and when $E = \mathbb{R}$, we shall omit it in these notations. We also introduce the following spaces :

- $\mathbb{S}_2^{t, \omega} := \left\{ Y \in \mathbb{L}^0(\mathbb{F}) : Y \text{ is continuous in time, } \mathbb{P}_0\text{-a.s. and } \mathbb{E}^{\mathbb{P}_0} \left[\sup_{0 \leq s \leq T-t} |Y_s|^2 \right] < \infty \right\}$;
- $\mathbb{S}^2 := \mathbb{S}_2^{0, 0}$;
- $C_2^0(\Theta) := \left\{ u \in C^0(\Theta) : u^{t, \omega} \in \mathbb{S}_2^{t, \omega} \text{ for all } (t, \omega) \in \Theta \right\}$;

- $\mathbb{H}^2 := \left\{ Z \in \mathbb{L}^0(\mathbb{F}, \mathbb{R}^d) : \mathbb{E}^{\mathbb{P}^0} \left[\int_0^T |Z_s|^2 ds \right] < \infty \right\}$;
- $\mathbb{I}^2 := \left\{ K \in \mathbb{S}^2 : K \text{ is increasing, } \mathbb{P}_\sigma\text{-a.s. and } K_0 = 0 \right\}$;

For any $A \in \mathcal{F}_T$, $\xi \in \mathbb{L}^0(\mathcal{F}_T, E)$, $X \in \mathbb{L}^0(\mathbb{F}, E)$, and $(t, \omega) \in \Theta$, define :

$$A^{t,\omega} := \{\omega' \in \Omega : \omega \otimes_t \omega' \in A\}, \quad \xi^{t,\omega}(\omega') := \xi(\omega \otimes_t \omega'), \quad X_s^{t,\omega}(\omega') := X(t+s, \omega \otimes_t \omega')$$

for all $\omega' \in \Omega$, where $(\omega \otimes_t \omega')_s := \omega_s \mathbf{1}_{[0,t]}(s) + (\omega_t + \omega'_{s-t}) \mathbf{1}_{(t,T]}(s)$, $0 \leq s \leq T$.

Following the standard arguments of monotone class, we have the following simple results.

Lemma III.1.1. *Let $0 \leq t \leq s \leq T$ and $\omega \in \Omega$. Then $A^{t,\omega} \in \mathcal{F}_{s-t}$ for all $A \in \mathcal{F}_s$, $\xi^{t,\omega} \in \mathbb{L}^0(\mathcal{F}_{s-t}, E)$ for all $\xi \in \mathbb{L}^0(\mathcal{F}_s, E)$, $X^{t,\omega} \in \mathbb{L}^0(\mathbb{F}, E)$ for all $X \in \mathbb{L}^0(\mathbb{F}, E)$, and $\tau^{t,\omega} - t \in \mathcal{T}_{s-t}$ for all $\tau \in \mathcal{T}_s$.*

We consider the semilinear path dependent PDE :

$$-\mathcal{L}u(\theta) - F(\cdot, u, \partial_\omega u)(\theta) = 0, \quad \text{where } \mathcal{L}u := \partial_t u + \frac{1}{2} \partial_{\omega\omega}^2 u. \quad (\text{III.1.1})$$

Remark III.1.2. In Ren, Touzi and Zhang [102], the comparison result can be proved for the equations of the form :

$$\left(-\partial u - \frac{1}{2} \text{Tr} \left[\sigma \sigma^T \partial_{\omega\omega}^2 u \right] - F(\cdot, u, \sigma^T \partial_\omega u) \right) (\theta) = 0,$$

where σ is Lipschitz-continuous in ω , namely

$$|\sigma_t(\omega) - \sigma_t(\omega')| \leq C \|\omega - \omega'\|_t \quad \text{for all } t \in [0, T], \omega, \omega' \in \Omega, \text{ for some } C \geq 0.$$

It is worth noting that σ is allowed to be degenerate. The main argument for the general case remains the same as that for $\sigma = I_d$, so we only discuss the equation (III.1.1) for the simplicity of notations.

Assumption III.1.3. (i) F is uniformly L_0 -Lipschitz continuous in (y, z) , for some $L_0 \geq 0$, i.e.

$$|F(\cdot, y, z) - F(\cdot, y', z')| \leq L_0 (|y - y'| + |z - z'|) \quad \text{for all } y, y' \in \mathbb{R}, z, z' \in \mathbb{R}^d,$$

(ii) There exists $F^0 \in C^0(\Theta)$ such that $|F(\cdot, 0, 0)| \leq F^0$.

Introduce a family of probability measure :

$$\mathcal{P}_L := \left\{ \mathbb{P}_\mu : \frac{d\mathbb{P}_\mu}{d\mathbb{P}_0} = \exp \left(\int_0^T \mu_t dB_t - \frac{1}{2} \int_0^T \mu_t^2 dt \right), \text{ for some } \mu \in \mathbb{L}^0(\mathbb{F}, \mathbb{R}^d), \|\mu\| \leq L \right\},$$

where L is a constant. Then the corresponding nonlinear expectations are defined as :

$$\bar{\mathcal{E}}_L[\cdot] := \sup_{\mathbb{P} \in \mathcal{P}_L} \mathbb{E}^\mathbb{P}[\cdot], \quad \underline{\mathcal{E}}_L[\cdot] := \inf_{\mathbb{P} \in \mathcal{P}_L} \mathbb{E}^\mathbb{P}[\cdot].$$

Also recall the definition of test functions in Chapter II (II.2.4), that of semijets in Chapter II (II.2.5), and that of \mathcal{P}_L -viscosity sub-/super-solutions in Chapter II Definition II.2.3. Slightly different from the notations in Chapter II, we denote the set of test functions by $\underline{\mathcal{A}}_L, \bar{\mathcal{A}}_L$ instead of $\underline{\mathcal{A}}_{\mathcal{P}_L}, \bar{\mathcal{A}}_{\mathcal{P}_L}$, for the simplicity of notations. Further, for the study of semilinear path dependent PDE, one does not need to take into account all test functions of the form of $\alpha t + \beta \omega_t + \frac{1}{2} \gamma \omega_t^2$, as we did in Chapter II for general second order path dependent PDEs. Instead, we only need to consider the test functions of the form of $\phi^{\alpha, \beta} := \alpha t + \beta \omega_t$, and thus define the semijets :

$$\underline{\mathcal{J}}_L u(\theta) = \left\{ (\alpha, \beta) : \phi^{\alpha, \beta} \in \underline{\mathcal{A}}_L u(\theta) \right\}, \quad \bar{\mathcal{J}}_L u(\theta) = \left\{ (\alpha, \beta) : \phi^{\alpha, \beta} \in \bar{\mathcal{A}}_L u(\theta) \right\}. \quad (\text{III.1.2})$$

The main objective of the chapter is to prove the following comparison result for continuous viscosity solutions.

Theorem III.1.4 (Comparison). *Let Assumption III.1.3 hold true, and $u, v \in C_2^0(\Theta)$ be \mathcal{P}_L -viscosity subsolution and supersolution, respectively, of PPDE (III.1.1) for some $L \geq L_0$. If $u_T \leq v_T$ on Ω , then $u \leq v$ on Θ .*

2 Optimal stopping under nonlinear expectation

Optimal stopping result is crucial in the current theory of viscosity solution to path dependent PDE. We will use the result of optimal stopping under nonlinear expectation for semicontinuous obstacles. In the existing literature, Kobylanski and Quenez [74] contains the desired result but only in the case of linear expectation. Peng and Xu studied in [?] reflected backward SDEs with L^2 obstacles, and they proved a crucial intermediate result which can lead to the optimal stopping result. However, since their main interest is reflected backward SDEs, there is no direct theorem that we may apply. In Ren [98], we give a new simple proof for the optimal stopping problem, by using the minimum condition of the Skorokhod decomposition.

Definition III.2.1. (i) A random variable X is $\bar{\mathcal{E}}_L$ -uniformly integrable if

$$\lim_{A \rightarrow \infty} \bar{\mathcal{E}}_L[|X|; |X| \geq A] \rightarrow 0.$$

(ii) A family of random variables $\{X_\alpha\}$ is $\bar{\mathcal{E}}_L$ -uniformly integrable if

$$\lim_{A \rightarrow \infty} \sup_{\alpha} \bar{\mathcal{E}}_L[|X_\alpha|; |X_\alpha| \geq A] \rightarrow 0.$$

Denote by \mathcal{T}_* the set of all \mathbb{F}^* -stopping times

Theorem III.2.2 (Optimal stopping for semicontinuous obstacle). *Let X be an \mathbb{F}^* -progressively measurable process such that*

- (i) X is upper semicontinuous (u.s.c.) in t , \mathbb{P}_0 -a.s. ;
- (ii) $\sup_t X_t^+$ is $\bar{\mathcal{E}}_L$ -uniformly integrable ;
- (iii) X_t^- is $\bar{\mathcal{E}}_L$ -uniformly integrable for all $t \in [0, T]$.

Define

$$V(\theta) = \sup_{\tau \in \mathcal{T}_*} \bar{\mathcal{E}}_L[X_\tau^\theta]. \quad (\text{III.2.1})$$

Then there exists a stopping time $\tau^* \in \mathcal{T}_*$ such that $V_0 = \bar{\mathcal{E}}_L[X_{\tau^*}]$ and $X_{\tau^*} = V_{\tau^*}$, \mathbb{P}_0 -a.s. Moreover, there exists a process Y such that $Y_\tau = V_\tau$, \mathbb{P}_0 -a.s. for all $\tau \in \mathcal{T}_*$, and there are $\mathbb{P}^* \in \mathcal{P}_L$, \mathbb{P}^* -martingale M starting from 0, and $K \in \mathbb{I}^2$ such that

$$Y = Y_0 + M - K \quad \text{and} \quad \int (Y - X)dK = 0, \quad \mathbb{P}_0 - \text{a.s.}$$

Denote

$$\bar{\mathcal{E}}_L[\cdot | \mathcal{F}_t] := \text{ess sup}_{\mathbb{P} \in \mathcal{P}_L} \mathbb{E}^{\mathbb{P}}[\cdot | \mathcal{F}_t].$$

We consider another version of the optimal stopping problem :

$$Y_t := \text{ess sup}_{\tau \in \mathcal{T}_*^t} \bar{\mathcal{E}}_L[X_\tau | \mathcal{F}_t] = \text{ess sup}_{\tau \in \mathcal{T}_*^t, \mathbb{P} \in \mathcal{P}_L} \mathbb{E}^{\mathbb{P}}[X_\tau | \mathcal{F}_t], \quad (\text{III.2.2})$$

where \mathcal{T}_*^t is the set of all the stopping times in \mathcal{T}_* larger than t .

Remark III.2.3. We consider the optimal stopping problem of Y instead of that of V , because it is easier to prove the dynamic programming for the former and it simply holds that $Y_\tau = V_\tau$, \mathbb{P}_0 -a.s. for all $\tau \in \mathcal{T}_*$.

2.1 Doob-Meyer decomposition

In most of the existing literature, authors only discuss the Doob-Meyer decomposition for RCLL supermartingale in class D. However, in our case, we need the decomposition under some weaker conditions. We find that the argument in Beiglböck, Schachermayer and Veliyev [8] can deduce a variation of the classical Doob-Meyer decomposition which serves well our purpose. In this subsection, we will quickly review their result and prove the decomposition theorem (Proposition III.2.5).

Let Y be a \mathbb{P} -supermartingale for some probability measure \mathbb{P} . Denote

$$\mathcal{D}_n := \left\{ \frac{j}{2^n} : j \in \mathbb{N}, \frac{j}{2^n} \leq T \right\} \quad \text{and} \quad \mathcal{D} := \cup_n \mathcal{D}_n.$$

For each n , we have the discrete time Doob-Meyer decomposition :

$$Y_t = Y_0 + M_t^n - A_t^n, \quad \text{for all } t \in \mathcal{D}^n, \quad \mathbb{P}\text{-a.s.}$$

According to Lemma 2.1 and 2.2 in [8], we have :

Lemma III.2.4. (i) *Let $\{f_n\}_{n \geq 1}$ be a \mathbb{P} -uniformly integrable sequence of functions. Then there exists functions $g_n \in \text{conv}(f_n, f_{n+1}, \dots)$ such that $\{g_n\}_{n \geq 1}$ converges in $\|\cdot\|_{L^1(\mathbb{P})}$.*

(ii) *Assume that $\{Y_\tau\}_{\tau \in \mathcal{T}_\mathcal{D}}$ is \mathbb{P} -uniformly integrable, where $\mathcal{T}_\mathcal{D}$ is the set of stopping times in \mathcal{T}_* taking values in \mathcal{D} . Then the sequence $\{M_T^n\}_{n \geq 1}$ is \mathbb{P} -uniformly integrable.*

Then following the same argument as in [8], we obtain the following result.

Proposition III.2.5. *Let Y be \mathbb{P} -supermartingale such that $\{Y_\tau\}_{\tau \in \mathcal{T}_\mathcal{D}}$ is \mathbb{P} -uniformly integrable. Then there exists a martingale M and an adapted non-decreasing process A both starting from 0 such that*

$$Y_t = Y_0 + M_t - A_t, \quad \text{for all } t \in \mathcal{D}, \quad \mathbb{P}\text{-a.s.} \quad (\text{III.2.3})$$

Proof For each n , extend M^n to a cadlag martingale on $[0, T]$ by setting $M_t^n := \mathbb{E}^\mathbb{P}[M_T^n | \mathcal{F}_t]$. By Lemma III.2.4, there exist $M \in L^1(\mathbb{P})$ and for each n convex weights $\lambda_n^n, \dots, \lambda_{N_n}^n$ such that with

$$\mathcal{M}^n := \lambda_n^n M^n + \dots + \lambda_{N_n}^n M^{N_n}$$

we have $\mathcal{M}_1^n \rightarrow M$ in $L^1(\mathbb{P})$. Then, by Jensen's inequality, $\mathcal{M}_t^n \rightarrow M_t := \mathbb{E}^\mathbb{P}[M | \mathcal{F}_t]$ for all

$t \in [0, T]$. For each n we extend A^n to $[0, T]$ by $A^n := \sum_{t \in \mathcal{D}_n} A_t^n 1_{(t - \frac{1}{2^n}, t]}$ and set :

$$\mathcal{A}^n := \lambda_n^n A^n + \dots + \lambda_{N_n}^n A^{N_n}.$$

Then the process $\bar{A} := M + Y_0 - Y$ satisfies for every $t \in \mathcal{D}$

$$\mathcal{A}_t^n = \mathcal{M}_t^n + Y_0 - Y_t \longrightarrow M_t + Y_0 - Y_t = \bar{A}_t \text{ in } L^1(\mathbb{P}).$$

Therefore, \bar{A} is a.s. non-decreasing on \mathcal{D} , \mathbb{P} -a.s. Finally, the process $A_t := \sup_{s \leq t, s \in \mathcal{D}} \bar{A}_s$ is non-decreasing on $[0, T]$, \mathbb{P} -a.s., and satisfies (III.2.3). \square

Remark III.2.6. In [8], by further assuming that Y is cadlag and in class D, we may get the decomposition on $[0, T]$, and prove that process A is previsible.

2.2 Skorokhod decomposition for lower semicontinuous functions

Lemma III.2.7. *Let $\lambda : [0, T] \rightarrow \mathbb{R}$ be lower semicontinuous (l.s.c.) with $\lambda_0 = 0$, and define*

$$\kappa_t := \max_{s \leq t} \lambda_s^- = - \min_{s \leq t} \lambda_s \quad \text{and} \quad \eta_t := \lambda_t + \max_{s \leq t} \lambda_s^-.$$

Then,

(i) η is non-negative and κ is non-decreasing, such that

$$\eta_0 = \kappa_0 = 0, \quad \lambda_t = \eta_t - \kappa_t \text{ for all } t \in [0, T].$$

(ii) η is l.s.c., κ is right continuous, and it holds that

$$\int_0^T 1_{\{\eta_t \neq 0\}} d\kappa_t = 0.$$

(iii) for all other non-negative function η' and non-decreasing function κ' satisfying (i), it holds

$$\kappa_t \leq \kappa'_t \text{ for all } t \in [0, T].$$

Proof (i) is trivial. We only prove (ii) and (iii).

(ii). First, we claim that

$$\min_{r \leq t} \lambda_r = \liminf_{s \rightarrow t} \min_{r \leq s} \lambda_r. \tag{III.2.4}$$

Since $\lambda_t = \liminf_{s \rightarrow t} \lambda_s$, it is clear that $\min_{r \leq t} \lambda_r \geq \liminf_{s \rightarrow t} \min_{r \leq s} \lambda_r$. On the other hand, we have

$$\min_{r \leq t - \epsilon} \lambda_r \leq \liminf_{s \rightarrow t} \min_{r \leq s} \lambda_r, \quad \text{for all } \epsilon > 0.$$

It implies that $\inf_{r < t} \lambda_r \leq \liminf_{s \rightarrow t} \min_{r \leq s} \lambda_r$. Again by $\lambda_t = \liminf_{s \rightarrow t} \lambda_s$, we obtain that $\inf_{r < t} \lambda_r \geq \min_{r \leq t} \lambda_r$. So we proved (III.2.4). Consequently, by the definition of κ , we have $\kappa_t = \limsup_{s \rightarrow t} \kappa_s$. Taking into account that κ is non-decreasing, we obtain that $\kappa_t = \lim_{s \downarrow t} \kappa_s$.

For any $\epsilon > 0$, take $t \in \{s : \eta_s > \epsilon\}$, i.e.

$$\lambda_t + a > \epsilon, \quad \text{where } a := \kappa_t.$$

Since λ is l.s.c., the set $\{s : \lambda_s > -a + \epsilon\}$ is open. Thus, there is an open neighborhood O_t of t on which $\lambda > -a + \epsilon$. We claim that

$$\lambda > -\kappa + \epsilon \quad \text{on } O_t. \quad (\text{III.2.5})$$

Suppose to the contrary, i.e. there exists $\bar{t} \in O_t$ such that $\lambda_{\bar{t}} \leq -\kappa_{\bar{t}} + \epsilon$. If $\bar{t} \geq t$, then $\lambda_{\bar{t}} \leq -\kappa_{\bar{t}} + \epsilon \leq -\kappa_t + \epsilon = -a + \epsilon$, which is a contradiction. Otherwise, if $\bar{t} < t$, since $-a + \epsilon < \lambda_{\bar{t}} \leq -\kappa_{\bar{t}} + \epsilon$, we obtain that $\kappa_{\bar{t}} < a$. However, since $\kappa_t = a$, there exists $\hat{t} \in [\bar{t}, t]$ such that $\lambda_{\hat{t}} = -a$, which is also a contradiction. So we proved (III.2.5). It follows that $\{s : \eta_s > \epsilon\}$ is open for all $\epsilon > 0$, and thus η is l.s.c.

On the other hand, since $\{s : \eta_s > \epsilon\}$ is open, it can be written as the union of a countable number of open intervals, i.e. $\{s : \eta_s > \epsilon\} = \cup_n (s_n, t_n)$. Since $(s_n, t_n) \subset \{s : \eta_s > \epsilon\}$, we clearly have $\kappa_{t_n} - \kappa_{s_n} = 0$. Further, we have

$$\int_0^T 1_{\{\eta_s > \epsilon\}} d\kappa_s = \sum_n (\kappa_{t_n} - \kappa_{s_n}) = 0.$$

Finally, it follows from the monotone convergence theorem that $\int_0^T 1_{\{\eta_s > 0\}} d\kappa_s = 0$.

(iii). Assume to the contrary, i.e. let $t \in (0, T]$ such that $\kappa_t > \kappa'_t$. Take $s^* := \sup\{s \leq t : \eta_s = 0\}$. Since η is non-negative and l.s.c., the set $\{\eta = 0\}$ is closed, and therefore, $\eta_{s^*} = 0$. Also, since $(s^*, t] \subset \{\eta > 0\}$, we have $\kappa_t - \kappa_{s^*} = 0$. Then,

$$\eta'_{s^*} = \eta_{s^*} - \kappa_{s^*} + \kappa'_{s^*} \leq \kappa'_t - \kappa_t < 0,$$

contradiction. □

2.3 Optimal stopping for upper semicontinuous barriers

One may easily prove the following three lemmas.

Lemma III.2.8 (Dominated convergence). *Let X_n be a sequence of r.v.'s such that $\{X_n\}_n$ is $\bar{\mathcal{E}}_L$ -uniformly integrable, and $X_n \rightarrow 0$, \mathbb{P}_0 -a.s. Then we have $\lim_{n \rightarrow \infty} \bar{\mathcal{E}}_L[|X_n|] = 0$.*

Lemma III.2.9 (Fatou's lemma). *Let X_n be a sequence of bounded r.v.'s. Then we have*

$$\limsup_{n \rightarrow \infty} \bar{\mathcal{E}}_L[X_n] \leq \bar{\mathcal{E}}_L[\limsup_{n \rightarrow \infty} X_n].$$

Lemma III.2.10 (Tower property). *For any $\bar{\mathcal{E}}_L$ -uniformly integrable r.v. X , it holds that*

$$\bar{\mathcal{E}}_L[X|\mathcal{F}_t] = \bar{\mathcal{E}}_L[\bar{\mathcal{E}}_L[X|\mathcal{F}_s]|\mathcal{F}_t], \quad \mathbb{P}_0\text{-a.s.}, \text{ for all } t \leq s.$$

Lemma III.2.11. *Y is an \mathbb{F}^* -adapted $\bar{\mathcal{E}}_L$ -supermartingale. Moreover, $\{Y_\tau\}_{\tau \in \mathcal{T}_D}$ is $\bar{\mathcal{E}}_L$ -uniformly integrable.*

Proof By standard argument, one may prove the first part of the lemma. We are going to prove the second part, by showing that $\{Y_\tau^+\}_{\tau \in \mathcal{T}_D}$ and $\{Y_\tau^-\}_{\tau \in \mathcal{T}_D}$ are both $\bar{\mathcal{E}}_L$ -uniformly integrable.

1. By the definition of Y , it is clear that $Y_t \leq \bar{\mathcal{E}}_L[\sup_{s \in [0, T]} X_s | \mathcal{F}_t]$. Further, by Jensen's inequality, it follows that $Y_t^+ \leq \bar{\mathcal{E}}_L[\sup_{s \in [0, T]} X_s^+ | \mathcal{F}_t]$. Then for all $\tau \in \mathcal{T}_D$ we have

$$Y_\tau^+ \leq \bar{\mathcal{E}}_L[\sup_{s \in [0, T]} X_s^+ | \mathcal{F}_\tau], \quad \mathbb{P}_0\text{-a.s.}$$

By (ii) of the assumptions of Theorem III.2.2, it is easy to prove that $\{Y_\tau^+\}_{\tau \in \mathcal{T}_D}$ is $\bar{\mathcal{E}}_L$ -uniformly integrable.

2. Since Y is a \mathbb{P} -supermartingale for all $\mathbb{P} \in \mathcal{P}$, Y^- is a \mathbb{P} -submartingale for all $\mathbb{P} \in \mathcal{P}$. Consequently, we have

$$Y_\tau^- \leq \bar{\mathcal{E}}_L[Y_T^- | \mathcal{F}_\tau] = \bar{\mathcal{E}}_L[X_T^- | \mathcal{F}_\tau].$$

By (iii) of the assumptions of Theorem III.2.2, one may easily prove that $\{Y_\tau^-\}_{\tau \in \mathcal{T}_D}$ is $\bar{\mathcal{E}}_L$ -uniformly integrable. \square

Remark III.2.12. In the previous proof, it is crucial to consider the $\bar{\mathcal{E}}_L$ -uniform integrability of $\{Y_\tau\}_{\tau \in \mathcal{T}_D}$ instead of $\{Y_\tau\}_{\tau \in \mathcal{T}_*}$.

Lemma III.2.13. *Y has a left continuous version.*

Proof 1. We first prove $\lim_{s \uparrow t} \underline{\mathcal{E}}_L[Y_s - Y_t] = 0$. Since Y is a supermartingale, it is sufficient to prove that

$$\limsup_{s \uparrow t} \underline{\mathcal{E}}_L[Y_s - Y_t] \leq 0. \quad (\text{III.2.6})$$

Since $Y \geq X$, \mathbb{P}_0 -a.s., it follows from Lemma III.2.10 that

$$\begin{aligned} \underline{\mathcal{E}}_L[Y_s - Y_t] &= \underline{\mathcal{E}}_L \left[\text{ess sup}_{\tau \in \mathcal{T}_*^s} \bar{\mathcal{E}}_L[X_\tau | \mathcal{F}_s] - \bar{\mathcal{E}}_L[Y_t | \mathcal{F}_s] \right] \\ &\leq \underline{\mathcal{E}}_L \left[\text{ess sup}_{\tau \in \mathcal{T}_*^s} \bar{\mathcal{E}}_L[X_\tau 1_{\{\tau < t\}} + Y_t 1_{\{\tau \geq t\}} | \mathcal{F}_s] - \bar{\mathcal{E}}_L[Y_t | \mathcal{F}_s] \right] \\ &\leq \bar{\mathcal{E}}_L \left[\text{ess sup}_{\tau \in \mathcal{T}_*^s} \bar{\mathcal{E}}_L[(X_\tau - Y_t)^+ | \mathcal{F}_s] \right] \\ &\leq \bar{\mathcal{E}}_L \left[\bar{\mathcal{E}}_L[(\bar{X}_s^t - Y_t)^+ | \mathcal{F}_s] \right] = \bar{\mathcal{E}}_L[(\bar{X}_s^t - X_t)^+], \end{aligned}$$

where $\bar{X}_s^t := \sup_{s \leq r \leq t} X_r$. Since X is u.s.c. in t , it holds that $\lim_{s \uparrow t} \bar{X}_s^t \leq X_t$. Further, in view of (ii) and (iii) of the assumptions of Theorem III.2.2, (III.2.6) follows from Lemma III.2.8.

2. It follows from Lemma III.2.11 that Y is a \mathbb{P}_0 -supermartingale in the continuous filtration \mathbb{F}^* . By classical martingale theory, we know that for any $t \in [0, T)$,

$$Y_{t-} := \lim_{s \uparrow t, s \in \mathcal{D}} Y_s \text{ exists } \mathbb{P}_0\text{-a.s.},$$

and that $\{Y_{t-}\}_t$ is left continuous and $Y_t = \mathbb{E}[Y_t | \mathcal{F}_{t-}^*] \leq Y_{t-}$, \mathbb{P}_0 -a.s. We next show that $Y_{t-} = Y_t$, \mathbb{P}_0 -a.s. Suppose to the contrary that $\mathbb{P}_0[Y_t < Y_{t-}] > 0$. Then, we have $\mathbb{E}^{\mathbb{P}_0}[\sqrt{Y_{t-} - Y_t}] > 0$, implying that $\underline{\mathcal{E}}_L[Y_{t-} - Y_t] > 0$. On the other hand, it follows from the result of Step 1 and Lemma III.2.11 that

$$0 = \lim_{s \uparrow t, s \in \mathcal{D}} \underline{\mathcal{E}}_L[Y_s - Y_t] = \underline{\mathcal{E}}_L[Y_{t-} - Y_t] > 0,$$

contradiction. □

Then following the discussion in Section 2.1, we can show that :

Lemma III.2.14. *For all $\mathbb{P} \in \mathcal{P}$, there exists a \mathbb{P} -martingale $M^\mathbb{P}$ and a non-decreasing process $A^\mathbb{P}$ such that*

$$Y_t = Y_0 + M_t^\mathbb{P} - A_t^\mathbb{P}, \text{ for all } t \in [0, T], \mathbb{P}_0\text{-a.s.} \quad (\text{III.2.7})$$

In particular, there exists Z such that $M^{\mathbb{P}^0} = \int_0^\cdot Z_t dB_t$, \mathbb{P}_0 -a.s. Moreover, for $\mathbb{P}_\mu \in \mathcal{P}$, it holds that $M^\mathbb{P} = M^{\mathbb{P}^0} - \int_0^\cdot \mu_t \cdot Z_t dt$. In particular, there exists $\mathbb{P}^* := \mathbb{P}_{\mu^*}$ such that $M^{\mathbb{P}^*}$ is a \mathbb{P} -supermartingale for all $\mathbb{P} \in \mathcal{P}$.

We next make use of the Skorokhod decomposition in Section 2.2. For the simplicity of notation, we denote $M^* := M^{\mathbb{P}^*}$ and $A^* := A^{\mathbb{P}^*}$. Consider the backward process :

$$\lambda_t = (M_{T-t}^* - X_{T-t}) - (M_T^* - X_T).$$

Then we can find a non-negative process η and a non-decreasing process κ such that the statements in Lemma III.2.7 holds. Denote the corresponding forward processes :

$$\bar{\eta}_t := \eta_{T-t} \quad \text{and} \quad \bar{\kappa}_t := \kappa_{T-t}.$$

Proposition III.2.15. *It holds that*

$$\bar{\kappa} = A_T^* - A^*, \quad \mathbb{P}_0\text{-a.s.}$$

Proof 1. It follows from the Doob-Meyer decomposition (III.2.7) that

$$Y_t - X_t - Y_0 + A_t^* - (M_T^* - X_T) = \lambda_{T-t} = \bar{\eta}_t - \bar{\kappa}_t, \quad \mathbb{P}_0\text{-a.s.}$$

Since $M_T^* - X_T = M_T^* - Y_T = A_T^* - Y_0$, \mathbb{P}_0 -a.s., it holds

$$(Y_t - X_t) - (A_T^* - A_t^*) = \bar{\eta}_t - \bar{\kappa}_t, \quad \mathbb{P}_0\text{-a.s.}$$

Note that $Y \geq X$ and A^* is non-decreasing, \mathbb{P}_0 -a.s. By (iii) of Lemma III.2.7, we obtain

$$\bar{\kappa} \leq A_T^* - A^*, \quad \mathbb{P}_0\text{-a.s.} \tag{III.2.8}$$

2. Recall that

$$\bar{\kappa}_t = - \min_{s \geq t} \left((M_s^* - X_s) - (M_T^* - X_T) \right).$$

Since $X_T = Y_T$, \mathbb{P}_0 -a.s., it follows from (III.2.7) that

$$\bar{\kappa}_t = - \min_{s \geq t} (M_s^* - X_s) - A_T^* + Y_0, \quad \mathbb{P}_0\text{-a.s.}$$

Taking nonlinear conditional expectation on both sides, we obtain

$$\underline{\mathcal{E}}_L[A_T^* - \bar{\kappa}_t | \mathcal{F}_t] = Y_0 - \bar{\mathcal{E}}_L\left[\max_{s \geq t}(X_s - M_s^*) | \mathcal{F}_t\right], \quad \mathbb{P}_0\text{-a.s.} \quad (\text{III.2.9})$$

Since by Lemma III.2.14 M^* is \mathbb{P} -supermartingale for all $\mathbb{P} \in \mathcal{P}$, we obtain

$$\bar{\mathcal{E}}_L\left[\max_{s \geq t}(X_s - M_s^* + M_t^*) | \mathcal{F}_t\right] \geq \text{ess sup}_{\tau \in \mathcal{T}_*^t} \bar{\mathcal{E}}_L[(X_\tau - M_\tau^* + M_t^*) | \mathcal{F}_t] \geq \text{ess sup}_{\tau \in \mathcal{T}_*^t} \bar{\mathcal{E}}_L[X_\tau | \mathcal{F}_t] = Y_t, \quad \mathbb{P}_0\text{-a.s.}$$

In view of (III.2.8) and (III.2.9), we get

$$A_t^* \leq \underline{\mathcal{E}}_L[A_T^* - \bar{\kappa}_t | \mathcal{F}_t] \leq Y_0 - Y_t + M_t^* = A_t^*, \quad \mathbb{P}_0\text{-a.s.}$$

It implies that $A_t^* = \underline{\mathcal{E}}_L[A_T^* - \bar{\kappa}_t | \mathcal{F}_t]$, \mathbb{P}_0 -a.s. Again by (III.2.8), we conclude that $A_t^* = A_T^* - \bar{\kappa}_t$, \mathbb{P}_0 -a.s. \square

Proof of Theorem III.2.2 We are going to prove that $\tau^* := \inf\{t : X_t = Y_t\} \in \mathcal{T}_*$ is an optimal stopping time. By Lemma III.2.14 and Proposition III.2.15, it holds

$$Y_0 = Y_{\tau^*} - M_{\tau^*}^* + A_{\tau^*}^* \quad \text{and} \quad A_{\tau^*}^* = \int_0^{\tau^*} 1_{\{t: X_t = Y_t\}} dA_t^* = 0.$$

Therefore $Y_0 = \mathbb{E}^{\mathbb{P}^*}[Y_{\tau^*}]$. Further, by (ii) of Lemma III.2.7, we may deduce that

$$A, Y \quad \text{are both left continuous, } \mathbb{P}_0\text{-a.s.}$$

Hence $A_{\tau^*}^* = 0$, \mathbb{P}_0 -a.s. Taking into account that X is pathwise u.s.c., we obtain that

$$Y_{\tau^*} = X_{\tau^*}, \quad \mathbb{P}_0\text{-a.s.}$$

Finally, we have

$$Y_0 = \mathbb{E}^{\mathbb{P}^*}[Y_{\tau^*}] = \mathbb{E}^{\mathbb{P}^*}[X_{\tau^*}].$$

This implies that τ^* is an optimal stopping time. \square

3 Equivalent definitions of viscosity solutions to semilinear path dependent PDEs

Definition III.3.1. For $u \in \mathbb{L}^0$, we define for each $\theta \in \Theta$:

$$\begin{aligned} \underline{\mathcal{J}}'_L u(\theta) &:= \left\{ (\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^d : u(\theta) = \max_{\tau \in \mathcal{J}_H} \bar{\mathcal{E}}_L[u_\tau^\theta - \alpha\tau - \beta B_\tau], \text{ for some } H \in \mathcal{J}^+ \right\}; \\ \bar{\mathcal{J}}'_L u(\theta) &:= \left\{ (\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^d : u(\theta) = \min_{\tau \in \mathcal{J}_H} \underline{\mathcal{E}}_L[u_\tau^\theta - \alpha\tau - \beta B_\tau], \text{ for some } H \in \mathcal{J}^+ \right\}. \end{aligned}$$

Comparing to the definition in (III.1.2), we replace the stopping time of the form of $H = \epsilon \wedge \inf\{t \geq 0 : |\omega_t| \geq \epsilon\}$ by a stopping time in \mathcal{J}^+ . The following result shows how to find a point of tangency in mean. This replaces the local compactness argument in the standard Crandall-Lions theory of viscosity solutions.

Lemma III.3.2. Assume $u \in \mathbb{L}^0(\mathbb{F})$ satisfying $u_{\cdot \wedge H} \in \mathbb{S}^2$ and $u_0 > \bar{\mathcal{E}}_L[u_H]$ for some $H \in \mathcal{J}^+$. Then there exists $\omega^* \in \Omega$ and $t^* < H(\omega^*)$ such that $(0, 0) \in \underline{\mathcal{J}}'_L u_{t^*}(\omega^*)$.

Proof Define the optimal stopping problem V by (III.2.1) with $X := u$. Let $\tau^* \in \mathcal{J}_H^*$ be the optimal stopping rule. Since by Theorem III.2.2 we have

$$\bar{\mathcal{E}}_L[u_{\tau^*}] = V_0 \geq u_0 > \bar{\mathcal{E}}_L[u_H] \quad \text{and} \quad \mathbb{P}_0[u_{\tau^*} = V_{\tau^*}] = 1,$$

and it follows that $\mathbb{P}_0[u_{\tau^*} = V_{\tau^*}, \tau^* < H] > 0$, then there exists $\omega^* \in \Omega$ such that $t^* := \tau^*(\omega^*) < H(\omega^*)$ and $u_{t^*}(\omega^*) = V_{t^*}(\omega^*)$. By the definition of V and $\underline{\mathcal{J}}'_L u$, this means that (t^*, ω^*) is the desired point. \square

Proposition III.3.3. Suppose that $u \in C^0(\Theta)$ and that the generator $F : (\theta, y, z) \mapsto \mathbb{R}$ satisfies Assumption III.1.3. Then u is a viscosity subsolution of Equation (III.1.1) if and only if

$$-\alpha - F(\theta, u(\theta), \beta) \leq 0, \quad \text{for all } \theta \in \Theta, (\alpha, \beta) \in \underline{\mathcal{J}}'_L u(\theta). \quad (\text{III.3.1})$$

The similar result holds for supersolutions.

Proof The 'if' part is trivial by the definitions. We will only prove the 'only if' part. Fix a $\theta \in \Theta$, and suppose $(\alpha, \beta) \in \underline{\mathcal{J}}'_L u(\theta)$, i.e.

$$u(\theta) = \max_{\tau \in \mathcal{J}_H} \bar{\mathcal{E}}_L[u_\tau^\theta - \alpha\tau - \beta B_\tau] \quad \text{for some } H \in \mathcal{J}^+.$$

For any $\delta > 0$, we may suppose $H < \hat{H}_\delta := \inf\{t' : d(\theta', 0) \geq \delta\}$. Then for any $\epsilon > 0$ it holds

$$u(\theta) > \bar{\mathcal{E}}_L[u_H^\theta - (\alpha + \epsilon)H - \beta B_H].$$

We next define a sequence of hitting time :

$$\bar{H}_0^n := 0, \quad \bar{H}_{k+1}^n := \left(\bar{H}_k^n + \frac{1}{n}\right) \wedge \inf\left\{t' \geq \bar{H}_k^n : |\omega'_{t'} - \omega'_{\bar{H}_k^n}| \geq \frac{1}{n}\right\}, \quad \text{for all } k \geq 0,$$

and define $H_n := \inf\{\bar{H}_k^n : \bar{H}_k^n > H\}$. Clearly $H_n \downarrow H$. Since $u \in C^0(\Theta)$, it follows from Fatou's Lemma (Lemma III.2.9) that

$$\begin{aligned} \bar{\mathcal{E}}_L[u_H^\theta - (\alpha + \epsilon)H - B_H] &\geq \bar{\mathcal{E}}_L\left[\limsup_{n \rightarrow \infty} (u_{H_n}^\theta - (\alpha + \epsilon)H_n - B_{H_n})\right] \\ &\geq \limsup_{n \rightarrow \infty} \bar{\mathcal{E}}_L[u_{H_n}^\theta - (\alpha + \epsilon)H_n - B_{H_n}]. \end{aligned}$$

So there exists n sufficiently large such that

$$u(\theta) > \bar{\mathcal{E}}_L[u_{H_n}^\theta - (\alpha + \epsilon)H_n - B_{H_n}].$$

By Lemma III.3.2, there exists $\theta^* \in \Theta$ such that $t^* < H_n(\omega^*)$ and

$$u(\theta^*) = \max_{\tau \in \mathcal{J}_{H_n}^{\theta^* - t^*}} \bar{\mathcal{E}}_L[u_\tau^{\theta^*} - (\alpha + \epsilon)\tau - B_\tau].$$

Note that if $\bar{H}_k^n(\theta^*) \leq t^* < \bar{H}_{k+1}^n(\theta^*)$, then $H_n^{\theta^*} - t^* \geq H^* := (\bar{H}_{k+1}^n)^{\theta^*} - t^*$. Further, for ϵ' sufficiently small, we clearly have $H_{\epsilon'} := \epsilon' \wedge \inf\{t \geq 0 : |\omega_t| \geq \epsilon'\} \leq H^*$. It follows that

$$u(\theta^*) = \max_{\tau \in \mathcal{J}_{H_{\epsilon'}}} \bar{\mathcal{E}}_L[u_\tau^{\theta^*} - (\alpha + \epsilon)\tau - B_\tau].$$

By the definition of \mathcal{P}_L -viscosity solution, we obtain that

$$-(\alpha + \epsilon) - F(\theta^*, u(\theta^*), \beta) \leq 0.$$

Finally, by letting $\delta, \epsilon \rightarrow 0$ and $n \rightarrow \infty$, we obtain : $-\alpha - F(\theta, u(\theta), \beta) \leq 0$. \square

Remark III.3.4. Due to the previous proposition, we will omit the superscript "", and still call the new semijets $\underline{\mathcal{J}}_L, \bar{\mathcal{J}}_L$ in rest of this chapter. The slight abuse of notation also applies for the sets of test functions $\underline{\mathcal{A}}_L, \bar{\mathcal{A}}_L$. However, it is worth noting that the equivalent definition is only proved in the context of semilinear path dependent PDEs instead of general second order

nonlinear path dependent PDEs.

4 Comparison result for the heat equation

In this section, we consider equations with nonlinearity $F = 0$, i.e.

$$-\mathcal{L}u(t, \omega) = 0 \quad t < T, \quad \omega \in \Omega. \quad (\text{III.4.1})$$

Our objective is to provide an easy proof of the comparison result of Theorem III.1.4 which requires standard tools from stochastic analysis. For simplicity, we specialize the comparison Theorem III.1.4 to the case $L = 0$, and call the corresponding viscosity solution as \mathbb{P}_0 -viscosity solution. We emphasize that the set of test processes is the largest possible with $L = 0$.

Definition III.4.1. *We say u is a pathwise \mathbb{P}_0 -submartingale (resp. supermartingale) if*

$$u_t(\omega) \leq (\text{resp. } \geq) \mathbb{E}^{\mathbb{P}_0}[u_\tau^{t, \omega}] \quad \text{for any } (t, \omega) \in \Theta \text{ and } \tau \in \mathcal{T}_{T-t}.$$

Remark III.4.2. It is clear that a pathwise \mathbb{P}_0 -submartingale (resp. supermartingale) is a \mathbb{P}_0 -submartingale (resp. supermartingale).

Theorem III.4.3. *For a process $u \in C_2^0(\Theta)$, the following are equivalent :*

- (i) u is a pathwise \mathbb{P}_0 -submartingale (resp. supermartingale) ;
- (ii) u is \mathbb{P}_0 -viscosity subsolution (resp. supersolution) of the path-dependent heat equation (III.4.1).

Proof (i) \implies (ii) : Assume to the contrary that, for some $(t, \omega) \in [0, T) \times \Omega$ and $\varphi \in \mathcal{A}_0 u_t(\omega)$ with localizing time $\mathbb{H} \in \mathcal{T}^+$, $-c := \mathcal{L}\varphi_t(\omega) < 0$. Without loss of generality, we assume that $(t, \omega) = (0, 0)$. Note that

$$(\varphi - u)_0 \leq \mathbb{E}^{\mathbb{P}_0}[(\varphi - u)_\tau] \quad \text{for all } \tau \in \mathcal{T}_{\mathbb{H}}.$$

Denote $\tau := \inf\{t : \mathcal{L}\varphi_t \geq -\frac{c}{2}\} \wedge \mathbb{H} \in \mathcal{T}^+$. Then, by (ii), we obtain the following desired contradiction :

$$0 \geq u_0 - \mathbb{E}^{\mathbb{P}_0}[u_\tau] \geq \varphi_0 - \mathbb{E}^{\mathbb{P}_0}[\varphi_\tau] = \mathbb{E}^{\mathbb{P}_0}\left[-\int_0^\tau \mathcal{L}\varphi_s ds\right] \geq \frac{c}{2}\mathbb{E}^{\mathbb{P}_0}[\tau] > 0.$$

(ii) \implies (i) : First, denote $u_t^\epsilon(\omega) := u_t(\omega) + \epsilon t$. It is easy to verify that u^ϵ is a \mathbb{P}_0 -viscosity

subsolution to the following equation :

$$-\mathcal{L}u_t^\epsilon(\omega) + \epsilon \leq 0.$$

We now show that u^ϵ is a pathwise \mathbb{P}_0 -submartingale. Suppose to the contrary that there exists a point (t, ω) at which the supermartingale property fails, and set $(t, \omega) = (0, 0)$ without loss of generality. Then, there exists a stopping time $\mathbb{H} \in \mathcal{T}_T^+$ such that $u_0^\epsilon > \mathbb{E}^{\mathbb{P}_0}[u_{\mathbb{H}}^\epsilon]$. By Lemma III.3.2, there exists (t^*, ω^*) such that $0 \in \underline{\mathcal{A}}_0 u_{t^*}^\epsilon(\omega^*)$, and it follows from the \mathbb{P}_0 -viscosity subsolution property of u^ϵ that $\epsilon \leq 0$, which is the required contradiction.

Hence, u^ϵ is a pathwise \mathbb{P}_0 -submartingale, namely $u_t(\omega) + \epsilon t \leq \mathbb{E}^{\mathbb{P}_0}[u_\tau^{t,\omega} + \epsilon(\tau + t)]$ for all $\tau \in \mathcal{T}_{T-t}$. Send $\epsilon \rightarrow 0$, we obtain immediately that u is a pathwise \mathbb{P}_0 -submartingale. \square

Theorem III.4.3 leads immediately to the comparison result.

Theorem III.4.4. *Let $u, v \in C_2^0(\Theta)$ be \mathbb{P}_0 -viscosity subsolution and \mathbb{P}_0 -viscosity supersolution, respectively, of path dependent heat equation (III.4.1). If $u_T \leq v_T$ on Ω , then $u \leq v$ on Θ .*

Remark III.4.5. By Theorem III.4.3 we see that our notion of \mathbb{P}_0 -viscosity solution reduces to the notion of stochastic viscosity solution introduced by Bayraktar and Sirbu [6, 7] in the Markovian case.

Remark III.4.6. (i) Theorem III.4.3 also provides the unique solution of the heat equation. Indeed it implies that a pathwise \mathbb{P}_0 -martingale is a viscosity solution. Since the final value is fixed by the boundary condition ξ , we are naturally lead to the candidate solution $u(t, \omega) := \mathbb{E}^{\mathbb{P}_0}[\xi^{t,\omega}]$, $(t, \omega) \in \Theta$. Therefore, if this process is in $C_2^0(\Theta)$, it is the unique viscosity solution of the heat equation.

(ii) For the heat equation, we can in fact prove the comparison principle without requiring the continuity (in ω) of the viscosity semi-solutions.

5 Punctual differentiability

5.1 Some useful lemmas

Follow the arguments in [26], we obtain the following result considering stopping times.

Proposition III.5.1. *Let $\tau \in \mathcal{T}^*$ be previsible, namely there exist $\tau_n \in \mathcal{T}^*$ such that $\tau_n < \tau$ and $\tau_n \uparrow \tau$. Then there exists $\bar{\tau} \in \mathcal{T}$ such that $\bar{\tau} = \tau$, \mathbb{P}_0 -a.s.*

Proof Denote by $\mathbb{F}^+ := \{\mathcal{F}_t^+\}_{0 \leq t \leq T}$ the right filtration. For each $n \geq 1$ and $r \in \mathbb{Q} \cap [0, T]$, denote $E_r^n := \{\tau_n < r\} \in \mathcal{F}_r^*$. Then there exists $\tilde{E}_r^n \in \mathcal{F}_r$ such that $\tilde{E}_r^n \subset E_r^n$ and $\mathbb{P}_0(E_r^n \setminus \tilde{E}_r^n) = 0$. Note that E_r^n is decreasing in n and increasing in r , without loss of generality we may assume that \tilde{E}_r^n has the same monotonicity. Define

$$\tilde{\tau}_n := \inf\{r \in \mathbb{Q} \cap [0, T] : \omega \in \tilde{E}_r^n\} \wedge T, \quad \tilde{\tau} := \lim_{n \rightarrow \infty} \tilde{\tau}_n.$$

One can easily check that $\tilde{\tau}_n$ and $\tilde{\tau}$ are \mathbb{F}^+ -stopping times, $\tilde{\tau}_n \uparrow \tilde{\tau}$, and $\mathbb{P}_0(\tilde{\tau} = \tau) = 1$. To construct the desired \mathbb{F} -stopping time, we modify $\tilde{\tau}_n$ and $\tilde{\tau}$ as follows.

$$\bar{\tau}_n := \left(\tilde{\tau}_n \mathbf{1}_{\{\tilde{\tau}_n < \tilde{\tau}\}} + T \mathbf{1}_{\{\tilde{\tau}_n = \tilde{\tau}\}} \right) \wedge \left(T - \frac{1}{n} \right), \quad \bar{\tau} := \lim_{n \rightarrow \infty} \bar{\tau}_n.$$

It is clear that $\bar{\tau}_n$ are also \mathbb{F}^+ -stopping times, $\bar{\tau}_n \uparrow \bar{\tau}$, $\bar{\tau} \geq \tilde{\tau}$, and $\mathbb{P}_0(\bar{\tau} = \tau) = 1$. Moreover, for each n , on $\{\tilde{\tau}_n < \tilde{\tau}\}$ we have $\bar{\tau}_n = \tilde{\tau}_n \wedge (T - \frac{1}{n}) < \tilde{\tau} \leq \bar{\tau}$; and on $\{\tilde{\tau}_n = \tilde{\tau}\}$, we have $\bar{\tau}_n = \tilde{\tau}$ for all $m \geq n$, thus $\bar{\tau}_m = T - \frac{1}{m}$, $\bar{\tau} = T$, and therefore $\bar{\tau}_n = T - \frac{1}{n} < \bar{\tau}$. So in both cases we have $\bar{\tau}_n < \bar{\tau}$. Then

$$\{\bar{\tau} \leq t\} = \bigcap_{n \geq 1} \{\bar{\tau}_n < t\} \in \mathcal{F}_t, \quad \text{for all } t \leq T.$$

That is, $\bar{\tau}$ is an \mathbb{F} -stopping time. □

Lemma III.5.2. *Assume $X \in \mathbb{L}^0(\mathbb{F})$ is continuous (in t), \mathbb{P}_0 -a.s. Then there exists $\tau \in \mathcal{T}$ such that $\tau = \inf\{t : X_t = 0\} \wedge T$, \mathbb{P}_0 -a.s.*

Proof If $X_0 = 0$, then $\tau := 0$ satisfies all the requirement. We thus assume $X_0 \neq 0$. Set $E := \{\omega : X(\omega) \text{ is continuous on } [0, T]\}$ and $\hat{X} := X \mathbf{1}_E + \mathbf{1}_{E^c}$. Then $\hat{X} \in \mathbb{L}^0(\mathbb{F}^*)$ is continuous for all ω and $\hat{X}_0 \neq 0$. Denote $\hat{\tau} := \inf\{t : \hat{X}_t = 0\} \wedge T \in \mathcal{T}^*$ and $\hat{\tau}_n := \inf\{t : |\hat{X}_t| \leq \frac{1}{n}\} \wedge (T - \frac{1}{n}) \in \mathcal{T}^*$. Clearly $\hat{\tau}_n < \hat{\tau}$ and $\hat{\tau}_n \uparrow \hat{\tau}$. By Proposition III.5.1, there exists $\tau \in \mathcal{T}$ such that $\hat{\tau} = \tau$, \mathbb{P}_0 -a.s. Note that $\tau = \inf\{t : X_t = 0\} \wedge T$ on $\{\hat{\tau} = \tau\} \cap E$. Since $\mathbb{P}_0[\hat{\tau} = \tau] = \mathbb{P}_0[E] = 1$, this concludes the proof. □

Similar to standard semimartingale under a fixed probability measure \mathbb{P} , we say u is an $\bar{\mathcal{E}}_L$ -submartingale (resp. supermartingale) if, for any t and any $\tau \in \mathcal{T}$ such that $\tau \geq t$,

$$u_t \leq (\text{resp. } \geq) \bar{\mathcal{E}}_L[u_\tau | \mathcal{F}_t] := \operatorname{ess\,sup}_{\mathbb{P} \in \mathcal{P}_L} \mathbb{E}^\mathbb{P}[u_\tau | \mathcal{F}_t], \quad \mathbb{P}_0\text{-a.s.} \quad (\text{III.5.1})$$

Notice that viscosity solutions are pathwise defined. We extend the above notion in a pathwise manner.

Definition III.5.3. We say u is a pathwise $\bar{\mathcal{E}}_L$ -submartingale (resp. supermartingale) if

$$u_t(\omega) \leq (\text{resp. } \geq) \bar{\mathcal{E}}_L[u_\tau^{t,\omega}] \quad \text{for any } (t, \omega) \in \Theta \text{ and } \tau \in \mathcal{T}_{T-t}.$$

Proposition III.5.4. Assume $u \in C_2^0(\Theta)$ is a pathwise $\bar{\mathcal{E}}_L$ -submartingale. Then,

- (i). u is an $\bar{\mathcal{E}}_L$ -submartingale;
- (ii). there exists $\mathbb{P}^* \in \mathcal{P}_L$ such that u is a \mathbb{P}^* -submartingale.

Proof (i) is trivial by the definitions. Further, one may easily prove (ii) by using the weak compactness of \mathcal{P}_L . \square

5.2 Punctual differentiability of viscosity semi-solutions

When $u \in C^{1,2}(\Theta, \mathbb{R})$, it is immediately seen that $(\mathcal{L}u_t(\omega), \partial_\omega u_t(\omega)) \in \text{cl}(\mathcal{J}_L u_t(\omega))$ for $L \geq L_0$. Moreover, in Proposition ??, we have showed that the following are equivalent :

- u is a classical subsolution at (t, ω) ;
- u is a viscosity subsolution at (t, ω) .

Following Caffarelli and Cabre [15], we introduce a notion of differentiation which is weaker than the path derivatives and will be crucial for the proof of our main comparison result.

Definition III.5.5. Let $\varphi \in \mathbb{L}^0(\mathbb{F})$. We say φ is \mathcal{P}_L -punctually $C^{1,2}$ at (t, ω) , if

$$\mathcal{J}_L \varphi_t(\omega) := \text{cl}\left(\underline{\mathcal{J}}_L \varphi_t(\omega)\right) \cap \text{cl}\left(\bar{\mathcal{J}}_L \varphi_t(\omega)\right) \neq \emptyset.$$

The following result is straightforward.

Proposition III.5.6. Let $u \in C^0(\Theta, \mathbb{R})$.

- (i). If $u \in C^{1,2}(\Theta, \mathbb{R})$, then u is \mathcal{P}_L -punctually $C^{1,2}$ at all (t, ω) with $(\mathcal{L}u_t(\omega), \partial_\omega u_t(\omega)) \in \mathcal{J}_L u_t(\omega)$;
- (ii). If u is \mathcal{P}_L -punctually $C^{1,2}$ at (t, ω) and is a \mathcal{P}_L -viscosity solution (resp. subsolution, supersolution) of the path-dependent PDE (III.1.1) at (t, ω) , then for any $(\alpha, \beta) \in \mathcal{J}_L u_t(\omega)$ we have

$$-\alpha - F(t, \omega, u_t(\omega), \beta) = (\text{resp. } \leq, \geq) 0.$$

We extend part of Theorem III.4.3 to this case.

Lemma III.5.7. *Let Assumption III.1.3 hold, and for some $L \geq L_0$, $u \in C_2^0(\Theta)$ be an L -subsolution of PPDE (III.1.1). Then, the process $\hat{u} := u + \int_0^\cdot (L_0|u_s| + F_s^0 + 1)ds$ is a pathwise $\bar{\mathcal{E}}_L$ -submartingale.*

Proof Suppose to the contrary that $\hat{u}_t(\omega) > \bar{\mathcal{E}}_L[\hat{u}_H^{t,\omega}]$ for some $(t, \omega) \in [0, T) \times \Omega$ and $H \in \mathcal{T}_{T-t}^+$. Then, it follows from Lemma III.3.2 that there exist $\omega^* \in \Omega$ and $t^* \in [t, t + H(\omega^*))$ such that $0 \in \underline{\mathcal{A}}_L \hat{u}_{t^*}(\omega^*)$, that is, there exists $H' \in \mathcal{T}_{T-t^*}^+$ such that

$$-\hat{u}_{t^*}(\omega^*) \leq \underline{\mathcal{E}}_L \left[-\hat{u}_\tau^{t^*, \omega^*} \right] \quad \text{for all } \tau \in \mathcal{T}_{H'}.$$

Rewriting it we have

$$-u_{t^*}(\omega^*) \leq \underline{\mathcal{E}}_L \left[\varphi_\tau - u_\tau^{t^*, \omega^*} \right] \quad \text{for all } \tau \in \mathcal{T}_{H'}, \quad \text{where } \varphi_t := - \int_0^t (L_0|u_s| + (F^0)_s + 1)ds.$$

Clearly $\varphi \in C^{1,2}(\Theta)$ with $\mathcal{L}\varphi_{t^*}(\omega^*) = -L_0|u_{t^*}(\omega^*)| - F_{t^*}^0(\omega^*) - 1$ and $\partial_\omega \varphi_{t^*}(\omega^*) = 0$. Then the above inequality implies that $\varphi \in \underline{\mathcal{A}}_L u_{t^*}(\omega^*)$. Now by the viscosity subsolution property of u and Assumption III.1.3, we have

$$\begin{aligned} 0 &\geq -\mathcal{L}\varphi_{t^*}(\omega^*) - F_{t^*}(\omega^*, u_{t^*}(\omega^*), \partial_\omega \varphi_{t^*}(\omega^*)) \\ &= L_0|u_{t^*}(\omega^*)| + F_{t^*}^0(\omega^*) + 1 - F_{t^*}(\omega^*, u_{t^*}(\omega^*), 0) \geq F_{t^*}^0(\omega^*) + 1 - F_{t^*}(\omega^*, 0, 0) \geq 1, \end{aligned}$$

contradiction. □

Unlike the heat equation case, the above property and the corresponding statement for a viscosity supersolution v does not lead to the comparison principle directly. Our main idea is the following punctual differentiability of u .

Proposition III.5.8. *Assume u is a \mathbb{P}_0 -semimartingale with decomposition $du_t = Z_t \cdot dB_t + dA_t$, where $Z \in \mathbb{H}^2$ and $A \in \mathbb{L}^0(\mathbb{F})$ is continuous and has finite variation, \mathbb{P}_0 -a.s. Then there exist a Borel set $\mathbb{T}^u \subset [0, T]$ and $\Omega_t^u \in \mathcal{F}_t$ for each $t \in \mathbb{T}^u$ such that, for any $L > 0$,*

$$Leb(\mathbb{T}^u) = T, \quad \mathbb{P}_0(\Omega_t^u) = 1, \quad \text{and } u \text{ is } \mathcal{P}_L\text{-punctually } C^{1,2} \text{ at } (t, \omega) \text{ for all } t \in \mathbb{T}^u, \omega \in \Omega_t^u. \quad (\text{III.5.2})$$

Proof Denote

$$\zeta_t := \limsup_{0 \downarrow h \in \mathbb{Q}} \frac{1}{h} \int_t^{t+h} |Z_s - Z_t| ds, \quad \dot{A}_t^+ := \limsup_{0 \downarrow h \in \mathbb{Q}} \frac{1}{h} [A_{t+h} - A_t], \quad \dot{A}_t^- := \liminf_{0 \downarrow h \in \mathbb{Q}} \frac{1}{h} [A_{t+h} - A_t].$$

Note that the processes ζ , \dot{A}^+ , and \dot{A}^- are \mathbb{F}^+ -measurable (with possible values ∞ and $-\infty$).

Denote

$$\begin{aligned} \Omega_0 &:= \left\{ \omega \in \Omega : \int_0^T |Z_t(\omega)| dt < \infty, \text{ and } A \text{ is continuous and has finite variation on } [0, T] \right\}; \\ \Theta_0 &:= \left\{ (t, \omega) \in \Theta : \zeta_t(\omega) = 0, \dot{A}_t^+(\omega) = \dot{A}_t^-(\omega) \in \mathbb{R} \right\} \in \mathcal{B}([0, T]) \times \mathcal{F}_T, \end{aligned} \quad (\text{III.5.3})$$

Then $\mathbb{P}_0(\Omega_0) = 1$, and, by the Lebesgue differentiation theorem (see e.g. [104] Theorem 7.7, p. 139),

$$\text{Leb} \left[t : (t, \omega) \in \Theta_0 \right] = T \quad \text{for all } \omega \in \Omega_0.$$

Applying Fubini Theorem there exists $\mathbb{T}^u \subset [0, T]$ such that

$$\text{Leb}[\mathbb{T}^u] = T \quad \text{and} \quad \mathbb{P}_0[\Omega_t^1] = 1 \quad \text{for all } t \in \mathbb{T}^u, \quad \text{where } \Omega_t^1 := \{\omega \in \Omega : (t, \omega) \in \Theta_0\} \quad (\text{III.5.4})$$

Note that $\Omega_t^1 \in \mathcal{F}_{t+} \subset \mathcal{F}_t^*$, thanks to the Blumenthal's 0-1 law. Moreover, for any $t \in \mathbb{T}^u$, one can easily see that there exists $\Omega_t^2 \in \mathcal{F}_t$ such that

$$\mathbb{P}_0[\Omega_t^2] = 1 \quad \text{and} \quad du_s^{t, \omega} = Z_s^{t, \omega} dB_s + dA_s^{t, \omega}, \quad 0 \leq s \leq T - t, \quad \mathbb{P}_0\text{-a.s. for all } \omega \in \Omega_t^2 \quad (\text{III.5.5})$$

Now define $\Omega_t := \Omega_t^1 \cap \Omega_t^2 \cap \Omega_0 \in \mathcal{F}_t^*$ for all $t \in \mathbb{T}^u$, then we may find $\Omega_t^u \subset \Omega_t$ such that

$$\Omega_t^u \in \mathcal{F}_t, \quad \mathbb{P}_0[\Omega_t^u] = 1, \quad \text{for all } t \in \mathbb{T}^u. \quad (\text{III.5.6})$$

Define $\dot{A}_t(\omega) := \dot{A}_t^+(\omega) = \dot{A}_t^-(\omega)$ for $(t, \omega) \in \Theta_0$. We claim that $(\dot{A}_t(\omega), Z_t(\omega)) \in \mathcal{J}_L u_t(\omega)$ for all $t \in \mathbb{T}^u$, $\omega \in \Omega_t^u$ and $L > 0$. Without loss of generality, we shall only show that

$$(\dot{A}_t(\omega) + \epsilon, Z_t(\omega)) \in \mathcal{J}_L u_t(\omega) \quad \text{for any } \epsilon > 0. \quad (\text{III.5.7})$$

Indeed, fix $t \in \mathbb{T}^u$ and $\omega \in \Omega_t^u$. First, since $A(\omega)$ is continuous, we have

$$\lim_{h \downarrow 0} \frac{1}{h} \int_t^{t+h} |Z_s(\omega) - Z_t(\omega)| ds = 0, \quad \lim_{h \downarrow 0} \frac{1}{h} [A_{t+h}(\omega) - A_t(\omega)] = \dot{A}_t(\omega).$$

Next, set $\delta := \frac{\epsilon}{2L}$. By Lemma III.5.2, there exists $\mathfrak{H} \in \mathcal{T}_{T-t}$ such that

$$\mathfrak{H} = \inf \left\{ s > 0 : \int_0^s |Z_r^{t,\omega} - Z_t(\omega)| dr \geq \delta s, \right. \\ \left. \text{or } A_s^{t,\omega} - A_t(\omega) \geq (\dot{A}_t(\omega) + \frac{\epsilon}{2})s \right\} \wedge (T - t), \quad \mathbb{P}_0\text{-a.s.}$$

By (III.5.3) we see that $\mathfrak{H} > 0$ and thus $\mathfrak{H} \in \mathcal{T}_{T-t}^+$. For any $\lambda \in \mathbb{L}_L(\mathbb{F})$ and $\tau \in \mathcal{T}_{\mathfrak{H}}$, by (III.5.5) we have

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}^\lambda} \left[u_\tau^{t,\omega} - Q_\tau^{\dot{A}_t(\omega) + \epsilon, Z_t(\omega)} \right] - u_t(\omega) \\ &= \mathbb{E}^{\mathbb{P}^\lambda} \left[u_\tau^{t,\omega} - u_0^{t,\omega} - (\dot{A}_t(\omega) + \epsilon)\tau - Z_t(\omega) \cdot B_\tau \right] \\ &= \mathbb{E}^{\mathbb{P}^\lambda} \left[\int_0^\tau [Z_s^{t,\omega} - Z_t(\omega)] \cdot dB_s + (A_\tau^{t,\omega} - A_0^{t,\omega}) - (\dot{A}_t(\omega) + \epsilon)\tau \right] \\ &= \mathbb{E}^{\mathbb{P}^\lambda} \left[\int_0^\tau [Z_s^{t,\omega} - Z_t(\omega)] \cdot \lambda_s ds + (A_\tau^{t,\omega} - A_0^{t,\omega}) - (\dot{A}_t(\omega) + \epsilon)\tau \right] \\ &\leq \mathbb{E}^{\mathbb{P}^\lambda} \left[L \int_0^\tau |Z_s^{t,\omega} - Z_t(\omega)| ds + (A_\tau^{t,\omega} - A_0^{t,\omega}) - (\dot{A}_t(\omega) + \epsilon)\tau \right] \\ &\leq \mathbb{E}^{\mathbb{P}^\lambda} \left[L\delta\tau + (\dot{A}_t(\omega) + \frac{\epsilon}{2})\tau - (\dot{A}_t(\omega) + \epsilon)\tau \right] = 0, \end{aligned}$$

Then (III.5.7) follows from the arbitrariness of λ and τ . \square

Lemma III.5.9. *Let Assumption III.1.3 hold, and for some $L \geq L_0$, $u \in C_2^0(\Theta)$ be an \mathcal{P}_L -viscosity subsolution of PPDE (III.1.1). Then there exist measurable sets $\mathbb{T}^u \subset [0, T]$ and $\Omega_t^u \in \mathcal{F}_t$ for each $t \in \mathbb{T}^u$ such that (III.5.2) holds. A similar result holds for \mathcal{P}_L -viscosity supersolution.*

Proof Combining Lemma III.5.7, Remark III.4.2, and Proposition III.5.4, we see that \hat{u} , and hence u , is a \mathbb{P}_0 -semimartingale. Then by Proposition III.5.8, there exist measurable sets $\mathbb{T}^u \subset [0, T]$ and $\Omega_t^u \in \mathcal{F}_t$ for each $t \in \mathbb{T}^u$ such that (III.5.2) holds. \square

6 Comparison result for general semilinear path dependent PDEs

6.1 Maximum principle

We next return to the general semilinear PPDE (III.1.1). The following maximum principle, as the so-called partial comparison in [102], [35] and [37], is a crucial step for our proof of the comparison result.

Proposition III.6.1 (Maximum principle). *Assume that $u \in C_2^0(\Theta)$ and 0 are \mathcal{P}_L -viscosity subsolution and supersolution of a path dependent PDE with a generator F satisfying Assumption III.1.3, respectively. Assume further that $u_T \leq 0$. Then we have $u \leq 0$ on Θ .*

Proof First, by possibly transforming the problem to the comparison of $\tilde{u}_t := e^{\lambda t} u_t$, it follows from the Lipschitz property of the nonlinearity F in y that we may assume without generality that F is strictly decreasing in y .

Suppose to the contrary that $u(\theta) > 0$ at some point $\theta \in \Theta$. Without loss of generality assume $\theta = (0, 0)$. Define the stopping time

$$\mathfrak{H} := \inf\{t \geq 0 : u_t \leq 0\}.$$

Then we have $\bar{\mathcal{E}}_L[u_{\mathfrak{H}}] = 0 < u_0$. It follows from Lemma III.3.2 that there is θ^* such that $t^* < \mathfrak{H}(\omega^*)$ and $(0, 0) \in \underline{\mathcal{J}}_L u(\theta^*)$. By the definition of \mathfrak{H} , we have $u(\theta^*) > 0$. Further, since u is \mathcal{P}_L -viscosity subsolution, we have

$$0 \geq -0 - F(\theta^*, u(\theta^*), 0) > -0 - F(\theta^*, 0, 0).$$

The second inequality is due to the strict decrease of F in y . The previous inequality is a contradiction to the fact that 0 is a classical supersolution. \square

6.2 Comparison result

We are now ready for the key step for the proof of Theorem III.1.4. We observe that this statement is an adaptation of the approach of Caffarelli and Cabre [15] to the comparison in the context of the standard Crandall-Lions theory of viscosity solutions in finite dimensional spaces. See their Theorem 5.3 p45.

Proposition III.6.2. *Let Assumption III.1.3 hold true, and $u, v \in C_2^0(\Theta)$ be \mathcal{P}_L -viscosity subsolution and supersolution, respectively, of PPDE (III.1.1) for some $L \geq L_0$. Then, $w := u - v$ is an L -viscosity subsolution of*

$$-\mathcal{L}w(t, \omega) - L|w_t(\omega)| - L|\partial_\omega w_t(\omega)| \leq 0. \quad (\text{III.6.1})$$

Before we prove this proposition, we use it to complete the proof of Theorem III.1.4.

Proof of Theorem III.1.4 By Proposition III.6.2, functional $u - v$ is a \mathcal{P}_L -viscosity subsolution of PPDE (III.6.1). Clearly, 0 is a classical supersolution of the same equation. Since

$(u - v)_T \leq 0$, we conclude from the maximum principle Proposition III.6.1 that $u - v \leq 0$ on Θ . \square

Proof of Proposition III.6.2 Without loss of generality, we only check the viscosity sub-solution property at $(t, \omega) = (0, 0)$. For an arbitrary $(\alpha, \beta) \in \mathcal{J}_L w_0$, we want to show that

$$-\alpha - L|w_0| - L|\beta| \leq 0. \quad (\text{III.6.2})$$

1. By definition, there exists $\mathsf{H} \in \mathcal{J}^+$ such that

$$w_0 = \max_{\tau \in \mathcal{J}_{\mathsf{H}}} \bar{\mathcal{E}}_L \left[(w - \phi^{\alpha, \beta})_{\tau} \right].$$

Recall that $\phi^{\alpha, \beta}(t, \omega) = \alpha t + \beta \omega_t$. Fix $\delta > 0$. By otherwise choosing a smaller H , we may assume without loss of generality that

$$|\varphi_t - \varphi_0| \leq \delta \quad \text{for } \varphi = B, u, v. \quad (\text{III.6.3})$$

Define the optimal stopping problem Y by (III.2.2) with

$$X := w - \phi^{\alpha + \delta, \beta}.$$

Clearly, since $\delta > 0$,

$$\bar{\mathcal{E}}_L [X_{\mathsf{H}}] < w_0 = X_0 \leq Y_0 \quad \text{and} \quad Y_{\mathsf{H}} = X_{\mathsf{H}}, \quad \mathbb{P}_0\text{-a.s.} \quad (\text{III.6.4})$$

Then, it follows from (III.6.4) and Theorem III.2.2 that there exist $\mathbb{P}^* \in \mathcal{P}_L$ and $K \in \mathbb{I}^2$ such that

$$0 > \bar{\mathcal{E}}_L [Y_{\mathsf{H}} - Y_0] \geq \mathbb{E}^{\mathbb{P}^*} [Y_{\mathsf{H}} - Y_0] = -\mathbb{E}^{\mathbb{P}^*} [K_{\mathsf{H}}] = -\mathbb{E}^{\mathbb{P}^*} \left[\int_0^{\mathsf{H}} \mathbf{1}_{\{Y_t = X_t\}} dK_t \right].$$

We shall prove in Step 3 below that

$$K \text{ is absolutely continuous, } \mathbb{P}_0 - a.s. \quad (\text{III.6.5})$$

Then, denoting by \dot{K} the derivative of K and noticing that \mathbb{P}^* is equivalent to \mathbb{P}_0 , we deduce from the previous inequalities that :

$$\mathbb{E}^{\mathbb{P}^*} \left[\int_0^{\mathsf{H}} \mathbf{1}_{\{Y_t = X_t\}} \dot{K}_t dt \right] > 0 \quad \text{and thus} \quad \text{Leb} \otimes \mathbb{P}_0 [t < \mathsf{H}, Y_t = X_t] > 0. \quad (\text{III.6.6})$$

By Lemma III.5.9, there exist measurable sets $\mathbb{T}^u \subset [0, T]$ and $\Omega_t^u \in \mathcal{F}_t$ for each $t \in \mathbb{T}^u$ such that (III.5.2) holds. Similarly, we may find \mathbb{T}^v and Ω_t^v such that (III.5.2) holds for v as well. Then (III.6.6) leads to :

$$\text{Leb} \otimes \mathbb{P}_0 \left[t \in [0, \mathbb{H}] \cap \mathbb{T}^u \cap \mathbb{T}^v, Y_t = X_t \right] > 0,$$

and thus there exists

$$t^* \in \mathbb{T}^v \cap \mathbb{T}^u \quad \text{such that} \quad \mathbb{P}_0 \left[t^* < \mathbb{H}, Y_{t^*} = X_{t^*} \right] > 0,$$

which implies further that, recalling the V defined in (III.2.1) and Theorem III.2.2,

$$\mathbb{P}_0 \left[\Omega_{t^*}^u \cap \Omega_{t^*}^v \cap \{t^* < \mathbb{H}, Y_{t^*} = X_{t^*}\} \cap \{Y_{t^*} = V_{t^*}\} \right] > 0.$$

Therefore, there exists $\omega^* \in \Omega$ such that

$$\begin{aligned} & \text{both } u \text{ and } v \text{ are } \mathcal{P}_L\text{-punctually } C^{1,2} \text{ at } (t^*, \omega^*), \\ & t^* < \mathbb{H}(\omega^*) \text{ and } X_{t^*}(\omega^*) = \sup_{\tau \in \mathcal{J}} \bar{\mathcal{E}}_L \left[X_{\tau \wedge (\mathbb{H}^{t^*, \omega^*} - t^*)}^{t^*, \omega^*} \right]. \end{aligned} \quad (\text{III.6.7})$$

2. Let $(\alpha^u, \beta^u) \in \mathcal{J}_L u(t^*, \omega^*) \subset cl(\bar{\mathcal{J}}_L u_{t^*}(\omega^*))$ and $(\alpha^v, \beta^v) \in \mathcal{J}_L v(t^*, \omega^*) \subset cl(\underline{\mathcal{J}}_L v_{t^*}(\omega^*))$. Then $(\alpha^u - \delta, \beta^u) \in \bar{\mathcal{J}}_L u_{t^*}(\omega^*)$ and $(\alpha^v + \delta, \beta^v) \in \underline{\mathcal{J}}_L v(t^*, \omega^*)$. Then, we clearly have

$$(\alpha', \beta') \in \bar{\mathcal{J}}_L X_{t^*}(\omega^*), \quad \text{where} \quad \alpha' := \alpha^u - \alpha^v - \alpha - 3\delta, \quad \beta' := \beta^u - \beta^v - \beta. \quad (\text{III.6.8})$$

Choose $\lambda \in \mathbb{L}_L(\mathbb{F})$ such that $\beta' \cdot \lambda = L|\beta'|$. Then, for any $\epsilon > 0$, letting $\mathbb{H}' = \mathbb{H}^{t^*, \omega^*} - t^*$, we have

$$\begin{aligned} X_{t^*}(\omega^*) & \leq \underline{\mathcal{E}}_L \left[X_{\mathbb{H}'}^{t^*, \omega^*} - \phi_{\mathbb{H}'}^{\alpha', \beta'} \right] \leq \mathbb{E}^{\mathbb{P}^\lambda} \left[X_{\mathbb{H}'}^{t^*, \omega^*} - \phi_{\mathbb{H}'}^{\alpha', \beta'} \right] \\ & = \mathbb{E}^{\mathbb{P}^\lambda} \left[X_{\mathbb{H}'}^{t^*, \omega^*} - \alpha' \mathbb{H}' - L|\beta'| \mathbb{H}' \right] \leq \bar{\mathcal{E}}_L [X_{\mathbb{H}'}^{t^*, \omega^*}] - (\alpha' + L|\beta'|) \mathbb{E}^{\mathbb{P}^\lambda} [\mathbb{H}']. \end{aligned}$$

This, together with the optimality in (III.6.7), implies that $\alpha' + L|\beta'| \leq 0$. Moreover, the semi-viscosity properties of u and v lead to

$$-\alpha^u - F_{t^*}(\omega^*, u_{t^*}(\omega^*), \beta^u) \leq 0, \quad -\alpha^v - F_{t^*}(\omega^*, v_{t^*}(\omega^*), \beta^v) \geq 0.$$

Then, recalling (III.6.8) and by (III.6.3),

$$\begin{aligned} 0 &\leq \alpha^u + F_{t^*}(\omega^*, u_{t^*}(\omega^*), \beta^u) - \alpha^v - F_{t^*}(\omega^*, v_{t^*}(\omega^*), \beta^v) - \alpha' - L|\beta'| \\ &\leq \alpha + 3\delta + L|w_{t^*}(\omega^*)| + L|\beta| \leq \alpha + L|w_0| + L|\beta| + (3 + L)\delta. \end{aligned}$$

Now send $\delta \rightarrow 0$, we obtain (III.6.2).

3. It remains to prove (III.6.5). By Proposition III.5.7 and Remark III.4.2, we know the process \hat{u} is an $\bar{\mathcal{E}}_L$ -submartingale. Then it follows from Proposition III.5.4 that there exist $\mathbb{P}^u \in \mathcal{P}_L$, a \mathbb{P}_0 -martingale M^u starting from 0 and $K^u \in \mathbb{I}^2$ such that

$$d\hat{u}_t = dM_t^u + dK_t^u + q_t^u dt, \quad \mathbb{P}_0\text{-a.s.},$$

where $q_t^u := \frac{d\langle M^u, B \rangle}{dt}$. This implies

$$du_t = dM_t^u + (q_t^u + L_0|u_t| + F_t^0 + 1)dt + dK_t^u, \quad \mathbb{P}_0\text{-a.s.}$$

Similarly, for some \mathbb{P}_0 -martingale M^v starting from 0, $q_t^v := \frac{d\langle M^v, B \rangle}{dt}$ and $K^v \in \mathbb{I}^2$,

$$dv_t = dM_t^v + (q_t^v + L_0|v_t| + F_t^0 + 1)dt + dK_t^v, \quad \mathbb{P}_0\text{-a.s.}$$

Thus, with a \mathbb{P}^* -martingale M^* starting from 0 and an appropriate process σ^X ,

$$dX_t = dM_t^* - \sigma_t^X dt + d(K_t^u + K_t^v), \quad \mathbb{P}_0\text{-a.s.} \quad (\text{III.6.9})$$

Now for any $0 \leq s \leq t \leq T$, define $\tau_s := \inf\{t \geq s \wedge H : X_t = Y_t\}$. Recalling Theorem III.2.2, we have $K_{\tau_s} = K_{s \wedge H}$, \mathbb{P}_0 -a.s. Then, by (III.6.9) we have

$$\begin{aligned} \mathbb{E}^{\mathbb{P}^*} \left[K_{t \wedge H} - K_{s \wedge H} \middle| \mathcal{F}_{s \wedge H} \right] &= \mathbb{E}^{\mathbb{P}^*} \left[K_{t \wedge H} - K_{\tau_s} \middle| \mathcal{F}_{s \wedge H} \right] = \mathbb{E}^{\mathbb{P}^*} \left[Y_{\tau_s} - Y_{t \wedge H} \middle| \mathcal{F}_{s \wedge H} \right] \leq \mathbb{E}^{\mathbb{P}^*} \left[X_{\tau_s} - X_{t \wedge H} \middle| \mathcal{F}_{s \wedge H} \right] \\ &= \mathbb{E}^{\mathbb{P}^*} \left[\int_{\tau_s}^{t \wedge H} (\sigma_r^X dr - dK_r^u - dK_r^v) \middle| \mathcal{F}_{s \wedge H} \right] \leq \mathbb{E}^{\mathbb{P}^*} \left[\int_{s \wedge H}^{t \wedge H} |\sigma_r^X| dr \middle| \mathcal{F}_{s \wedge H} \right]. \end{aligned}$$

This implies that $dK_t \leq |\sigma_t^X| dt$, \mathbb{P}^* -a.s. and hence also \mathbb{P}_0 -a.s. □

Chapitre IV

Semilinear path dependent PDE : Existence via Perron's method

In Chapter III, we focus on the semilinear path dependent PDE (III.1.1) and prove the comparison result for continuous viscosity solutions, in the spirit of the work of Caffarelli and Cabré [15] in the context of PDEs. In [35, 102] it is also proved that the solutions of corresponding backward SDEs are viscosity solutions, instead, we are interested in proving the existence of viscosity solutions to semilinear path dependent PDEs by PDE-type arguments, that is, by Perron's method. It is worth noting that in the fully nonlinear case, one may no longer depend on backward SDEs for finding viscosity solutions for path dependent PDEs, and thus the Perron method will be necessary. Although our result cannot be applied to the fully nonlinear case directly, many arguments in this paper could be useful. Also, the Perron method is not only useful in proving the existence of viscosity solutions, but also has applications in various contexts, for example, the wellposedness of envelope viscosity solution (see [2]), the uniqueness of martingale problems [18], etc. In the proof of Perron's method, we follow the same idea as the classical literature on viscosity solutions of PDEs, but the arguments turn out to be different and nontrivial.

It is well understood in PDE literature that the comparison result for continuous viscosity solutions is not sufficient for the existence of solutions. In Perron's method, we need a comparison result for semicontinuous viscosity solutions. However, the argument in Chapter III cannot be adapted into this context directly, because it is not clear whether upper semicontinuous submartingales are almost everywhere punctually differentiable (a crucial intermediate result in Chapter III). In the current chapter, we apply a regularization on semicontinuous viscosity solutions so as to mollify them to be continuous. Let u be a viscosity subsolution, and u^n be

its regularized version. A reasonable regularization should satisfy :

$$u^n \text{ is continuous; } \quad u^n \rightarrow u, \text{ as } n \rightarrow \infty; \quad u^n \text{ is still a viscosity subsolution.}$$

The regularization we propose involves a backward distance for paths, is new in literature, satisfies all the above conditions and helps to prove the comparison result. It is worth mentioning that a regularization is probably inevitable in the study of the comparison result for fully nonlinear path dependent PDEs. The regularization we find might shed light on the future research.

In this chapter, we continue to using the notations introduced in Chapter III.

1 Comparison for semicontinuous viscosity solutions

We are going to show the following main result.

Assumption IV.1.1. *The generator function $F(\theta, y, z)$ satisfies the following assumptions.*

(i) *F is uniformly Lipschitz continuous in (y, z) , i.e. there exists a constant L such that*

$$|F(\cdot, y, z) - F(\cdot, y', z')| \leq L|y - y'| + L|z - z'|.$$

(ii) *There exists $F^0 \in C^0(\Theta)$ such that $|F(\cdot, 0, 0)| \leq F^0$.*

(iii) *There exists a function $\rho^F : (\theta, x, y) \in \Theta \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ such that ρ^F is continuous in (θ, x, y) and non-decreasing in γ , $\rho^F(\theta, 0, y) = 0$ for all $(\theta, y) \in \Theta \times \mathbb{R}$, and*

$$|F(\theta, y, \cdot) - F(\theta', y, \cdot)| \leq \rho^F(\theta, d(\theta, \theta'), y), \quad \text{for all } \theta, \theta' \in \Theta.$$

Definition IV.1.2. *A function $u : \Theta \rightarrow \mathbb{R}$ belongs to USC_b (resp. LSC_b), if u is bounded and satisfies*

$$u(\theta) \geq \limsup_{d(\theta, \theta') \rightarrow 0} u(\theta') \quad (\text{resp. } \leq \liminf_{d(\theta, \theta') \rightarrow 0} u(\theta')).$$

Theorem IV.1.3. *Let Assumption IV.1.1 hold true, and $u \in \text{USC}_b(\Theta), v \in \text{LSC}_b(\Theta)$ be viscosity subsolution and supersolution of path dependent PDE (III.1.1), respectively. If $u_T \leq v_T$, then $u \leq v$ on Θ .*

1.1 Regularization

We first introduce a backward pseudo-distance on Θ :

$$\overleftarrow{d}(\theta, \theta') := |t - t'| + \sup_{s \geq 0} |\omega_{(t-s) \vee 0} - \omega'_{(t'-s) \vee 0}|.$$

The following lemma explains the relation between $d(\cdot, \cdot)$ and $\overleftarrow{d}(\cdot, \cdot)$.

Lemma IV.1.4. *For all $\theta, \theta' \in \Theta$, we have*

$$\left| d(\theta, \theta') - \overleftarrow{d}(\theta, \theta') \right| \leq \bar{\rho}(\theta, |t - t'|), \quad \text{where } \bar{\rho}(\theta, \delta) := \sup_{|s-s'| \leq \delta} |\omega_{t \wedge s} - \omega_{t \wedge s'}|. \quad (\text{IV.1.1})$$

In particular, a function $f : \Theta \rightarrow \mathbb{R}$ is continuous in $d(\cdot, \cdot)$ if and only if f is continuous in $\overleftarrow{d}(\cdot, \cdot)$.

Proof Define $\omega_s = 0$ for $s < 0$. The first claim follows from the simple observation :

$$\left| |\omega_{t \wedge s} - \omega'_{t' \wedge s}| - |\omega_{t \wedge (t-t'+s)} - \omega'_{t' \wedge s}| \right| \leq |\omega_{t \wedge s} - \omega_{t \wedge (t-t'+s)}| \leq \bar{\rho}(\theta, |t - t'|).$$

The second claim is a trivial corollary. \square

For a viscosity subsolution $u \in \text{USC}_b(\Theta)$ and a viscosity supersolution $v \in \text{LSC}_b(\Theta)$, we define $M := \sup_{\theta \in \Theta} (|u(\theta)| \vee |v(\theta)|)$, and

$$u^n(\theta) := \sup_{\theta' \in \Theta} \left(u(\theta') - n \overleftarrow{d}(\theta, \theta') \right), \quad v^n(\theta) := \inf_{\theta' \in \Theta} \left(v(\theta') + n \overleftarrow{d}(\theta, \theta') \right), \quad (\text{IV.1.2})$$

Lemma IV.1.5. *For each n , u^n is bounded, Lipschitz continuous in $\overleftarrow{d}(\cdot, \cdot)$, and continuous in $d(\cdot, \cdot)$. Moreover, u^n is decreasing in n and $\lim_{n \rightarrow \infty} u^n(\theta) = u(\theta)$, for all $\theta \in \Theta$. The similar result holds true for v^n .*

Proof Clearly, u^n is bounded and Lipschitz continuous in $\overleftarrow{d}(\cdot, \cdot)$, for each n . By Lemma IV.1.4, u^n is also continuous in $d(\cdot, \cdot)$. Also, it is clear that u^n is decreasing in n and $u^n \geq u$ for each n . Define $u^\infty := \lim_{n \rightarrow \infty} u^n$. Then, $u^\infty \geq u$. On the other hand, since u is bounded, we have

$$u^n(\theta) := \sup_{\overleftarrow{d}(\theta', \theta) \leq \frac{2M}{n}} \left(u(\theta') - n \overleftarrow{d}(\theta, \theta') \right)$$

In particular, there exists θ^n such that

$$\overleftarrow{d}(\theta^n, \theta) \leq \frac{2M}{n} \quad \text{and} \quad u^n(\theta) \leq u(\theta^n) + \frac{1}{n}.$$

Therefore, $u^\infty(\theta) \leq \limsup_{n \rightarrow \infty} u(\theta^n)$. Since $u \in \text{USC}_b(\Theta)$, it follows that

$$u^\infty(\theta) \leq \limsup_{n \rightarrow \infty} u(\theta^n) \leq u(\theta).$$

□

1.2 Generator $F(\theta, y, z)$ independent of y

In this subsection we suppose that there is no dependence on y in the generator $F(\theta, y, z)$. Let $u \in \text{USC}_b(\Theta)$ be a viscosity subsolution of the path dependent PDE with the generator $F(\theta, y, z) = F_0(\theta, z)$, and $v \in \text{LSC}_b(\Theta)$ be a viscosity supersolution of the path dependent PDE with the generator $F(\theta, y, z) = F_0(\theta, z) + \delta(\theta)$. We suppose that Assumption [IV.1.1](#) holds true for both generators F_0 and $F_0 + \delta$. In particular, we denote $\rho^0 := \rho^{F_0} \vee \rho^{F_0 + \delta}$.

Remark IV.1.6. Recall Lemma [III.3.2](#). By looking into the proof, we note that the only important argument is using the result of optimal stopping problem. That is why the result of the lemma still holds true if $u_{\cdot \wedge \mathbb{H}}$ only satisfies the assumptions in Theorem [III.2.2](#).

Proposition IV.1.7. *For each n , u^n is a viscosity subsolution of the following path dependent PDE :*

$$-\mathcal{L}u^n(\theta) - F_0(\theta, \partial_\omega u^n(\theta)) - \rho^0(\theta, \epsilon_n(\theta)) \leq 0, \quad (\text{IV.1.3})$$

where $\epsilon_n(\theta) := \frac{2M+1}{n} + \bar{\rho}(\theta, \frac{2M}{n})$. Similarly, v^n is a viscosity supersolution of :

$$-\mathcal{L}v^n(\theta) - F_0(\theta, \partial_\omega v^n(\theta)) + \delta(\theta) + \rho^0(\theta, \epsilon_n(\theta)) \geq 0. \quad (\text{IV.1.4})$$

Proof We only prove the result for u^n . Let $(\alpha, \beta) \in \mathcal{J}_L u^n(\theta)$, i.e.

$$u^n(\theta) = \max_{\tau \in \mathcal{T}_\mathbb{H}} \bar{\mathcal{E}}_L \left[(u^n)_\tau^\theta - \alpha\tau - \beta B_\tau \right], \quad \text{for some } \mathbb{H} \in \mathcal{T}^+.$$

Without loss of generality, we may assume that $\mathbb{H}(\omega') \leq \mathbb{H}^n(\omega') := \inf\{t' : |t'| + \|\omega'_{t' \wedge \cdot}\| \geq \frac{1}{n}\}$ for all $\omega' \in \Omega$. For any $\epsilon > 0$, we have

$$u^n(\theta) - c > \bar{\mathcal{E}}_L \left[(u^n)_{\mathbb{H}}^{\theta} - (\alpha + \epsilon)_{\mathbb{H}} - \beta B_{\mathbb{H}} \right], \quad \text{for some } c > 0. \quad (\text{IV.1.5})$$

By the definition of u^n and $|u| \leq M$, there exists $\theta^n = (t^n, \omega^n) \in \Theta$ such that

$$\overleftarrow{d}(\theta, \theta^n) \leq \frac{2M}{n} \quad \text{and} \quad u^n(\theta) - c \leq u(\theta^n) - n \overleftarrow{d}(\theta, \theta^n). \quad (\text{IV.1.6})$$

Further, since $(u^n)_{\mathbb{H}}^{\theta} \geq u_{\mathbb{H}}^{\theta^n} - n \overleftarrow{d}((t + \mathbb{H}, \omega \otimes_t B), (t^n + \mathbb{H}, \omega^n \otimes_{t^n} B)) = u_{\mathbb{H}}^{\theta^n} - n \overleftarrow{d}(\theta, \theta^n)$, it follows from (IV.1.5) and (IV.1.6) that

$$u(\theta^n) > \bar{\mathcal{E}}_L \left[u_{\mathbb{H}}^{\theta^n} - (\alpha + \epsilon)_{\mathbb{H}} - \beta B_{\mathbb{H}} \right]$$

By Lemma III.3.2 and $\mathbb{H} \leq \mathbb{H}^n$, we may find $\bar{\theta}^n \in \Theta$ such that

$$(\alpha + \epsilon, \beta) \in \underline{\mathcal{J}}_L u(\bar{\theta}^n) \quad \text{and} \quad d(\theta^n, \bar{\theta}^n) \leq \frac{1}{n}.$$

Since u is a viscosity subsolution, we have

$$-(\alpha + \epsilon) - F_0(\bar{\theta}^n, \beta) \leq 0. \quad (\text{IV.1.7})$$

Further, by Assumption IV.1.1, we obtain that

$$|F_0(\bar{\theta}^n, \beta) - F_0(\theta, \beta)| \leq \rho^0(\theta, d(\theta, \bar{\theta}^n)) \leq \rho^0(\theta, d(\theta, \theta^n) + d(\theta^n, \bar{\theta}^n)) \leq \rho^0\left(\theta, d(\theta, \theta^n) + \frac{1}{n}\right). \quad (\text{IV.1.8})$$

By Lemma IV.1.4 and (IV.1.6), we have

$$d(\theta, \theta^n) \leq \overleftarrow{d}(\theta, \theta^n) + \bar{\rho}(\theta, |t - t^n|) \leq \frac{2M}{n} + \bar{\rho}\left(\theta, \frac{2M}{n}\right).$$

It follows from (IV.1.7) and (IV.1.8) that

$$-(\alpha + \epsilon) - F_0(\theta, \beta) - \rho^0(\theta, \epsilon_n(\theta)) \leq 0.$$

Finally, by letting $\epsilon \rightarrow 0$, we show that u^n is a viscosity subsolution of the path dependent PDE (IV.1.3). \square

In Proposition III.6.2, we proved that if u, v are viscosity subsolution and supersolution of

the same path dependent PDE, then $u - v$ is a viscosity subsolution of the equation $-\mathcal{L}w - L|w| - L|\partial_\omega w| = 0$. Here, although u^n, v^n are corresponding to two different equations, one may follow the same argument and prove that :

Proposition IV.1.8. *Denote $w^n := u^n - v^n$. Then $w^n \in \text{USC}_b$ is a viscosity subsolution of the path dependent PDE :*

$$-\mathcal{L}w^n(\theta) - L|\partial_\omega w^n(\theta)| \leq 2\rho^0(\theta, \epsilon_n(\theta)) + \delta(\theta). \quad (\text{IV.1.9})$$

Proposition IV.1.9. *Denote $w := u - v$. Then $w = \lim_{n \rightarrow \infty} w^n$ and is a viscosity subsolution of*

$$-\mathcal{L}w(\theta) - L|\partial_\omega w(\theta)| \leq \delta(\theta). \quad (\text{IV.1.10})$$

Proof By Lemma IV.1.5, we have $w = \lim_{n \rightarrow \infty} w^n$. Suppose $(\alpha, \beta) \in \underline{\mathcal{J}}_L w(\theta)$. Then by Lemma IV.2.5, for any n and $\epsilon > 0$, there exists $N_n \geq n$ and $\hat{\theta}^n$ such that

$$d(\hat{\theta}^n, \theta) \leq \frac{1}{n} \quad \text{and} \quad (\alpha + \epsilon, \beta) \in \underline{\mathcal{J}}_L w^{N_n}(\hat{\theta}^n)$$

By Proposition IV.1.8, w^n is a viscosity subsolution of equation (IV.1.9). Therefore,

$$-(\alpha + \epsilon) - L|\beta| \leq 2\rho^0(\hat{\theta}^n, \epsilon_n(\hat{\theta}^n)) + \delta(\hat{\theta}^n).$$

Let $n \rightarrow \infty$ and then $\epsilon \rightarrow 0$. It follows that $-\alpha - L|\beta| \leq 0$. So we verified that w is a viscosity subsolution of equation (IV.1.10). \square

1.3 Maximum principle

In this section, we study the equation corresponding to the Pucci's extremal operator :

$$-\mathcal{L}u - Lu^+ - L|\partial_\omega u| = 0. \quad (\text{IV.1.11})$$

Proposition IV.1.10 (Maximum principle). *Let $u \in \text{USC}_b(\Theta)$ be a viscosity subsolution of Equation (IV.1.11), and suppose that $u_T \leq 0$. Then, we have $u \leq 0$ on Θ .*

In preparation of the proof of Proposition IV.1.10, we need some observations. Recall the sup-convolution defined in (IV.1.2). Since $u \leq u^m$, we clearly have :

Lemma IV.1.11. *If $u \in \text{USC}_b(\Theta)$ is a viscosity subsolution of Equation (IV.1.11), then u is also a viscosity subsolution of :*

$$-\mathcal{L}u - L(u^m)^+ - L|\partial_\omega u| \leq 0. \quad (\text{IV.1.12})$$

For Equation (IV.1.12), the generator is :

$$F^m(\theta, z) = L(u^m(\theta))^+ - L|z|.$$

Further, we may estimate :

$$|F^m(\theta, z) - F^m(\theta', z)| \leq L(u^m(\theta) - u^m(\theta'))^+ \leq Lm \overleftarrow{d}(\theta, \theta') \leq Lm \left(d(\theta, \theta') + \bar{\rho}(\theta, d(\theta, \theta')) \right).$$

Therefore, generator F^m satisfies Assumption IV.1.1 and is among the generators independent of y discussed in the previous section.

Proof of Proposition IV.1.10 By using the same argument as in the proof of Proposition IV.1.7, we can prove that u^n is a viscosity subsolution of

$$-\mathcal{L}u^n(\theta) - L(u^m(\theta))^+ - L|\partial_\omega u^n(\theta)| - \rho^{n,m}(\theta) \leq 0,$$

where $\rho^{n,m}(\theta) := Cm \left(\frac{1}{n} + \bar{\rho}(\theta, \frac{C}{n}) \right)$ and C is a sufficiently large constant. Clearly, u^n is also a viscosity subsolution of :

$$-\mathcal{L}w(\theta) - L(w(\theta))^+ - L|\partial_\omega w(\theta)| \leq \rho^{n,m}(\theta) + L(u^m(\theta) - u^n(\theta))^+. \quad (\text{IV.1.13})$$

Now we introduce a function $v^{n,m}$:

$$v^{n,m}(\theta) := \bar{\mathcal{E}}_L \left[\int_0^{T-t} e^{Ls} \left((\rho^{n,m})_s^\theta + L((u^m)_s^\theta - (u^n)_s^\theta)^+ \right) ds + e^{L(T-t)} ((u^n)_{T-t}^\theta)^+ \right].$$

As a value function of a stochastic optimal control problem, one may easily prove that $v^{n,m}$ is viscosity supersolution of Equation (IV.1.13). Further it is clear that $v^{n,m} \in C(\Theta)$ and $v_T^{n,m} = (u_T^n)^+$. Then by Theorem III.1.4, we obtain that $u^n \leq v^{n,m}$ on Θ . Now let $n \rightarrow \infty$, we have

$$u(\theta) \leq \bar{\mathcal{E}}_L \left[\int_0^{T-t} e^{Ls} L((u^m)_s^\theta - u_s^\theta)^+ ds \right] \quad \text{for all } \theta \in \Theta,$$

where we used the fact $u_T \leq 0$. Finally, let $m \rightarrow \infty$, we get $u \leq 0$ on Θ . \square

1.4 Comparison result for general generators

In this section we are going to prove the comparison result for equations in the general form (III.1.1) under Assumption IV.1.1. Similar to Proposition 3.14 in [37] which provides a change of variable for continuous viscosity solutions, we show the following result on a change of variable for semi-continuous viscosity solutions.

Lemma IV.1.12. *Let $u \in \text{USC}_b(\Theta)$ be a viscosity subsolution of Equation (III.1.1). Define $\tilde{u}_t(\omega) := e^{-Lt}u_t(\omega)$. Then $\tilde{u} \in \text{USC}_b(\Theta)$ is a viscosity subsolution of the equation :*

$$-\mathcal{L}\tilde{u}(\theta) - L\tilde{u}(\theta) - e^{-Lt}F(\theta, e^{Lt}\tilde{u}(\theta), e^{Lt}\partial_\omega\tilde{u}(\theta)) = 0.$$

The similar result holds for viscosity supersolutions.

Proof Without loss of generality, we only verify the viscosity subsolution property at 0. Let $(\alpha, \beta) \in \underline{\mathcal{J}}_L\tilde{u}(0)$, i.e.

$$\tilde{u}_0 = \max_{\tau \in \mathcal{J}_H} \bar{\mathcal{E}}_L[\tilde{u}_\tau - \alpha\tau - \beta B_\tau] \quad \text{for some } H \in \mathcal{J}^+.$$

It means that

$$u_0 = \max_{\tau \in \mathcal{J}_H} \bar{\mathcal{E}}_L[e^{-L\tau}u_\tau - \alpha\tau - \beta B_\tau]. \quad (\text{IV.1.14})$$

Since we have

$$\lim_{t \rightarrow 0} \frac{e^{-Lt} - 1}{t} = -L \quad \text{and} \quad \limsup_{t \rightarrow 0} u_t \leq u_0,$$

for $\epsilon > 0$ we may assume that

$$e^{-Lt} - 1 + Lt \geq -\epsilon t \quad \text{and} \quad u_t \leq u_0 + \epsilon, \quad \text{for all } t \leq H.$$

From (IV.1.14), we obtain that for all $\tau \in \mathcal{J}_H$

$$\begin{aligned} u_0 &\geq \bar{\mathcal{E}}_L[(e^{-L\tau} - 1 + L\tau)u_\tau + u_\tau - L\tau u_\tau - \alpha\tau - \beta B_\tau] \\ &\geq \bar{\mathcal{E}}_L[\epsilon C\tau + u_\tau - L(u_0 + \epsilon)\tau - \alpha\tau - \beta B_\tau]. \end{aligned}$$

This implies that $(\alpha + Lu_0 + (L - C)\epsilon, \beta) \in \underline{\mathcal{J}}_L u(0)$. Thus

$$-\alpha - Lu_0 - (L - C)\epsilon - F(0, u_0, \beta) \leq 0.$$

By letting $\epsilon \rightarrow 0$, we obtain the desired result. \square

Remark IV.1.13. For continuous viscosity solutions, the previous result holds true for the change of variables of the form of $\tilde{u}_t(\omega) := e^{\lambda t} u_t(\omega)$ for all $\lambda \in \mathbb{R}$. However, as showed in the previous lemma, the same result only holds true for $\lambda \leq 0$ in the context of semi-continuous viscosity solutions.

Due to the previous lemma, without loss of generality we may assume that the generator $F : (\theta, y, z) \mapsto \mathbb{R}$ is non-decreasing in y .

Proof of Theorem IV.1.3 Since $u^n \geq u$, u is a viscosity subsolution of the equation :

$$-\mathcal{L}u(\theta) - F(\theta, u^n(\theta), \partial_\omega u(\theta)) \leq 0.$$

Similarly, v is a viscosity supersolution of the equation :

$$-\mathcal{L}v(\theta) - F(\theta, u^n(\theta), \partial_\omega u(\theta)) + L(u^n(\theta) - v^n(\theta))^+ \geq -\mathcal{L}v(\theta) - F(\theta, v^n(\theta), \partial_\omega u(\theta)) \geq 0.$$

Consider the generator $F^n(\theta, z) := F(\theta, u^n(\theta), z)$, and observe that

$$\begin{aligned} |F^n(\theta, z) - F^n(\theta', z)| &= |F(\theta, u^n(\theta), z) - F(\theta', u^n(\theta'), z)| \\ &\leq Ln \overleftarrow{d}(\theta, \theta') + \rho^F(\theta, d(\theta, \theta'), u^n(\theta)) \\ &\leq Ln \left(d(\theta, \theta') + \bar{\rho}(\theta, d(\theta, \theta')) \right) + \rho^F(\theta, d(\theta, \theta'), u^n(\theta)) \\ &=: \rho^{F^n}(\theta, d(\theta, \theta')). \end{aligned}$$

Therefore, the generator F^n is of the type discussed in the previous section. So by setting $\delta(\theta) := L(u^n(\theta) - v^n(\theta))^+$, we obtain from Proposition IV.1.9 that $w := u - v$ is a viscosity subsolution of the equation :

$$-\mathcal{L}w(\theta) - L|\partial_\omega w(\theta)| \leq L(u^n(\theta) - v^n(\theta))^+, \quad \text{for each } n.$$

Further, by letting $n \rightarrow \infty$, we have that w is a viscosity subsolution of Equation (IV.1.11). Finally, by the maximum principle (Proposition IV.1.10) we conclude that $w = u - v \leq 0$ on Θ . \square

2 Existence via Perron's method

Due to Proposition 3.14 in [37], we may equivalently study the existence of viscosity solution for the equation corresponding to the change of variable : $\tilde{u}_t := e^{-Lt}u_t$. It follows from the Lipschitz property of the nonlinearity F in y that we may assume without loss of generality that F is increasing in y .

Assumption IV.2.1. *The generator function $F(\theta, y, z)$ satisfies (i) of Assumptions IV.1.1 and :*

- (i) F is continuous in θ .
- (ii) F is non-decreasing in y .

For a function w on Θ , we define its USC and LSC envelopes :

$$w^*(\theta) := \limsup_{d(\theta, \theta') \rightarrow 0} w(\theta') \quad \text{and} \quad w_*(\theta) := \liminf_{d(\theta, \theta') \rightarrow 0} w(\theta').$$

The main result of this section is :

Theorem IV.2.2. *Let Assumption IV.2.1 and the comparison result of Theorem IV.1.3 hold true. Assume further that there is a viscosity subsolution $\underline{u} \in \text{USC}_b(\Theta)$ and a supersolution $\bar{v} \in \text{LSC}_b(\Theta)$ of Equation (III.1.1) which satisfy the boundary condition $(\underline{u})_T = \bar{v}_T^* = \xi$. Denote*

$$\mathcal{D} := \left\{ \phi : \phi \in \text{USC}_b(\Theta) \text{ is a viscosity subsolution of Equation (III.1.1) and } \underline{u} \leq \phi \leq \bar{v} \right\}.$$

Then $u(\theta) := \sup\{\phi(\theta) : \phi \in \mathcal{D}\}$ is a continuous viscosity solution of Equation (III.1.1), and satisfies the boundary condition $u_T = \xi$.

We will prove in the following subsections the two propositions :

Proposition IV.2.3. $u^* \in \text{USC}_b(\Theta)$ is a viscosity subsolution of Equation (III.1.1).

Proposition IV.2.4. $u_* \in \text{LSC}_b(\Theta)$ is a viscosity supersolution of Equation (III.1.1).

Then the comparison result allows to complete the proof.

Proof of Theorem IV.2.2 Since $u \geq \underline{u}$, we have $u_* \geq \underline{u}_*$, in particular, $(u_*)_T \geq \xi$. On the other hand, since $u \leq \bar{v}$, we have $u^* \leq \bar{v}^*$, in particular, $u_T^* \leq \xi$. Therefore, $u_T^* \leq (u_*)_T$, and it follows from the comparison result that $u^* \leq u_*$. We conclude that $u^* = u = u_*$, and thus u is a bounded continuous viscosity solution of Equation (III.1.1). \square

2.1 Stability of viscosity solutions

As in the theory of viscosity solutions for PDEs, the stability of solutions is crucial for the result of Perron's method.

Lemma IV.2.5. *Let $u \in \mathbb{L}^0(\mathbb{F})$ be bounded, and $u^n \in \mathbb{L}^0(\mathbb{F})$ be bounded and pathwise u.s.c. \mathbb{P}_0 -a.s. Fix $\theta \in \Theta$, and suppose that*

(i) *there exists a sequence $\{\theta^n\} \subset \Theta$ such that*

$$d(\theta^n, \theta) \rightarrow 0 \quad \text{and} \quad u(\theta) = \lim_{n \rightarrow \infty} u^n(\theta^n);$$

(ii) *for any $\bar{\theta} \in \Theta$ and any sequence $\{\bar{\theta}^n\} \subset \Theta$ such that $d(\bar{\theta}^n, \bar{\theta}) \rightarrow 0$, it holds*

$$u(\bar{\theta}) \geq \limsup_{n \rightarrow \infty} u^n(\bar{\theta}^n).$$

Then, for any $(\alpha, \beta) \in \underline{\mathcal{J}}_L u(\theta)$, $n \in \mathbb{N}$ and $\epsilon > 0$, there exists $N_n \geq n$ and $\hat{\theta}^n$ such that

$$d(\theta^{N_n}, \hat{\theta}^n) \leq \frac{1}{n} \quad \text{and} \quad (\alpha + \epsilon, \beta) \in \text{cl}\left(\underline{\mathcal{J}}_L u^{N_n}(\hat{\theta}^n)\right).$$

Proof Since $(\alpha, \beta) \in \underline{\mathcal{J}}_L u(\theta)$, there exists $\mathbb{H} \in \mathcal{T}^+$ such that $u(\theta) = \max_{\tau \in \mathcal{T}_\mathbb{H}} \bar{\mathcal{E}}_L[u_\tau^\theta - \alpha\tau - \beta B_\tau]$. Denote $\mathbb{H}^n(\omega') := \mathbb{H}(\omega') \wedge \inf\{t' : \|\omega'_{t'}\| > \frac{1}{n}\}$. Then for any $\epsilon > 0$, it holds

$$u(\theta) > \bar{\mathcal{E}}_L[u_{\mathbb{H}^n}^\theta - (\alpha + \epsilon)\mathbb{H}^n - \beta B_{\mathbb{H}^n}].$$

Further, by (i) and (ii), we obtain

$$\lim_{m \rightarrow \infty} u^m(\theta^m) > \bar{\mathcal{E}}_L\left[\limsup_{m \rightarrow \infty} (u^m)_{\mathbb{H}^n}^{\theta^m} - (\alpha + \epsilon)\mathbb{H}^n - \beta B_{\mathbb{H}^n}\right] \geq \limsup_{m \rightarrow \infty} \bar{\mathcal{E}}_L\left[(u^m)_{\mathbb{H}^n}^{\theta^m} - (\alpha + \epsilon)\mathbb{H}^n - \beta B_{\mathbb{H}^n}\right].$$

Therefore, for each n , there exists $N_n \geq n$ such that

$$u^{N_n}(\theta^{N_n}) > \bar{\mathcal{E}}_L\left[(u^{N_n})_{\mathbb{H}^n}^{\theta^{N_n}} - (\alpha + \epsilon)\mathbb{H}^n - \beta B_{\mathbb{H}^n}\right].$$

Then, by Lemma III.3.2, we may find $\hat{\theta}^n$ such that

$$d(\hat{\theta}^n, \theta^{N_n}) \leq \frac{1}{n} \quad \text{and} \quad (\alpha + \epsilon, \beta) \in \underline{\mathcal{J}}_L u^{N_n}(\hat{\theta}^n).$$

□

2.2 Representation of solution to a particular equation

We study a special path dependent PDE, and give one of its viscosity solutions by a stochastic representation. Let u be a bounded process and $\mathsf{H} \in \mathcal{H}$, and define a function :

$$\eta(\theta) := \underline{\mathcal{E}}_L \left[(u_{\mathsf{H}})^\theta - \alpha(\mathsf{H}^\theta - t) - \beta B_{\mathsf{H}^\theta - t} \right].$$

Proposition IV.2.6. (i) η is a viscosity subsolution of the path dependent PDE :

$$-\mathcal{L}\eta(\theta) + \alpha + L|\beta - \partial_\omega \eta(\theta)| = 0.$$

(ii) If u is Lipschitz continuous, then η is continuous on $\{\theta : t \leq \mathsf{H}(\omega)\}$.

In preparation to the proof of Proposition IV.2.6, we study the processes :

$$\bar{\eta}_t := \underline{\mathcal{E}}_L \left[u_{\mathsf{H}} - \alpha \mathsf{H} - \beta B_{\mathsf{H}} \middle| \mathcal{F}_t \right] := \operatorname{ess\,inf}_{\mathbb{P} \in \mathcal{P}_L} \mathbb{E}^{\mathbb{P}} \left[u_{\mathsf{H}} - \alpha \mathsf{H} - \beta B_{\mathsf{H}} \middle| \mathcal{F}_t \right].$$

Similar to Proposition 6.5 in [102], one may easily prove the following result of dynamic programming.

Lemma IV.2.7. *There exists $Z \in \mathbb{H}^2$ such that*

$$\bar{\eta}_t = u_{\mathsf{H}} - a(\mathsf{H} - t) + \int_t^{\mathsf{H}} L|\beta - Z_s| ds - \int_t^{\mathsf{H}} Z_s dB_s.$$

Moreover, it holds $\mathbb{P}_0 \left[\eta_\tau = \bar{\eta}_\tau \right] = 1$ for all $\tau \in \mathcal{T}_{\mathsf{H}}$. In particular, we have

$$\eta_0 = \underline{\mathcal{E}}_L \left[\eta_\tau - \alpha\tau - \beta B_\tau \right] \quad \text{for all } \tau \in \mathcal{T}_{\mathsf{H}}. \quad (\text{IV.2.1})$$

Proof of Proposition IV.2.6 Without loss of generality, we only need to verify the properties at $\theta = (0, 0)$.

(i) By Lemma IV.2.7, η is \mathbb{F}^* -adapted. Take $(\alpha', \beta') \in \mathcal{J}\eta_0$, i.e.

$$\eta_0 = \max_{\tau \in \mathcal{T}_{\mathsf{H}'}} \bar{\mathcal{E}}_L \left[\eta_\tau - \alpha'\tau - \beta' B_\tau \right] \quad \text{for some } \mathsf{H}' \in \mathcal{T}^+.$$

In view of (IV.2.1), we obtain that

$$\mathbb{E}^{\mathbb{P}^\mu} \left[\eta_\tau - \alpha\tau - \beta B_\tau \right] \geq \eta_0 \geq \mathbb{E}^{\mathbb{P}^\mu} \left[\eta_\tau - \alpha'\tau - \beta' B_\tau \right], \quad \text{for all } \mathbb{P}^\mu \in \mathcal{P}_L \text{ and } \tau \in \mathcal{T}_{\mathsf{H} \wedge \mathsf{H}'}$$

So, $\mathbb{E}^{\mathbb{P}^\mu}[-(\alpha' - \alpha)\tau - (\beta' - \beta)B_\tau] \leq 0$ for all $\tau \in \mathcal{T}_{\mathbb{H} \wedge \mathbb{H}'}$. It follows that

$$-\alpha' + \alpha - (\beta' - \beta) \cdot \mu \leq 0.$$

By taking $\mu^* := -L(\text{sgn}(\beta'_i - \beta_i))_{1 \leq i \leq d}$, we obtain that

$$-\alpha' + \alpha + L|\beta' - \beta| \leq 0.$$

(ii) Since u is Lipschitz continuous, one may easily estimate that

$$\begin{aligned} |\eta(\theta) - \eta(\theta')| &\leq C \left(\bar{\mathcal{E}}_L[|\mathbb{H}^\theta - \mathbb{H}^{\theta'}| + \|B_{(\mathbb{H}^\theta - t)^\wedge} - B_{(\mathbb{H}^{\theta'} - t')^\wedge}\|] + d(\theta, \theta') \right) \\ &\leq C' \left(\bar{\mathcal{E}}_L[|\mathbb{H}^\theta - \mathbb{H}^{\theta'}|] + d(\theta, \theta') \right). \end{aligned}$$

We applied BDG inequality for the last inequality. Since $\mathbb{H} \in \mathcal{H}$, we may suppose

$$\mathbb{H} = T_0 \wedge \mathbb{H}_0, \quad \mathbb{H}_0 := \inf\{t : \omega_t \notin O\} \text{ for some bounded open set } O.$$

Then it is clear that $|\mathbb{H}^\theta - \mathbb{H}^{\theta'}| \leq |t - t'| + |\mathbb{H}_0^\theta - \mathbb{H}_0^{\theta'}|$. Further it is proved in [99] that

$$\lim_{d(\theta, \theta') \rightarrow 0} \bar{\mathcal{E}}_L[|\mathbb{H}_0^\theta - \mathbb{H}_0^{\theta'}|] = 0.$$

Therefore function η is continuous. □

2.3 Subsolution property

Proof of Proposition IV.2.3 Fix any $\theta \in \Theta$. By the definition of u and u^* , there is a sequence of functions $\{\phi^n\} \subset \mathcal{D}$ and a sequence $\{\theta^n\} \subset \Theta$ such that

$$d(\theta^n, \theta) \rightarrow 0 \quad \text{and} \quad u^*(\theta) = \lim_{n \rightarrow \infty} \phi^n(\theta^n).$$

Then by Lemma IV.2.5, for any $(\alpha, \beta) \in \underline{\mathcal{J}}_L u(\theta)$, $n \in \mathbb{N}$ and $\epsilon > 0$, there is $N_n \geq n$ and $\hat{\theta}^n$ such that

$$d(\theta^{N_n}, \hat{\theta}^n) \leq \frac{1}{n} \quad \text{and} \quad (\alpha + \epsilon, \beta) \in \text{cl}\left(\underline{\mathcal{J}}_L \phi^{N_n}(\hat{\theta}^n)\right).$$

Further, since $\phi^n (\leq u)$ is a viscosity subsolution of Equation (III.1.1) for each n , we deduce from the non-decrease of F in y that

$$-(\alpha + \epsilon) - F(\hat{\theta}^n, u(\hat{\theta}^n), \beta) \leq -(\alpha + \epsilon) - F(\hat{\theta}^n, \phi^{N_n}(\hat{\theta}^n), \beta) \leq 0.$$

Then since $\limsup_{n \rightarrow \infty} u(\hat{\theta}^n) \leq u^*(\theta)$, by letting $n \rightarrow \infty$ we obtain that

$$-(\alpha + \epsilon) - F(\theta, u^*(\theta), \beta) \leq 0.$$

Finally, by letting $\epsilon \rightarrow 0$, we get the desired result. \square

Proposition IV.2.8. *It holds that $u = u^* \in \text{USC}_b(\Theta)$ is a viscosity subsolution of Equation (III.1.1).*

Proof By the previous proposition, we know that $u^* \in \mathcal{D}$, and thus $u^* \leq u$. On the other hand, by the definition of u^* , it holds that $u^* \geq u$. Therefore, $u = u^*$. \square

2.4 Supersolution property

Proof of Proposition IV.2.4 1. Suppose that u_* is not a viscosity supersolution. Then by Proposition III.3.3, there is $\theta^0 = (t^0, \omega^0) \in \Theta$ and $(\alpha, \beta) \in \bar{\mathcal{J}}'_L u_*(\theta^0)$, i.e. $u_*(\theta^0) = \min_{\tau \in \mathcal{T}_H} \underline{\mathcal{E}}_L[(u_*)_{\tau}^{\theta^0} - \alpha\tau - \beta B_{\tau}]$ for some $H \in \mathcal{H}$, such that

$$-\alpha - F(\theta^0, u_*(\theta^0), \beta) =: -2\delta < 0. \quad (\text{IV.2.2})$$

Since $F(\theta, y, z)$ is non-decreasing in y and $u_* \in \text{LSC}_b(\Theta)$, it follows from (IV.2.2) that

$$-\alpha + \delta - F(\cdot, u_*, \beta) < 0 \quad \text{on } O_{9\epsilon_0} := \{\theta : d(\theta^0, \theta) < 9\epsilon_0\} \quad \text{for some small } \epsilon_0 > 0. \quad (\text{IV.2.3})$$

Without loss of generality, we may assume that H is in the form of :

$$H(\omega) = 3\epsilon_1 \wedge \inf\{s : |\omega_s| \geq 3\epsilon_1\} \quad \text{for some } \epsilon_1 > 0 \text{ such that } 3\epsilon_1 < 3\epsilon_0 \wedge \rho_{\theta^0}^{-1}(3\epsilon_0),$$

where ρ_{θ^0} is an invertible modulus of continuity of the path ω^0 , and $\rho_{\theta^0}^{-1}$ is the inverse function. Further, take a small neighborhood O_{ϵ_2} of θ^0 , where

$$\epsilon_2 < \epsilon_1 \wedge \rho_{\theta^0}^{-1}(\epsilon_1).$$

We next introduce two stopping times :

$$H_0(\omega) := \inf\{t \geq 0 : \theta \in O_{\epsilon_2}\} \quad \text{and} \quad H_1(\omega) := \inf\{t \geq H_0(\omega) : |\omega_t - \omega_{t^0}^0| \geq 3\epsilon_1\} \wedge (t^0 + 3\epsilon_1),$$

together with the set :

$$Q := \{\theta \in \Theta : H_0(\omega) \leq t \leq H_1(\omega)\}.$$

We claim and will prove in Step 5 that

$$O_{\epsilon_2} \subset Q \subset O_{9\epsilon_0}.$$

In particular, we have $H_0(\omega^0) < t^0 < H_1(\omega^0)$, and thus $H_1^{\theta^0} - t^0 = H$. Since $(\alpha, \beta) \in \bar{\mathcal{J}}_L u_*(\theta)$, we have

$$u_*(\theta^0) < \underline{\mathcal{E}}_L \left[(u_*)_{H_1^{\theta^0} - t^0}^{\theta^0} - (\alpha - \delta)(H_1^{\theta^0} - t^0) - \beta B_{H_1^{\theta^0} - t^0} \right].$$

We next define the inf-convolution of u_* :

$$\underline{u}^n(\theta) := \inf_{\theta' \in \Theta} \{u_*(\theta') + nd(\theta', \theta)\} \quad \text{for all } \theta \in \Theta. \quad (\text{IV.2.4})$$

Notice that \underline{u}^n is Lipschitz continuous. Since $u_* \in \text{LSC}_b(\Theta)$, it is easy to show that $\underline{u}^n \uparrow u_*$. Thus, by (IV.2.4), we deduce that for n sufficiently large

$$u_*(\theta^0) < \underline{\mathcal{E}}_L \left[(\underline{u}^n)_{H_1^{\theta^0} - t^0}^{\theta^0} - (\alpha - \delta)(H_1^{\theta^0} - t^0) - \beta B_{H_1^{\theta^0} - t^0} \right].$$

By defining $\varphi(\theta) := \underline{\mathcal{E}}_L \left[(\underline{u}^n)_{H_1^\theta - t}^\theta - (\alpha - \delta)(H_1^\theta - t) - \beta B_{H_1^\theta - t} \right]$ for all $\theta \in \Theta$, we have

$$\varphi(\theta^0) > u_*(\theta^0). \quad (\text{IV.2.5})$$

We finally define

$$U := (\varphi \vee u)1_Q + u1_{Q^c}.$$

2. In this step, we show that φ is viscosity subsolution of the equation :

$$-\mathcal{L}w - F(\cdot, \varphi \vee u, \partial_\omega w) \leq 0, \quad \text{on } \{\theta : t < H_1(\omega)\}. \quad (\text{IV.2.6})$$

It follows from Proposition IV.2.6 that for all $(\alpha', \beta') \in \underline{\mathcal{J}}_L \varphi(\theta)$, it holds that

$$-\alpha' + \alpha - \delta + L|\beta - \beta'| \leq 0.$$

Further, by (IV.2.3) we obtain that

$$-\alpha' - F(\theta, (\varphi \vee u)(\theta), \beta') \leq -\alpha' - F(\theta, u_*(\theta), \beta) + L|\beta - \beta'| \leq 0.$$

So the desired result follows.

3. In this step, we prove that U is a viscosity subsolution of Equation (III.1.1). First, for $\theta \in Q^o := \{\theta : \mathbf{H}_0(\omega) \leq t < \mathbf{H}_1(\omega)\}$, it is clear that both φ and u are viscosity subsolutions of Equation (IV.2.6). Then take any $(\alpha', \beta') \in \underline{\mathcal{J}}_L U(\theta)$, i.e.

$$U(\theta) = \max_{\tau \in \mathcal{T}_{\mathbf{H}'}} \bar{\mathcal{E}}_L[U_\tau^\theta - \alpha'\tau - \beta'B_\tau] \quad \text{for some } \mathbf{H}' \in \mathcal{T}^+.$$

If $u(\theta) \leq \varphi(\theta)$, then it follows that

$$\varphi(\theta) \geq \bar{\mathcal{E}}_L[U_\tau^{\theta'} - \alpha'\tau - \beta'B_\tau] \geq \bar{\mathcal{E}}_L[\varphi_\tau^{\theta'} - \alpha'\tau - \beta'B_\tau] \quad \text{for all } \tau \in \mathcal{T}_{\mathbf{H}'}$$

Thus $(\alpha', \beta') \in \underline{\mathcal{J}}_L \varphi(\theta)$. Otherwise, if $u(\theta) > \varphi(\theta)$, we may similarly get $(\alpha', \beta') \in \underline{\mathcal{J}}_L u(\theta)$. In both cases, it follows that

$$-\alpha' - F(\theta, (\varphi \vee u)(\theta), \beta') \leq 0.$$

So we have proved that U is a viscosity subsolution of Equation (III.1.1) on Q^o .

On the other hand, for $\theta \in (Q^o)^c$, we have $U(\theta) = u(\theta)$, because whenever $t = \mathbf{H}_1(\omega)$ we have $\varphi(\theta) = \underline{u}^n(\theta) \leq u_*(\theta) \leq u(\theta)$. Then it becomes trivial to verify that U is a viscosity subsolution of Equation (III.1.1) on $(Q^o)^c$.

4. Our objective is to construct a viscosity subsolution in $\text{USC}_b(\Theta)$. Since we did not prove Q is closed, we do not know whether $U \in \text{USC}_b(\Theta)$ itself. We next prove that the USC envelop U^* is still a viscosity subsolution of Equation (III.1.1). Take any $(\alpha', \beta') \in \underline{\mathcal{J}}_L U^*(\theta)$. By the definition of U^* , there exists a sequence $\{\theta^n\} \subset \Theta$ such that

$$d(\theta^n, \theta) \rightarrow 0, \quad \text{and} \quad \lim_{n \rightarrow \infty} U(\theta^n) = U^*(\theta).$$

Further, by (ii) of Proposition IV.2.6, U is pathwise u.s.c. Consequently, we can apply Lemma

IV.2.5 and obtain that for any $n \in \mathbb{N}$ and $\epsilon' > 0$, there exists $N_n \geq n$ and $\hat{\theta}^n$ such that

$$d(\theta^{N_n}, \hat{\theta}^n) \leq \frac{1}{n} \quad \text{and} \quad (\alpha' + \epsilon', \beta') \in \text{cl}\left(\mathcal{J}_L U(\hat{\theta}^n)\right).$$

Since U is a viscosity subsolution of Equation (III.1.1) and F is non-decreasing in y , we have

$$-\alpha' - \epsilon' - F(\hat{\theta}^n, U^*(\hat{\theta}^n), \beta') \leq -\alpha' - \epsilon' - F(\hat{\theta}^n, U(\hat{\theta}^n), \beta') \leq 0.$$

Letting $n \rightarrow \infty$ and $\epsilon' \rightarrow 0$, we get

$$-\alpha' - F(\theta, U^*(\theta), \beta') \leq 0.$$

Then it is clear that $U^* \in \mathcal{D}$, so $U^* \leq u$ on Θ . On the other hand, there exists a sequence $\{\theta^n\} \subset O_{\epsilon_2}$ such that $u_*(\theta^0) = \lim_{n \rightarrow \infty} u(\theta^n)$. Also, by Proposition IV.2.6, φ is continuous on $Q \supset O_{\epsilon_2}$. Then by (IV.2.5) we have

$$\liminf_{n \rightarrow \infty} (U^* - u)(\theta^n) \geq \lim_{n \rightarrow \infty} (\varphi - u)(\theta^n) = \varphi(\theta^0) - u_*(\theta^0) > 0.$$

Therefore, there is θ^n such that $U^*(\theta^n) > u(\theta^n)$. That is a contradiction to $U^* \in \mathcal{D}$.

5. We finally complete the proof of $O_{\epsilon_2} \subset Q \subset O_{9\epsilon_0}$. First, for all $\theta \in O_{\epsilon_2}$, it is clear that $H_0(\omega) \leq t$. We denote $t_0 := H_0(\omega)$ and then consider $s \in [t_0, t]$. Since $|t_0 - t^0| \leq \epsilon_2$ and $|t - t^0| \leq \epsilon_2$, we have $|s - t^0| \leq \epsilon_2$. Further, since $\theta \in O_{\epsilon_2}$, we have

$$|\omega_s - \omega_{t^0 \wedge s}^0| \leq d(\theta, \theta^0) \leq \epsilon_2 \leq \epsilon_1,$$

and

$$|\omega_s - \omega_{t^0}^0| \leq |\omega_s - \omega_{t^0 \wedge s}^0| + |\omega_{t^0 \wedge s}^0 - \omega_{t^0}^0| \leq \epsilon_1 + \rho_{\theta^0}(t^0 - t^0 \wedge s) < 2\epsilon_1.$$

It follows that $H_1(\omega) \geq t$, and thus $\theta \in Q$.

Next, take any $\theta \in Q$. Still denote $t_0 := H_0(\omega)$. For $s \leq t_0$, since $(t_0, \omega) \in O_{\epsilon_2}$, it is clear that

$$|\omega_s - \omega_{t^0 \wedge s}^0| \leq d((t_0, \omega), \theta^0) \leq \epsilon_2.$$

On the other hand, for $s \in [t_0, t]$, since $s \leq t \leq H_1(\omega)$, it holds

$$|t^0 - s| \leq 3\epsilon_1 < 3\epsilon_0 \quad \text{and} \quad |\omega_s - \omega_{t^0 \wedge s}^0| \leq |\omega_s - \omega_{t^0}^0| + |\omega_{t^0 \wedge s}^0 - \omega_{t^0}^0| \leq 3\epsilon_1 + \rho_{\theta^0}(t^0 - t^0 \wedge s) < 6\epsilon_0.$$

It follows that $d(\theta, \theta^0) < 9\epsilon_0$, and thus $\theta \in O_{9\epsilon_0}$. □

Chapitre V

Monotone scheme for fully nonlinear path dependent PDEs

In their seminal work [4], Barles and Souganidis proved a convergence theorem for monotone numerical schemes for viscosity solutions of fully nonlinear PDEs. Assuming that a strong comparison principle holds true for viscosity solutions of a PDE, they show that for all numerical schemes satisfying the three properties, “monotonicity”, “consistency” and “stability”, the numerical solutions converge locally uniformly to the unique viscosity solution of the PDE as the discretization parameters converge to zero. They mainly use the stability of viscosity solutions of PDEs and the local compactness of the state space. Due to their result, one only needs to check some local properties of a numerical scheme in order to get a global convergence result. Also, their result and method are widely used in the numerical analysis of viscosity solutions to PDEs. It would be interesting to extend the convergence theorem of Barles and Souganidis [4] in the context of PPDE.

1 Notation

Unlike the previous two chapters, we study the general second order path dependent PDEs in this chapter. We consider the following path dependent PDE

$$-\partial_t u(t, \omega) - G(\cdot, u, \partial_\omega u, \partial_{\omega\omega}^2 u)(t, \omega) = 0, \quad \text{for all } (t, \omega) \in \Theta, \quad (\text{V.1.1})$$

with the terminal condition $u(T, \cdot) = \xi$. We next introduce the nonlinear expectation corresponding to the fully nonlinear path dependent PDEs. As in [37], we fix a constant $L > 0$, and denote by \mathcal{P} the collection of all continuous semimartingale measures \mathbb{P} on Ω whose drift and diffusion coefficients are bounded by L . More precisely, a probability measure $\mathbb{P} \in \mathcal{P}$ if under \mathbb{P} , the canonical process B is a semimartingale with natural decomposition $B = A^\mathbb{P} + M^\mathbb{P}$, where

$A^{\mathbb{P}}$ is a process of finite variation, $M^{\mathbb{P}}$ is a continuous martingale with quadratic variation $\langle M^{\mathbb{P}} \rangle$, such that $A^{\mathbb{P}}$ and $\langle M^{\mathbb{P}} \rangle$ are absolutely continuous in t , and

$$\|\mu^{\mathbb{P}}\|_{\infty}, \|a^{\mathbb{P}}\|_{\infty} \leq L, \quad \text{where } \mu_t^{\mathbb{P}} := \frac{dA_t^{\mathbb{P}}}{dt}, \quad a_t^{\mathbb{P}} := \frac{d\langle M^{\mathbb{P}} \rangle_t}{dt}, \quad \mathbb{P}\text{-a.s.} \quad (\text{V.1.2})$$

We then define the nonlinear expectations :

$$\bar{\mathcal{E}}[\cdot] := \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}[\cdot] \quad \text{and} \quad \underline{\mathcal{E}}[\cdot] := \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}[\cdot]. \quad (\text{V.1.3})$$

Recall the definition of \mathcal{P} -viscosity sub-/super-solutions in Definition II.2.3, as well as that of jets in (II.2.5).

2 Monotone condition

Definition V.2.1. Let $h > 0$, K be a subset of a metric space, $\mathbb{F}_h : K \times [0, 1] \rightarrow \mathbb{R}$ be a Borel measurable function. Let $\{U_i, i \geq 1\}$ be a sequence of independent random variables defined on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Every U_i follows the uniform distribution on $[0, 1]$. Denote the filtration $\tilde{\mathbb{F}} := \{\tilde{\mathcal{F}}_i, i \in \mathbb{N}\}$, where $\tilde{\mathcal{F}}_n := \sigma\{U_i, i \leq n\}$. Let $\mathcal{K} = \mathbb{L}^0(\tilde{\mathbb{F}}, K)$ denote the collection of all $\tilde{\mathbb{F}}$ -adapted control processes taking values in K . For all $\nu \in \mathcal{K}$, we define

$$X_{(i+1)h}^{h,\nu} = X_{ih}^{h,\nu} + \mathbb{F}_h(\nu_{ih}, U_i). \quad (\text{V.2.1})$$

Further, we denote by $\hat{X}^{h,\nu} : [0, T] \times \tilde{\Omega} \rightarrow \Omega$ the linear interpolation of the discrete process $\{X_{ih}^{h,\nu}, i \in \mathbb{N}\}$ such that $\hat{X}_{ih}^{h,\nu} = X_{ih}^{h,\nu}$ for all i . Finally, for any function $\varphi \in \mathbb{L}^0(\mathcal{F})$, we define the nonlinear expectation :

$$\underline{\mathcal{E}}_h[\varphi] := \inf_{\nu \in \mathcal{U}} \tilde{\mathbb{E}}\left[\varphi(\hat{X}^{h,\nu})\right] \quad \text{and} \quad \bar{\mathcal{E}}_h[\varphi] := \sup_{\nu \in \mathcal{U}} \tilde{\mathbb{E}}\left[\varphi(\hat{X}^{h,\nu})\right]. \quad (\text{V.2.2})$$

We next introduce the numerical schemes \mathbb{T} . Let $(t, \omega) \in [0, T] \times \Omega$ and $0 < h \leq T - t$, $\mathbb{T}_h^{t,\omega}$ be a family of functions from $\mathbb{L}^0(\mathcal{F}_{t+h})$ to \mathbb{R} . We then define

$$u^h(t, \omega) := \mathbb{T}_h^{t,\omega} u_{t+h}^h,$$

and assume that \mathbb{T} satisfies the following conditions.

Assumption V.2.2. (i) *Consistency* : for every $(t, \omega) \in [0, T) \times \Omega$ and $\varphi \in C_0^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d)$,

$$\lim_{(t', \omega', h, c) \rightarrow (t, 0, 0, 0)} \frac{[c + \varphi](t', \omega \otimes_t \omega') - \mathbb{T}_h^{t', \omega \otimes_t \omega'} [[c + \varphi](t' + h, \cdot)]}{h} = \mathcal{L}^{t, \omega} \varphi_0.$$

(ii) *Monotonicity* : there exist K, \mathbb{F}_h, U_i as in Definition V.2.1 such that, for any $\varphi, \psi \in \mathbb{L}^0(\mathcal{F}_{t+h})$, it holds that

$$\mathbb{T}_h^{t, \omega}[\varphi] - \mathbb{T}_h^{t, \omega}[\psi] \geq \inf_{0 \leq \alpha \leq L} \underline{\mathcal{E}}_h [e^{\alpha h} (\varphi - \psi)^{t, \omega}] - h\rho(h). \quad (\text{V.2.3})$$

Moreover, Φ_h satisfies that for all $v \in K$,

$$|\tilde{\mathbb{E}}[\mathbb{F}_h(v, U)]| \leq Lh, \quad \text{Var}[\mathbb{F}_h(v, U)] \leq Lh \quad \text{and} \quad \tilde{\mathbb{E}}[\mathbb{F}_h(v, U)^3] \leq Lh^{3/2}. \quad (\text{V.2.4})$$

(iii) *Stability* : u^h is uniformly bounded and uniformly continuous in (t, ω) , uniformly on h .

Our main theorem is the following convergence result of the monotone scheme for path dependent PDE (V.1.1).

Theorem V.2.3. *Assume that*

- *path dependent PDE (V.1.1) is parabolic, i.e. $G(t, \omega, y, z, \gamma)$ is nondecreasing in γ ,*
- *the nonlinearity G of path dependent PDE (V.1.1) and the terminal condition ξ are continuous in all arguments, and $G(t, \omega, y, z, \gamma)$ is uniformly Lipschitz in y ,*
- *the comparison principle of viscosity solutions of (V.1.1) holds, i.e. if $u, v \in \text{BUC}(\Theta)$ are \mathcal{P} -viscosity subsolution and supersolution of path dependent PDE (V.1.1), respectively, and $u(T, \cdot) \leq v(T, \cdot)$, then $u \leq v$ on Θ .*

If the numerical scheme \mathbb{T} satisfies Assumption V.2.2, then path dependent PDE (V.1.1) admits a unique bounded viscosity solution u , and

$$u^h \rightarrow u \quad \text{locally uniformly, as } h \rightarrow 0. \quad (\text{V.2.5})$$

Remark V.2.4. A comparison result of viscosity solutions of fully nonlinear path dependent PDEs is proved in [38] for path dependent PDE (V.1.1) under certain conditions. Further, in the case of semilinear path dependent PDEs, a comparison result is proved in Ren, Touzi and Zhang [102] under very general assumptions.

Remark V.2.5 (Comparison with Zhang and Zhuo [114]). Let us compare our Assumption V.2.2 with that in [114]. Our condition (i) is weaker and thus easier to verify comparing to that

in [114]. The essential difference is between our condition (ii) and theirs. Our condition (ii), although stated in a complicated way, is satisfied by all (to the best of our knowledge) classical monotone scheme in PDE context. Moreover, by the interpretation of the finite difference scheme for stochastic control problem as controlled Markov chains (see Kushner and Dupuis [78]), this condition is consistent with the classical one in [4].

The next section is devoted to proving Theorem V.2.3.

3 Convergence theorem

3.1 Preliminary results

In preparation of the proof of Theorem V.2.3, we prove the following lemmas.

Lemma V.3.1 (Fatou's Lemma). *Assume that the random variables $X^n \in C^0(\mathcal{F})$ are bounded. Then we have*

$$\liminf_{n \rightarrow \infty} \underline{\mathcal{E}}[X^n] \geq \underline{\mathcal{E}}\left[\liminf_{n \rightarrow \infty} X^n\right]$$

Proof In order to prove the Fatou lemma, it is enough to show the monotone convergence theorem, i.e. given a sequence $\{X^n : n \in \mathbb{N}\}$ of increasing random variables, we have

$$\lim_{n \rightarrow \infty} \underline{\mathcal{E}}[X^n] = \underline{\mathcal{E}}[\lim_{n \rightarrow \infty} X^n]. \quad (\text{V.3.1})$$

Since $X^n \in C^0(\mathcal{F})$ for each n , it follows from Theorem 31 in [28] that (V.3.1) holds true. \square

Recall the nonlinear expectation $\underline{\mathcal{E}}_h$ defined in (V.2.2).

Lemma V.3.2. *Let $\varphi : \Omega \rightarrow \mathbb{R}$ be bounded uniformly continuous. Then there exists a modulus continuity $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which depends only on the continuity modulus of φ and $|\varphi|_0$, such that*

$$\underline{\mathcal{E}}[\varphi] \leq \underline{\mathcal{E}}_h[\varphi] + \rho(h).$$

Proof Denote $\rho' : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ as a continuity modulus of φ . Let $\nu \in \mathcal{K}$ and $X^{h,\nu}$ be defined by (V.2.1) and $\widehat{X}^{h,\nu}$ its linear interpolation on $[0, T]$. Then under the condition (V.2.4), it follows from Lemma 4.8 of Tan [110] (see also Dolinsky [33]) that we can construct a process $\widehat{X}^{h,\nu}$ and another process \bar{X} in the same probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, such that the image measure of \bar{X}

lies in \mathcal{P} , and for some constant C independent of h ,

$$\tilde{\mathbb{P}}\left(\left|\widehat{X}^{h,\nu} - \bar{X}\right| \geq h^{1/8}\right) \leq Ch^{1/8}.$$

Let $\rho(h) := \rho'(h^{1/8}) + 2\|\varphi\|_\infty h^{1/8}$, then it follows that

$$\underline{\mathcal{E}}[\varphi] \leq \tilde{\mathbb{E}}[\varphi(\bar{X})] \leq \tilde{\mathbb{E}}[\varphi(\widehat{X}^{h,\nu})] + \rho(h),$$

which concludes the proof by the arbitrariness of $\nu \in \mathcal{K}$. □ □

Lemma V.3.3. *Let $\varphi : \Omega \rightarrow \mathbb{R}$ be lower semicontinuous and bounded from below, then it holds for all $(t, \omega) \in \Theta$ that*

$$\liminf_{h \rightarrow 0} \underline{\mathcal{E}}_h[\varphi] \geq \underline{\mathcal{E}}[\varphi].$$

In particular, by defining $\mathbb{H} := \inf\{t \geq 0 : |B_t| \geq x\}$ for some $x > 0$, we have

$$\limsup_{h \rightarrow 0} \bar{\mathcal{E}}_h[1_{\{\mathbb{H} \leq \delta\}}] \leq \bar{\mathcal{E}}[1_{\{\mathbb{H} \leq \delta\}}] \quad \text{for any } \delta > 0.$$

Proof Define the approximation for the function φ :

$$\varphi^n(\omega) := \inf_{\omega' \in \Omega} \left\{ \varphi(\omega') + n\|\omega - \omega'\| \right\}.$$

Clearly, for each $n \in \mathbb{N}$, function φ^n is Lipschitz continuous, and $\varphi^n \uparrow \varphi$. By Lemma V.3.2, we obtain that

$$\liminf_{h \rightarrow 0} \underline{\mathcal{E}}_h[\varphi] \geq \limsup_{h \rightarrow 0} \underline{\mathcal{E}}_h[\varphi^n] \geq \underline{\mathcal{E}}[\varphi^n], \quad \text{for all } n \in \mathbb{N}.$$

Since $\varphi^n \uparrow \varphi$, by Fatou's lemma we have

$$\liminf_{n \rightarrow \infty} \underline{\mathcal{E}}[\varphi^n] \geq \underline{\mathcal{E}}[\varphi].$$

Therefore

$$\liminf_{h \rightarrow 0} \underline{\mathcal{E}}_h[\varphi] \geq \underline{\mathcal{E}}[\varphi]. \tag{V.3.2}$$

Then we easily get the symmetric result for upper semicontinuous function ψ , i.e.

$$\limsup_{h \rightarrow 0} \bar{\mathcal{E}}_h[\psi] \leq \bar{\mathcal{E}}[\varphi].$$

To conclude, it remains to prove that the function $\omega \mapsto 1_{\{\mathbf{H}(\omega) \leq \delta\}}$ is upper semicontinuous. Note that

$$\{\mathbf{H} \leq \delta\} = \left\{ \max_{t \in [0, \delta]} |B_t| \geq x \right\}$$

Since the function $\varphi : \omega \mapsto \max_{t \in [0, \delta]} |B_t(\omega)|$ is continuous, the set $\{\mathbf{H} \leq \delta\}$ is closed. Consequently, the function $1_{\{\mathbf{H} \leq \delta\}}$ is upper semicontinuous. \square \square

Lemma V.3.4. *For any $\delta > 0$ and $\epsilon > 0$, define $x(\delta) = Ld\sqrt{\delta}(\sqrt{\delta} + \sqrt{-2 \ln \frac{\epsilon\delta}{4d}})$ and $\mathbf{H}^{\delta, x} = \inf\{t \geq 0 : |B_t| \geq x\}$. Then, for δ small enough we have*

$$\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P}[\mathbf{H}^\delta \leq \delta] \leq \epsilon\delta. \tag{V.3.3}$$

Proof Note that

$$\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P}[\mathbf{H}^\delta \leq \delta] = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P}\left[\max_{t \in [0, \delta]} |B_t| \geq x\right] \leq d \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P}\left[\max_{t \in [0, \delta]} |B_t^1| \geq \frac{x}{d}\right]$$

By the definition of \mathcal{P} above (V.1.2), for all $\mathbb{P} \in \mathcal{P}$, the canonical process B admits the canonical decomposition $B = A^\mathbb{P} + M^\mathbb{P}$, where $A^\mathbb{P} = (A^1, \dots, A^d)$ is a finite variation process and $M = (M^1, \dots, M^d)$ is a \mathbb{P} -martingale. Moreover, for each $i = 1, \dots, d$,

$$\mathbb{P}\left[\max_{t \in [0, \delta]} |B_t^i| \geq \frac{x}{d}\right] = \mathbb{Q}\left[\max_{t \in [0, \delta]} |A_t^i + M_t^i| \geq \frac{x}{d}\right] \leq \mathbb{Q}\left[\max_{t \in [0, \delta]} |M_t^i| \geq \frac{x}{d} - L\delta\right].$$

Further, by the time-change for martingales (see e.g. Theorem 4.6 on page 174 of [69]), there is a scalar Brownian motion W defined on a probability space $(\Omega, \mathbf{F}, \mathbf{P})$ such that

$$\begin{aligned} \mathbb{P}\left[\max_{t \in [0, \delta]} |M_t^i| \geq \frac{x}{d} - L\delta\right] &= \mathbf{P}\left[\max_{t \in [0, \delta]} |W_{\langle M^1 \rangle_t}| \geq \frac{x}{d} - L\delta\right] \\ &\leq \mathbf{P}\left[\max_{t \in [0, L^2\delta]} |W_t| \geq \frac{x}{d} - L\delta\right] \\ &= 4\mathbf{P}\left[W_1 \geq \frac{x/d - L\delta}{L\sqrt{\delta}}\right] \end{aligned}$$

Since $\eta := \frac{x/d-L\delta}{L\sqrt{\delta}} = \sqrt{-2 \ln \frac{\epsilon\delta}{4d}} > 1$ when δ is small enough, we have

$$4\mathbb{P}[W_1 \geq \eta] \leq 4e^{-\frac{\eta^2}{2}} = \frac{\epsilon\delta}{d}.$$

We then conclude that $\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P}[\mathbb{H}^\delta \leq \delta] \leq \epsilon\delta$. \square \square

3.2 Equivalent definition of viscosity solutions

There are equivalent definitions of viscosity solution of path dependent PDE, for example in [101] we may find the definition in which one uses smooth test functions in the time-path space Θ . Here we are going to introduce another equivalent definition using constant localization and test functions in $C_0^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d)$, i.e. the class of all $C^{1,2}$ scalar functions φ of which the partial derivatives $\partial_t \varphi, \partial_x \varphi, \partial_{xx}^2 \varphi$ are of compact support. Consider the set of test functions :

$$\underline{\mathcal{A}}u(t, \omega) := \left\{ \varphi \in C_0^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d) : (u^{t,\omega} - \varphi)_0 = \max_{\tau \in \mathcal{J}_\delta} \bar{\mathcal{E}}[(u^{t,\omega} - \varphi)_\tau], \text{ for some } \delta > 0 \right\},$$

$$\bar{\mathcal{A}}u(t, \omega) := \left\{ \varphi \in C_0^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d) : (u^{t,\omega} - \varphi)_0 = \min_{\tau \in \mathcal{J}_\delta} \underline{\mathcal{E}}[(u^{t,\omega} - \varphi)_\tau], \text{ for some } \delta > 0 \right\}.$$

Proposition V.3.5. *Assume that $G(t, \omega, y, z, \gamma)$ is continuous in (t, ω) . A function u is a \mathcal{P} -viscosity subsolution (resp. supersolution) of Equation (V.1.1), if and only if at any point $(t, \omega) \in [0, T) \times \Omega$ it holds for all $\varphi \in \underline{\mathcal{A}}u(t, \omega)$ (resp. $\bar{\mathcal{A}}u(t, \omega)$) that*

$$\mathcal{L}^{t,\omega} \varphi_0 := -\partial_t \varphi_0 - G(t, \omega, u(t, \omega), \partial_x \varphi_0, \partial_{xx}^2 \varphi_0) \leq (\text{resp. } \geq) 0. \quad (\text{V.3.4})$$

Proof We only discuss the case of subsolution. The result about the supersolution follows similarly.

1. We first prove the only if part. Let $(t, \omega) \in [0, T) \times \Omega$ and $(\alpha, \beta, \gamma) \in \underline{\mathcal{J}}u(t, \omega)$ with a localizing time \mathbb{H}_δ . Clearly, there is a function $\varphi \in C_0^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d)$ such that $\varphi = \phi^{\alpha, \beta, \gamma}$ on the set $[0, \delta] \times \{x \in \mathbb{R}^d : |x| \leq x(\delta)\}$, where $x(\cdot)$ is defined as in Lemma V.3.4. Thus,

$$(\varphi - u)_0 = \max_{\tau \in \mathcal{J}_{\bar{\mathbb{H}}^\delta}} \bar{\mathcal{E}}[(\varphi - u)_\tau],$$

where $\bar{\mathbb{H}}^\delta := \delta \wedge \mathbb{H}^\delta$ with \mathbb{H}^δ be defined as in Lemma V.3.4. We have

$$(\varphi - u)_0 \geq \bar{\mathcal{E}}[(\varphi - u)_\delta] - \bar{\mathcal{E}}[(\varphi - u)_\delta - (\varphi - u)_{\bar{\mathbb{H}}^\delta}]. \quad (\text{V.3.5})$$

For the second term on the right hand side of (V.3.5), we have

$$\begin{aligned}\bar{\mathcal{E}}[(\varphi - u)_\delta - (\varphi - u)_{\bar{H}^\delta}] &\leq \bar{\mathcal{E}}\left[|(\varphi - u)_\delta - (\varphi - u)_{\bar{H}^\delta}|; \mathbf{H}^\delta \leq \delta\right] \\ &\leq C \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P}[\mathbf{H}^\delta \leq \delta].\end{aligned}$$

Take $\epsilon > 0$. By Lemma V.3.4, it holds for δ sufficiently small that $\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P}[\mathbf{H}^\delta \leq \delta] < \frac{\epsilon\delta}{2C}$. Then it follows from (V.3.5) that

$$(\varphi - u)_0 > \bar{\mathcal{E}}[(\varphi - u)_\delta] - \frac{\epsilon\delta}{2}.$$

We next consider the optimal stopping problem :

$$Y_t(\omega) = \sup_{\tau \in \mathcal{T}_{\delta-t}} \bar{\mathcal{E}}[(\varphi - u)_\tau^{t,\omega} - \epsilon\tau].$$

According to Ekren, Touzi and Zhang [36], $\tau^* := \inf\{t : Y_t = \varphi_t - u_t - \epsilon t\}$ is an optimal stopping rule. Suppose that we always have $\bar{H}^\delta \leq \tau^* \leq \delta$. Then we obtain that

$$\begin{aligned}\bar{\mathcal{E}}[(\varphi - u)_{\tau^*} - \epsilon\tau^*] &\leq \bar{\mathcal{E}}[(\varphi - u)_\delta - \epsilon\delta] + \bar{\mathcal{E}}[(\varphi - u)_{\tau^*} - (\varphi - u)_\delta - \epsilon(\tau^* - \delta)] \\ &\leq \bar{\mathcal{E}}[(\varphi - u)_\delta - \epsilon\delta] + \bar{\mathcal{E}}[|(\varphi - u)_{\tau^*} - (\varphi - u)_\delta - \epsilon(\tau^* - \delta)|; \mathbf{H}^\delta \leq \delta] \\ &\leq \bar{\mathcal{E}}[(\varphi - u)_\delta - \epsilon\delta] + C \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P}[\mathbf{H}^\delta \leq \delta] \\ &\leq \bar{\mathcal{E}}[(\varphi - u)_\delta] - \frac{\epsilon\delta}{2} < (\varphi - u)_0.\end{aligned}$$

It is a contradiction against the fact that τ^* is optimal. Therefore, there is ω^* such that $t^* := \tau^*(\omega^*) < \bar{H}^\delta(\omega^*)$ and

$$(\varphi - u)_{t^*}(\omega^*) = \max_{\tau \in \mathcal{T}_{\delta-t^*}} \bar{\mathcal{E}}[(\varphi - u)_\tau^{t^*,\omega^*} - \epsilon\tau].$$

So we have

$$\left(-\partial_t \varphi + \epsilon - G(\cdot, u, \partial_x \varphi, \partial_{xx}^2 \varphi)\right)(t^*, \omega^*) \leq 0.$$

By letting $\delta \rightarrow 0$ and then $\epsilon \rightarrow 0$, we obtain

$$\left(-\partial_t \varphi - G(\cdot, u, \partial_x \varphi, \partial_{xx}^2 \varphi)\right)(0, 0) \leq 0.$$

Finally, since $\alpha = \partial_t \varphi_0, \beta = \partial_x \varphi_0, \gamma = \partial_{xx}^2 \varphi_0$, it holds that $-\alpha - G(0, u_0, \beta, \gamma) \leq 0$.

2. We next prove the if part. Let $(t, \omega) \in [0, T) \times \Omega$ and $\varphi \in \mathcal{A}u(t, \omega)$ with a localizing time $\delta \in \mathbb{R}^+$. Without loss of generality, we assume that $(t, \omega) = (0, 0)$ and $(\varphi - u)_0 = 0$. Denote

$$\alpha := \partial_t \varphi_0, \quad \beta := \partial_x \varphi_0, \quad \text{and} \quad \gamma := \partial_{xx}^2 \varphi_0. \quad (\text{V.3.6})$$

For any $\epsilon > 0$, since φ is smooth, by otherwise choosing a stopping time $H_{\delta'} < \delta$ we may assume

$$|\partial_t \varphi_t - \alpha| \leq \epsilon, \quad |\partial_x \varphi_t - \beta| \leq \epsilon, \quad |\partial_{xx}^2 \varphi_t - \gamma| \leq 2\epsilon, \quad 0 \leq t \leq H_{\delta'}.$$

Denote $\alpha_\epsilon := \alpha + [1 + 2L]\epsilon$. Then, for all $\tau \in \mathcal{T}_{H_{\delta'}}$,

$$\begin{aligned} & \bar{\mathcal{E}}[(u - \phi^{\alpha_\epsilon, \beta, \gamma})_\tau] - u_0 = \bar{\mathcal{E}}[(u - u_0 - \phi^{\alpha_\epsilon, \beta, \gamma})_\tau] \\ & \leq \bar{\mathcal{E}}[(u - \varphi)_\tau] + \bar{\mathcal{E}}[(\varphi - \varphi_0 - \phi^{\alpha_\epsilon, \beta, \gamma})_\tau] \\ & \leq \bar{\mathcal{E}}\left[\int_0^\tau (\partial_t \varphi_s - \alpha_\epsilon) ds + (\partial_x \varphi_s - \beta) \cdot dB_s + \frac{1}{2}(\partial_{xx}^2 \varphi_s - \gamma) : d\langle B \rangle_s\right]. \end{aligned}$$

where the last inequality is due to the Itô's formula. Note that, for any $\|\mu\|_\infty, \|a\|_\infty \leq L$, we have

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}_{\mu, \sigma}}\left[\int_0^\tau (\partial_t \varphi_s - \alpha_\epsilon) ds + (\partial_x \varphi_s - \beta) \cdot dB_s + \frac{1}{2}(\partial_{xx}^2 \varphi_s - \gamma) : d\langle B \rangle_s\right] \\ & = \mathbb{E}^{\mathbb{Q}_{\mu, \sigma}}\left[\int_0^\tau \left(\partial_t \varphi_s - \alpha + (\partial_x \varphi_s - \beta) \cdot \mu_s + \frac{1}{2}(\partial_{xx}^2 \varphi_s - \gamma) : a_s\right) ds - [1 + 2L]\epsilon\tau\right] \leq 0. \end{aligned}$$

By the arbitrariness of μ, σ , we see that

$$\bar{\mathcal{E}}[(u - \phi^{\alpha_\epsilon, \beta, \gamma})_\tau] - u_0 \leq 0.$$

That is, $(\alpha_\epsilon, \beta) \in \underline{\mathcal{J}}u_0$. Since u is a \mathcal{P} -viscosity subsolution, it follows that

$$-\alpha_\epsilon - G(0, 0, u_0, \beta, \gamma) \leq 0.$$

Let $\epsilon \rightarrow 0$, then the desired result follows. \square

3.3 Proof of the convergence theorem

We first introduce two functions :

$$\underline{u}(t, \omega) = \liminf_{h \rightarrow 0} u^h(t, \omega) \quad \text{and} \quad \bar{u}(t, \omega) = \limsup_{h \rightarrow 0} u^h(t, \omega). \quad (\text{V.3.7})$$

Note that \underline{u}, \bar{u} inherit the uniform modulus of continuity of u^h , so $\underline{u}, \bar{u} \in \text{BUC}(\Theta)$. It is also clear that $\underline{u} \leq \bar{u}$ and $\underline{u}_T = \bar{u}_T$. Then it is enough to prove that \underline{u} is a \mathcal{P} -viscosity supersolution and \bar{u} is a \mathcal{P} -viscosity subsolution, so that by the comparison principle we may obtain $\bar{u} \leq \underline{u}$, to conclude the proof of Theorem [V.2.3](#).

Proposition V.3.6. *The functions \underline{u} and \bar{u} defined in (V.3.7) are \mathcal{P} -viscosity supersolution and subsolution, respectively.*

Proof We only prove the result for \underline{u} . The corresponding result for \bar{u} can be proved similarly.

1. Without loss of generality, we only verify the viscosity supersolution property at the point $(0, 0)$. Let function $\varphi \in \bar{\mathcal{A}}\underline{u}(0, 0)$, and by adding a constant to φ , we assume that $\underline{u}(0, 0) > \varphi(0, 0)$, so that

$$0 < \eta := (\underline{u} - \varphi)_0 = \min_{\tau \in \mathcal{J}_\delta} \underline{\mathcal{E}}[(\underline{u} - \varphi)_\tau], \quad \text{for some } \delta > 0. \quad (\text{V.3.8})$$

Assume that \underline{u} and φ are both bounded by a constant $M \geq 0$. Take a subsequence still named as u^h such that $\underline{u}_0 = \lim_{h \rightarrow 0} u_0^h$. Now fix a constant $\epsilon > 0$, and denote $\varphi^\epsilon(t, x) = \varphi(t, x) - \epsilon t$. By Lemma [V.3.4](#), there is a constant $C(\epsilon) \in (0, 1/L)$ such that for all $0 < \delta < C(\epsilon)$, we have

$$\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P}[\mathbb{H}^{\delta, x} \leq \delta] \leq \frac{\epsilon}{32(2M + \epsilon)} \delta. \quad (\text{V.3.9})$$

Since u^h is uniformly continuous uniformly in h , by considering δ small enough we may assume that $u^h - \varphi^\epsilon > 0$ on $[0, \bar{\mathbb{H}}^{\delta, x}]$, where $\bar{\mathbb{H}}^{\delta, x} := \delta \wedge \mathbb{H}^{\delta, x}$. It follows from (V.3.8) that

$$(\underline{u} - \varphi^\epsilon)_0 \leq \underline{\mathcal{E}}[(\underline{u} - \varphi)_\delta] = \underline{\mathcal{E}}[(\underline{u} - \varphi^\epsilon)_\delta] - \epsilon \delta. \quad (\text{V.3.10})$$

In Step **2** we will show that

$$\underline{\mathcal{E}}[(\underline{u} - \varphi^\epsilon)_\delta] \leq \liminf_{h \rightarrow 0} \underline{\mathcal{E}}_h[(u^h - \varphi^\epsilon)_\delta]. \quad (\text{V.3.11})$$

It follows that for h sufficiently small

$$(u^h - \varphi^\epsilon)_0 \leq \underline{\mathcal{E}}_h[(u^h - \varphi^\epsilon)_\delta] - \frac{3\epsilon\delta}{4}. \quad (\text{V.3.12})$$

Then by the optimal stopping argument in Step **3**, we may find $(t^*, \omega^*) \in \Theta$ such that $\bar{H}^{\delta,x}(\omega^*) \wedge (\delta - h) > t^* \in \Delta_h$ and

$$(u^h - \varphi^\epsilon)_0^{t^*, \omega^*} = \min_{\tau \in \mathcal{T}_{\delta-t^*}^h, \beta \in \mathcal{B}^h} \underline{\mathcal{E}}_h[\beta_\tau (u^h - \varphi^\epsilon)_\tau^{t^*, \omega^*}], \quad (\text{V.3.13})$$

where $\Delta_h := \{kh : k \in \mathbb{N}\}$, $\mathcal{T}_{\delta-t^*}^h := \{\tau \in \mathcal{T}_{\delta-t^*} : \tau \text{ takes values in } \Delta_h\}$ and \mathcal{B}^h is the collection of all processes β defined by $\beta_t := e^{\sum_{i=0}^{\lfloor t/h \rfloor - 1} \alpha_i h}$ for some \mathcal{F}_{ih} -measurable α_i taking value in $[0, L]$. In particular, (V.3.13) implies that

$$(u^h - \varphi^\epsilon)(t^*, \omega^*) \leq \inf_{0 \leq \alpha \leq L} \underline{\mathcal{E}}_h[e^{\alpha h} (u^h - \varphi^\epsilon)_h^{t^*, \omega^*}]$$

By (ii) of Assumption V.2.2, we obtain

$$(u^h - \varphi^\epsilon)(t^*, \omega^*) \leq \mathbb{T}_h^{t^*, \omega^*}[u^h] - \mathbb{T}_h^{t^*, \omega^*}[\varphi^\epsilon] + h\rho(h).$$

Since $u^h(t^*, \omega^*) = \mathbb{T}_h^{t^*, \omega^*}[u^h]$, it follows that

$$\frac{\mathbb{T}_h^{t^*, \omega^*}[\varphi^\epsilon] - \varphi^\epsilon(t^*, \omega^*)}{h} \leq \rho(h).$$

Further, by (i) of Assumption V.2.2, letting $h \rightarrow 0$, we obtain

$$-\partial_t \varphi(t^*, \omega_{t^*}^*) + \epsilon - G(\cdot, \varphi, \partial_x \varphi, \partial_{xx}^2 \varphi)(t^*, \omega_{t^*}^*) \geq 0. \quad (\text{V.3.14})$$

We next let $\delta \rightarrow 0$. Since $t^* < \bar{H}^{\delta,x}(\omega^*)$, we have $(t^*, \omega^*) \rightarrow 0$ as $\delta \rightarrow 0$. Therefore, it follows from (V.3.14) that

$$-\partial_t \varphi_0 + \epsilon - G(0, \underline{u}_0 - \eta, \partial_x \varphi_0, \partial_{xx}^2 \varphi_0) \geq 0.$$

Finally, we can conclude the proof by letting $\epsilon \rightarrow 0$ and then $\eta \rightarrow 0$.

2. For the simplification of notation, we denote $X := (\underline{u} - \varphi^\epsilon)_\delta$ and $X^h := (u^h - \varphi^\epsilon)_\delta$. It follows from (iii) of Assumption V.2.2 that $\{X^h : h > 0\}$ is uniformly bounded and uniformly continuous uniformly in h , and note that $X = \liminf_{h \rightarrow 0} X^h$. By Lemma V.3.1 and V.3.2, we

obtain that

$$\begin{aligned} \liminf_{h \rightarrow 0} \underline{\mathcal{E}}_h[X^h] &\geq \liminf_{h \rightarrow 0} \underline{\mathcal{E}}[X^h] + \liminf_{h \rightarrow 0} (\underline{\mathcal{E}}_h[X^h] - \underline{\mathcal{E}}[X^h]) \\ &\geq \underline{\mathcal{E}}[X] + \liminf_{h \rightarrow 0} \inf_{\ell > 0} (\underline{\mathcal{E}}_h[X^\ell] - \underline{\mathcal{E}}[X^\ell]) \geq \underline{\mathcal{E}}[X] + \liminf_{h \rightarrow 0} \rho(h) = \underline{\mathcal{E}}[X]. \end{aligned}$$

3. We consider the mixed control and optimal stopping problem in finite discrete-time :

$$Y_t^h(\omega) := \inf_{\tau \in \mathcal{T}_{\delta-t}^h, \beta \in \mathcal{B}} \underline{\mathcal{E}}_h[\beta_\tau (Z^h)_\tau^{t,\omega}], \quad \text{where } Z_t^h := (u^h - \varphi^\epsilon)_t, \quad t \in \mathbb{D}_h. \quad (\text{V.3.15})$$

By standard argument, we have

$$Y_0^h = \inf_{\beta \in \mathcal{B}} \underline{\mathcal{E}}_h[\beta_{\tau^*} Z_{\tau^*}^h], \quad \text{where } \tau^* := \inf\{t \in \Delta_h : Y_t^h = Z_t^h\}.$$

Recall that $Z^h > 0$ on $[0, \mathbb{H}^{\delta,x}]$ for h small enough. Then since $\tau^* \leq \delta$, we have

$$\begin{aligned} \underline{\mathcal{E}}_h[Z_{\tau^*}^h] &\leq \underline{\mathcal{E}}_h[Z_{\tau^*}^h; \mathbb{H}^{\delta,x} > \delta] + \bar{\mathcal{E}}_h[Z_{\tau^*}^h; \mathbb{H}^{\delta,x} \leq \delta] \\ &= \inf_{\beta \in \mathcal{B}} \underline{\mathcal{E}}_h[\beta_{\tau^*} Z_{\tau^*}^h; \mathbb{H}^{\delta,x} > \delta] + \bar{\mathcal{E}}_h[Z_{\tau^*}^h; \mathbb{H}^{\delta,x} \leq \delta] \\ &\leq \inf_{\beta \in \mathcal{B}} \underline{\mathcal{E}}_h[\beta_{\tau^*} Z_{\tau^*}^h] + \sup_{\beta \in \mathcal{B}} \bar{\mathcal{E}}[\beta_{\tau^*} | Z_{\tau^*}^h; \mathbb{H}^{\delta,x} \leq \delta] + \bar{\mathcal{E}}_h[|Z_{\tau^*}^h|; \mathbb{H}^{\delta,x} \leq \delta] \\ &\leq Y_0^h + (1 + e^{L\delta}) \bar{\mathcal{E}}_h[|Z_{\tau^*}^h|; \mathbb{H}^{\delta,x} \leq \delta] \\ &\leq Y_0^h + (1 + e^{L\delta})(2M + \epsilon) \bar{\mathcal{E}}_h[1_{\{\mathbb{H}^{\delta,x} \leq \delta\}}]. \end{aligned}$$

Further, we obtain from Lemma V.3.3 that for h small enough it holds

$$\bar{\mathcal{E}}_h[1_{\{\mathbb{H}^{\delta,x} \leq d\}}] < \bar{\mathcal{E}}[1_{\{\mathbb{H}^{\delta,x} \leq d\}}] + \frac{\epsilon\delta}{8(4M + 2\epsilon)}, \quad (\text{V.3.16})$$

So we get

$$\underline{\mathcal{E}}_h[Z_{\tau^*}^h] \leq Y_0^h + (1 + e^{L\delta})(2M + \epsilon) \bar{\mathcal{E}}[1_{\{\mathbb{H}^{\delta,x} \leq \delta\}}] + \frac{\epsilon\delta}{8}.$$

Further, it follows from (V.3.9) that $(1 + e^{L\delta})(2M + \epsilon) \bar{\mathcal{E}}[1_{\{\mathbb{H}^{\delta,x} \leq \delta\}}] \leq \frac{\epsilon\delta}{8}$. Therefore,

$$Y_0^h \geq \underline{\mathcal{E}}_h[Z_{\tau^*}^h] - \frac{\epsilon\delta}{4}.$$

Suppose that

$$\bar{H}^{\delta,x}(\omega) \wedge (\delta - h) \leq \tau^*(\omega) \leq \delta, \quad \text{for all } \omega. \quad (\text{V.3.17})$$

Note that

$$\underline{\mathcal{E}}_h[Z_{\tau^*}^h] \geq \underline{\mathcal{E}}_h[Z_{\delta}^h] + \underline{\mathcal{E}}_h[Z_{\tau^*}^h - Z_{\tau^* \vee (\delta-h)}^h] + \underline{\mathcal{E}}_h[Z_{\tau^* \vee (\delta-h)}^h - Z_{\delta}^h]. \quad (\text{V.3.18})$$

Further it follows from (V.3.9) and (V.3.16) that

$$\begin{aligned} I_1 &:= \underline{\mathcal{E}}_h[Z_{\tau^*}^h - Z_{\tau^* \vee (\delta-h)}^h] = \underline{\mathcal{E}}_h[Z_{\tau^*}^h - Z_{\delta-h}^h; \tau^* < \delta - h] \\ &\geq -(4M + 2\epsilon)\bar{\mathcal{E}}_h[1_{\{H^{\delta,x} \leq \delta\}}] > -(4M + 2\epsilon)\bar{\mathcal{E}}[1_{\{H^{\delta,x} \leq \delta\}}] - \frac{\epsilon\delta}{8} \geq -\frac{\epsilon\delta}{4}. \end{aligned}$$

On the other hand, we have

$$I_2 := \underline{\mathcal{E}}_h[Z_{\tau^* \vee (\delta-h)}^h - Z_{\delta}^h] \geq -\bar{\mathcal{E}}_h[(\rho_u + \rho_{\varphi})(h + 2\|B_{(\delta-h)\wedge\cdot} - B_{\delta\wedge\cdot}\|)] - \epsilon h,$$

where ρ_u, ρ_{φ} are module of continuity of function u, φ , and are chosen to be bounded and continuous. Again by Lemma V.3.3, we have for h sufficiently small that

$$\begin{aligned} \bar{\mathcal{E}}_h[(\rho_u + \rho_{\varphi})(h + 2\|B_{(\delta-h)\wedge\cdot} - B_{\delta\wedge\cdot}\|)] &< \bar{\mathcal{E}}[(\rho_u + \rho_{\varphi})(h + \\ &2\|B_{(\delta-h)\wedge\cdot} - B_{\delta\wedge\cdot}\|)] + \frac{\epsilon\delta}{8} = \bar{\mathcal{E}}[(\rho_u + \rho_{\varphi})(h + 2\|B_{h\wedge\cdot}\|)] + \frac{\epsilon\delta}{8}, \end{aligned}$$

It follows that $\lim_{h \rightarrow 0} \bar{\mathcal{E}}[(\rho_u + \rho_{\varphi})(h + 2\|B_{h\wedge\cdot}\|)] = 0$ and therefore

$$I_2 > -\frac{\epsilon\delta}{4}, \quad \text{for } h \text{ sufficiently small.}$$

Finally, by (V.3.18) and (V.3.12) we have

$$Y_0^h \geq \underline{\mathcal{E}}_h[Z_{\delta}^h] - \frac{\epsilon\delta}{4} + I_1 + I_2 > \underline{\mathcal{E}}_h[Z_{\delta}^h] - \frac{3\epsilon\delta}{4} \geq Z_0^h,$$

which contradicts the definition of Y in (V.3.15). Therefore, (V.3.17) is wrong, i.e. there is ω^* such that $t^* := \tau^*(\omega^*) < \bar{H}^{\delta,x}(\omega^*) \wedge (\delta - h)$. Further, since $Y_{t^*}(\omega^*) = Z_{t^*}^h(\omega^*)$, we obtain (V.3.13). \square \square

4 Examples of monotone schema

We discuss here some classical monotone numerical schemes in the stochastic control context, and provide some sufficient conditions to make Assumption V.2.2 hold true. Let us first add some assumptions on the functions G and ξ for path dependent PDE (V.1.1).

Assumption V.4.1. *The terminal condition ξ is Lipschitz in ω , G is increasing in γ , and G is Lipschitz in (y, z, γ) : i.e. there is some constant C such that for all $(t, \omega) \in \Theta$ and $(y, z, \gamma), (y', z', \gamma') \in \mathbb{R} \times \mathbb{R}^d \times S_d$,*

$$\left| G(t, \omega, y, z, \gamma) - G(t, \omega, y, z', \gamma') \right| \leq C \left(|y - y'| + |z - z'| + |\gamma - \gamma'| \right).$$

In this section, we denote $t_k := hk$ for $h = \Delta t > 0$. Given $\mathbf{x} = (\mathbf{x}_{t_0}, \mathbf{x}_{t_1}, \dots, \mathbf{x}_{t_k})$ a sequence of points in \mathbb{R}^d , we denote by $\hat{\mathbf{x}} \in \Omega$ the linear interpolation of \mathbf{x} such that $\hat{\mathbf{x}}_{t_i} = \mathbf{x}_{t_i}$ for all i . Further, for $(t, \omega) \in \Theta$, $h > 0$ and $z \in \mathbb{R}^d$, we define a path

$$(\omega \otimes_t^h z) := \omega \otimes_t z^h, \quad \text{where } z_s^h := \begin{cases} \frac{s}{h}z, & \text{for } 0 \leq s \leq h; \\ z, & \text{for } s > h. \end{cases}$$

Let E be some normed vector space, then for maps $\psi : \Theta \rightarrow E$, we introduce the norm $|\psi|_0$ and $|\psi|_1$ by

$$|\psi|_0 := \sup_{(t, \omega) \in \Theta} |\psi(t, \omega)| \quad \text{and} \quad |\psi|_1 := \sup_{(t, \omega) \neq (t', \omega')} \frac{|\psi(t, \omega) - \psi(t', \omega')|}{(|\omega_{t \wedge \cdot} - \omega'_{t' \wedge \cdot}| + |t - t'|)^{1/2}}.$$

4.1 Finite difference scheme

For simplicity, we assume that the state space is of dimension one ($d = 1$). Let $\Delta x > 0$ be the space discretization size. For every $(t, \omega) \in \Theta$, $h > 0$ and \mathcal{F}_{t+h} -measurable random variable $\psi : \Omega \rightarrow \mathbb{R}$, we define the discrete derivatives

$$\mathcal{D}_h \psi(t, \omega) := \left(\mathcal{D}_h^0 \psi, \mathcal{D}_h^1 \psi, \mathcal{D}_h^2 \psi \right)(t, \omega),$$

where

$$\begin{aligned} \mathcal{D}_h^0 \psi(t, \omega) &:= \psi(\omega_{t \wedge \cdot}), & \mathcal{D}_h^1 \psi(t, \omega) &:= \frac{\psi(\omega \otimes_t^h \Delta x) - \psi(\omega_{t \wedge \cdot})}{\Delta x}, \\ \text{and } \mathcal{D}_h^2 \psi(t, \omega) &:= \frac{\psi(\omega \otimes_t^h \Delta x) - 2\psi(\omega_{t \wedge \cdot}) + \psi(\omega \otimes_t^h (-\Delta x))}{\Delta x^2}. \end{aligned}$$

Then an explicit finite difference scheme is given by

$$\mathbb{T}_h^{t,\omega}[u_{t+h}^h] := u^h(t+h, \omega_{t\wedge\cdot}) + hG(t, \omega, \mathcal{D}_h u_{t+h}^h(t, \omega)). \quad (\text{V.4.1})$$

Proposition V.4.2. *Suppose that Assumption V.4.1 holds true and G is Lipschitz in ω , i.e. there is a constant C such that for all $\omega, \omega' \in \Omega$ and all $(t, y, z, \gamma) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times S_d$,*

$$\left| G(t, \omega, y, z, \gamma) - G(t, \omega', y, z, \gamma) \right| \leq C \|\omega_{t\wedge\cdot} - \omega'_{t\wedge\cdot}\|.$$

Assume in addition the CFL (Courant-Friedrichs-Lewy) condition, i.e.

$$\varepsilon \leq \frac{\Delta t |\nabla_\gamma G|_0}{\Delta x^2} \leq \frac{1}{2} - \varepsilon, \quad (\text{V.4.2})$$

and that $\nabla_\gamma G \geq \varepsilon$ for some small constant $\varepsilon > 0$. Then Assumption V.2.2 holds true for finite difference scheme (V.4.1). In particular, the numerical solution u^h is $\frac{1}{2}$ -Hölder in t and Lipschitz in ω , uniformly on h .

Proof We will check each condition in Assumption V.2.2. For the simplicity of presentation, we assume that G is independent of y . Clearly, the argument still works if G is Lipschitz in y .

(i) The consistency condition (Assumption V.2.2 (i)) is obviously satisfied by (V.4.1) as in the no path-dependent case.

(ii) For the monotonicity in Assumption V.2.2 (ii), let us consider two different bounded functions φ and ψ . Denote $\phi := \varphi - \psi$, then by direct computation,

$$\mathbb{T}_h^{t,\omega}[\varphi] - \mathbb{T}_h^{t,\omega}[\psi] = \phi(\omega^0) + h \left(G_y \mathcal{D}_h^{0,t} \phi + G_z \mathcal{D}_h^{1,t} \phi + G_\gamma \mathcal{D}_h^{2,t} \phi \right),$$

where G_y , G_z and G_γ is some function depending on (t, ω) and (φ, ψ) , but uniformly bounded by the Lipschitz constant L of G . Let $b \in [-L, L]$ and $\varepsilon \leq a \leq |\nabla_\gamma G|_0$ be two constants, and $\zeta^{a,b}$ be a random variable defined on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ such that

$$\begin{aligned} \tilde{\mathbb{P}}(\zeta^{a,b} = 0) &= 1 - b \frac{\Delta t}{\Delta x} - 2a \frac{\Delta t}{\Delta x^2}, \\ \tilde{\mathbb{P}}(\zeta^{a,b} = \Delta x) &= b \frac{\Delta t}{\Delta x} + a \frac{\Delta t}{\Delta x^2} \quad \text{and} \quad \tilde{\mathbb{P}}(\zeta^{a,b} = -\Delta x) = a \frac{\Delta t}{\Delta x^2}. \end{aligned}$$

The law of $\zeta^{a,b}$ is well defined for $\Delta t = h$ small enough, because every term above is positive

and the sum of all terms equals to 1 under condition (V.4.2). Further, we have

$$\tilde{\mathbb{E}}[\zeta^{a,b}] = bh, \quad \text{Var}[\zeta^{a,b}] = ah, \quad \text{and} \quad \tilde{\mathbb{E}}[|\zeta^{a,b}|^3] \leq |\Delta x|^3 \leq Ch^{3/2}, \quad (\text{V.4.3})$$

where the last terms follows by $\mathbb{D}x \approx h^{\frac{1}{2}}$.

Then let $F_h(a, b, \cdot) : \mathbb{R} \rightarrow [0, 1]$ be the distribution function of $\zeta^{a,b}$ and $\mathbb{F}_h(a, b, \cdot) : [0, 1] \rightarrow \mathbb{R}$ be the generalized inverse function of $F_h(a, b, \cdot)$, i.e.

$$\mathbb{F}_h(a, b, x) := \inf\{y : F_h(a, b, y) > x\}. \quad (\text{V.4.4})$$

In view of (V.4.3), the monotonicity condition of Assumption V.2.2 (ii) holds true.

(iii) To prove Assumption V.2.2 (iii), we will prove that there is a constant C independent of h such that

$$|u^h(t, \omega) - u^h(t', \omega')| \leq C \left(\|\omega_{t\wedge\cdot} - \omega'_{t'\wedge\cdot}\| + \sqrt{|t' - t|} \right), \quad \forall (t, \omega), (t', \omega') \in \Theta. \quad (\text{V.4.5})$$

Let us first prove that u^h is Lipschitz in ω . Denote

$$L_t^h := \sup_{(t, \omega), (t', \omega') \in \Theta} \frac{u^h(t, \omega) - u^h(t', \omega')}{\|\omega_{t\wedge\cdot} - \omega'_{t'\wedge\cdot}\|} \mathbf{1}_{\{\|\omega_{t\wedge\cdot} - \omega'_{t'\wedge\cdot}\| > 0\}}.$$

By direct computation, we have

$$u^h(t, \omega) - u^h(t, \omega') = hG_\omega \|\omega_{t\wedge\cdot} - \omega'_{t\wedge\cdot}\| + \tilde{D}_h^G u_{t+h}^h(t, \omega) - \tilde{D}_h^G u_{t+h}^h(t, \omega'), \quad (\text{V.4.6})$$

where

$$\tilde{D}_h^G u_{t+h}^h := \left((1 + hG_y) \mathcal{D}_h^0 + hG_z \mathcal{D}_h^1 + hG_\gamma \mathcal{D}_h^2 \right) u_{t+h}^h,$$

with G_y , G_z and G_γ uniformly bounded by L . Then there is a constant C independent of h such that

$$L_t^h \leq (1 + Ch)L_{t+h}^h + Ch.$$

Notice that the terminal condition ξ is Lipschitz, it follows by the discrete Gronwall inequality, we have $L_t^h \leq Ce^{CT}$ for a constant C independent of h . Hence, there is a constant C' independent

of h such that

$$|u^h(t, \omega) - u^h(t, \omega')| \leq C' \|\omega_{t\wedge\cdot} - \omega'_{t\wedge\cdot}\|, \quad \forall t \in [0, T], \omega, \omega' \in \Omega. \quad (\text{V.4.7})$$

We next consider the regularity of u^h in t . Let $t := ih$ and $t' := jh > t$. Note that

$$u^h(t, \omega) = u^h(t+h, \omega_{t\wedge\cdot}) + hG(t, \omega, 0, 0, 0) + h\left(G(t, \omega, \mathcal{D}_h u_{t+h}^h(t, \omega)) - G(t, \omega, 0, 0, 0)\right).$$

By a direct computation, we have

$$u^h(t, \omega) = \tilde{\mathbb{E}} \left[\sum_{k=i}^{j-1} G(t_k, \omega \otimes_t \widehat{X}^h, 0, 0, 0) h + u^h(t', \omega \otimes_t \widehat{X}^h) \right], \quad (\text{V.4.8})$$

where X^h is a discrete process defined as $X_0^h := 0$,

$$X_{t_{k+1}}^h := X_{t_k}^h + \Phi_h(\nabla_\gamma G, \nabla_z G, U_{k+1}),$$

with Φ_h be given by (V.4.4), and \widehat{X}^h is the linear interpolation of X^h . Define

$$A_0^h := 0, \quad A_{t_k}^h := \sum_{i=0}^{k-1} \tilde{\mathbb{E}} \left[\Phi_h(\nabla_\gamma G, \nabla_z G, U_{i+1}) \middle| \mathcal{F}_i \right], \quad \text{and} \quad M^h := X^h - A^h.$$

Clearly, M^h is a martingale and A^h is a predictable process. Further, it follows from the property of \mathbb{F}_h in (V.4.3) that

$$\tilde{\mathbb{E}} \left| A_{t_{k+1}}^h - A_{t_k}^h \right| \leq Lh \quad \text{and} \quad \text{Var} \left[M_{t_{k+1}}^h - M_{t_k}^h \right] \leq Lh.$$

Then by (V.4.8), we have

$$|u(t, \omega) - u(t', \omega_{t\wedge\cdot})| \leq C(t' - t) + C \tilde{\mathbb{E}} \left[\sup_{i \leq k \leq j} |M_{t_k}^h| \right]. \quad (\text{V.4.9})$$

Further, by Doob's inequality, it follows that

$$\tilde{\mathbb{E}} \left[\sup_{i \leq k \leq j} |M_{t_k}^h| \right] \leq \sqrt{\tilde{\mathbb{E}} \left[\sup_{i \leq k \leq j} |M_{t_k}^h|^2 \right]} \leq 2 \left(\sqrt{\tilde{\mathbb{E}} \left[(M_{t_j}^h)^2 \right]} \right) \leq C \sqrt{t_j - t_i}.$$

Finally, combining the above estimation with (V.4.7) and (V.4.9), we obtain (V.4.5). \square

Remark V.4.3. We here assume that the path dependent PDE is non-degenerate ($\nabla_\gamma G \geq \varepsilon >$

0). When $\nabla_\gamma G = 0$ and $\nabla_z G \geq 0$, the scheme is still monotone. When $\nabla_\gamma G = 0$ and $\nabla_z G \leq 0$, it is possible to redefine the first order discrete derivative by

$$\mathcal{D}_h^1 \psi(t, \omega) := \frac{\psi(\omega_{t \wedge \cdot}) - \psi(\omega \otimes_t^h(-\Delta x))}{\Delta x}$$

to obtain a monotone scheme.

Remark V.4.4. In the multidimensional case, $\nabla_\gamma G$ is a matrix. If $\nabla_\gamma G$ is diagonal dominated, then following Kushner and Dupuis [78], it is easy to construct a monotone scheme under similar CFL condition (V.4.2). When $\nabla_\gamma G$ is not diagonal dominated, it is possible to use the generalized finite difference scheme proposed by Bonnans, Ottenwaelter and Zidani [12].

4.2 The trinomial tree scheme of Guo-Zhang-Zhuo

We consider the path dependent PDE of the form (V.1.1). Let σ_0 be some symmetric $d \times d$ matrix, denote

$$F(t, \omega, y, z, \gamma) := G(t, \omega, y, z, \gamma) - \frac{1}{2} \sigma_0^2 : \gamma, \quad \tilde{G}_\gamma := \sigma_0^{-1} G_\gamma \sigma_0^{-1}.$$

Let $\zeta = (\zeta_1, \dots, \zeta_d)$ a random vector defined on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that $\zeta_i, i = 1, \dots, d$ are i.i.d and

$$\tilde{\mathbb{P}}(\zeta_i = \frac{1}{\sqrt{p}}) = \frac{p}{2}, \quad \tilde{\mathbb{P}}(\zeta_i = -\frac{1}{\sqrt{p}}) = \frac{p}{2}, \quad \tilde{\mathbb{P}}(\zeta_i = 0) = 1 - p, \quad \text{with } p \in (0, 1).$$

For every \mathcal{F}_{t+h} -measurable function $\psi : \Omega \rightarrow \mathbb{R}$, let us define $\mathcal{D}_h^i \psi(t, \omega) := \tilde{\mathbb{E}} \left[\psi(\omega \otimes_t^h (\sqrt{h} \sigma_0 \zeta)) K_i(\zeta) \right]$ with

$$K_0 := 1, \quad K_1 := \frac{\sigma_0^{-1} \zeta}{\sqrt{h}}, \quad K_2 := \frac{\sigma_0^{-1} [(1-p)\zeta \zeta^T - (1-3p)\text{Diag}[\zeta \zeta^T] - 2pI_d] \sigma_0^{-1}}{(1-p)h},$$

where for any matrix $\gamma = [\gamma_{i,j}]_{1 \leq i,j \leq d} \in S_d$, $\text{Diag}[\gamma]$ denotes the diagonal matrix whose (i, i) -th component is γ_{ii} . Then the numerical scheme is defined as

$$\mathbb{T}_h^{t,\omega} [u^h(t+h, \cdot)] := \mathcal{D}_h^0 u^h(t, \omega) + hF(\cdot, \mathcal{D}_h u_{t+h}^h)(t, \omega). \quad (\text{V.4.10})$$

Proposition V.4.5. *Let Assumptions V.4.1 hold true and G is Lipschitz in ω . Suppose in*

addition that Assumption 3.3 in Guo, Zhang and Zhuo [59] holds true (where we replace their notation \tilde{G}_γ by $\nabla_\gamma \tilde{G}$ in our context). Then the trinomial tree scheme (V.4.10) satisfies Assumption V.2.2.

Proof The consistency and monotonicity condition in Assumption V.2.2 (i) and (ii) can be justified by almost the same argument as in [59]. Further, using the same argument as in Proposition V.4.2, it is easy to show that

$$|u^h(t, \omega) - u^h(t', \omega')| \leq C \left(\|\omega_{t \wedge \cdot} - \omega'_{t' \wedge \cdot}\| + \sqrt{|t' - t|} \right), \quad \forall (t, \omega), (t', \omega') \in \Theta,$$

for some constant C independent of h , which implies in particular (iii) of Assumption V.2.2. \square

Remark V.4.6. As a path dependent PDE degenerates to be a classical PDE, the conditions in Proposition V.4.5 turns to be exactly the same conditions in Theorem 3.10 of [59].

4.3 The probabilistic scheme of Fahim-Touzi-Warin

We consider path dependent PDE (V.1.1) in which G is in the form of

$$G(t, \omega, y, z, \gamma) = \mu(t, \omega) \cdot z - \frac{1}{2} \sigma \sigma^T(t, \omega) : \gamma - F(t, \omega, y, z, \gamma).$$

Before introducing the numerical scheme, we first define a random vector

$$X_h^{(t, \omega)} := \mu(t, \omega)h + \sigma(t, \omega)W_h,$$

where $W_h \sim N(0, hI_d)$ is a Gaussian vector. For every bounded function $\psi \in \mathbb{L}^0(\mathcal{F}_{t+h})$, we define

$$\mathcal{D}_h \psi(t, \omega) := \mathbb{E} \left[\psi(\omega \otimes_t \hat{X}^{(t, \omega)}) H_h(t, \omega) \right],$$

where $H_h(t, \omega) = (H_0^h, H_1^h, H_2^h)^T$ with

$$H_0^h := 1, \quad H_1^h := (\sigma^T(t, \omega))^{-1} \frac{W_h}{h}, \quad H_2^h := (\sigma^T(t, \omega))^{-1} \frac{W_h W_h^T - hI_d}{h^2} \sigma^{-1}(t, \omega).$$

Then the probabilistic scheme is given by

$$\mathbb{T}_h^{t, \omega} [u^h(t+h, \cdot)] := \mathbb{E} \left[u^h(t+h, \hat{X}^{(t, \omega)}) \right] + hF(\cdot, \mathcal{D}_h u_{t+h}^h)(t, \omega). \quad (\text{V.4.11})$$

Remark V.4.7. The probabilistic scheme in [48] is inspired by the second order BSDE theory of Cheridito, Soner, Touzi and Victoir [16], and extends the classical numerical scheme of BSDE (see e.g. Bouchard and Touzi [13], Zhang [113]). In practice, one can use the simulation-regression method to estimate the conditional expectation in the above scheme (see e.g. Gobet, Lemor and Warin [58]). We refer to Guyon and Henry-Labordère [60] for more details on the use of the scheme, to Tan [109] for an extension to a degenerate case, and to Tan [110] for an extension to path-dependent control problems.

Assumption V.4.8. (i) *The nonlinearity F is Lipschitz w.r.t. (ω, u, z, γ) uniformly in t and $|F(\cdot, \cdot, 0, 0, 0)|_0 < \infty$.*

(ii) *F is elliptic and dominated by the diffusion term of X , that is,*

$$\nabla_\gamma F \leq \sigma \sigma^T, \quad \text{on } \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d. \quad (\text{V.4.12})$$

(iii) *$\nabla_p F \in \text{Image}(\nabla_\gamma F)$ and $|(\nabla_p F)^T (\nabla_\gamma F)^{-1} \nabla_p F|_0 < \infty$.*

(iv) *$|\mu|_1, |\sigma|_1 < \infty$ and σ is invertible and ξ is bounded Lipschitz.*

Proposition V.4.9. *Suppose that Assumption V.4.8 holds true. Then the probabilistic numerical scheme (V.4.11) satisfies Assumption V.2.2.*

Proof (i) Assumption V.2.2 (i) is obviously satisfied in view of Lemma 3.11 of [48].

(ii) Further, using probabilistic interpretation of this scheme in Tan [110, Section 3.2], we may verify (ii) of Assumption V.2.2. See also the estimation given by Lemma 3.1 of [110].

(iii) For (iii) of Assumption V.2.2, we shall prove that the numerical solution u^h is Lipschitz in ω and 1/2-Hölder in t . In [48], the authors proved this property in the case of PDEs. Their arguments for the Lipschitz continuity in ω can be easily adapted to this path-dependent case. For the regularity of u^h in t , they used a regularization technique, which seems impossible to be adapted to the path-dependent case. However, we can still use similar arguments as in Proposition V.4.2, i.e. use the discrete-time controlled semimartingale interpretation, to prove the Hölder property of u^h in t . \square

Remark V.4.10. As a path dependent PDE degenerates to be a PDE, the conditions in Assumption V.4.8 turns to be exactly the same conditions imposed in [48] (see their Theorem 3.6).

4.4 The semi-Lagrangian scheme

For the semi-Lagrangian scheme, we shall consider the path dependent PDE (V.1.1) of the Bellman-Issac type, i.e. the function G is in the form of

$$G(t, \omega, y, z, \gamma) = \inf_{k_1 \in K_1} \sup_{k_2 \in K_2} \left(\frac{1}{2} a^{k_1, k_2}(\cdot) : \gamma + b^{k_1, k_2}(\cdot) \cdot z + c^{k_1, k_2}(\cdot) y + f^{k_1, k_2}(\cdot) \right) (t, \omega),$$

where K_1 and K_2 are some sets, $(a^{k_1, k_2}, b^{k_1, k_2}, c^{k_1, k_2}, f^{k_1, k_2})$ are functionals defined on Θ .

Let ζ be a random vector satisfying

$$\mathbb{E}[\zeta] = 0, \quad \text{Var}[\zeta] = I_d \quad \text{and} \quad \mathbb{E}[|\zeta|^3] < \infty. \quad (\text{V.4.13})$$

Then the semi-Lagrangian scheme is defined as

$$\begin{aligned} \mathbb{T}_h^{t, \omega}[u^h(t+h, \cdot)] &:= \inf_{k_1 \in K_1} \sup_{k_2 \in K_2} \left\{ u^h(t+h, \omega \otimes_t (\sigma^{k_1, k_2}(t, \omega) \zeta \sqrt{h} + b^{k_1, k_2}(t, \omega) h)) \right. \\ &\quad \left. + u^h(t+h, \omega) c^{k_1, k_2}(t, \omega) h + f^{k_1, k_2}(t, \omega) h \right\}. \end{aligned} \quad (\text{V.4.14})$$

Proposition V.4.11. *Suppose that $|a|_1 + |b|_1 + |c|_1 + |f|_1 < \infty$, and (V.4.13) holds true. Then the semi-Lagrangian scheme (V.4.14) for the Bellman-Issac path-dependent equation satisfies Assumption V.2.2.*

Proof (i) The consistency condition (Assumption V.2.2 (i)) is easy to check.

(ii) Let E be a set, $e : K_1 \times K_2 \rightarrow E$ be an arbitrary mapping, and $\psi, \varphi : E \rightarrow \mathbb{R}$ be two bounded functions. Note that

$$\inf_{k_1 \in K_1} \sup_{k_2 \in K_2} \psi(e(k_1, k_2)) - \inf_{k_1 \in K_1} \sup_{k_2 \in K_2} \varphi(e(k_1, k_2)) \leq \sup_{k_1 \in K_1, k_2 \in K_2} (\psi - \varphi)(e(k_1, k_2)). \quad (\text{V.4.15})$$

Notice that \mathbb{R}^d is isomorphic to \mathbb{R} , we can always consider the random vector $\sigma^{k_1, k_2} \zeta \sqrt{h} + b^{k_1, k_2} h$ as a one-dimensional random variable. By consider the inverse function of its distribution function, then there is a family $\Phi_h(k_1, k_2, \cdot)$ such that $\Phi^h(k_1, k_2, U) \sim \sigma^{k_1, k_2} \zeta \sqrt{h} + b^{k_1, k_2} h$ in law with $U \sim \mathcal{U}([0, 1])$, for all $(k_1, k_2) \in K_1 \times K_2$. Then it follows from (V.4.15) that the monotonicity condition in Assumption V.2.2 (ii) holds true with $\Phi_h(k_1, k_2, \cdot)$ and $K = K_1 \times K_2$. (iii) Finally, by the same arguments as in Proposition V.4.2, we can easily deduce that u^h is Lipschitz in ω and $1/2$ -Hölder in t , uniformly on h , and hence complete the proof for the stability condition in Assumption V.2.2. \square

Remark V.4.12. Solutions of path dependent Bellman-Issac equations can characterize value

functions of stochastic differential games (see e.g. Pham and Zhang [96]).

Remark V.4.13. (i) For Bellman-Issac PDE, Debrabant and Jakobsen [25] studied the semi-Lagrangian scheme with a random variable ζ following a discrete distribution, together with an interpolation technique for the implementation.

(ii) For Bellman equation (PDE), Kharroubi, Langrené and Pham [72] propose a semi-Lagrangian type numerical scheme with $\zeta \sim N(0, 1)$, and provide a simulation-regression technique for the implementation. It is worth of mentioning that [72] provides a convergence rate for the scheme, while we only prove in this paper a general convergence theorem as in Barles and Souganidis [4].

5 Numerical examples

In this section, we provide two toy examples of numerical implementation in low-dimensional case. For more numerical examples (in high-dimensional case), we would like to refer to [48, 59, 60, 72, 109], etc.

A first numerical example For a first numerical example, we consider the PPDE

$$-\partial_t u - \min_{\mu \in [\underline{\mu}, \bar{\mu}]} \mu \partial_\omega u - \max_{a \in [\underline{a}, \bar{a}]} \frac{a}{2} \partial_{\omega\omega}^2 u = f(t, \omega, \bar{\omega}), \quad u(T, \omega) = g(\omega_T, \bar{\omega}_T). \quad (\text{V.5.1})$$

where $d = 1$, $\bar{\omega}_t := \int_0^t \omega_s ds$, $f : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are two functions.

The above PPDE (V.5.1) is motivated by a stochastic differential game :

$$u_0 = \inf_{\underline{\mu} \leq \mu_t \leq \bar{\mu}} \sup_{\underline{a} \leq a_t \leq \bar{a}} \mathbb{E} \left[\int_0^T f(t, X_t^{\mu, a}, \bar{X}^{\mu, a}) dt + g(X_T^{\mu, a}, \bar{X}_T^{\mu, a}) \right],$$

where $X^{\mu, \sigma}$ is controlled diffusion such that

$$X_t^{\mu, a} = \int_0^t \mu_s ds + \int_0^t \sqrt{a_s} dW_s, \quad \text{with } W \text{ a Brownian motion,}$$

and $\bar{X}_t^{\mu, a} = \int_0^t X_s^{\mu, a} ds$ (see e.g. Pham and Zhang [96] for more details).

We choose the terminal condition $g(x, y) = \cos(x + y)$ and the function

$$\begin{aligned} f(t, x, y) = & - (x - \underline{\mu}) \left(\sin(x - y) \right)^- + (x + \bar{\mu}) \left(\sin(x - y) \right)^+ \\ & + \frac{a}{2} \left(\cos(x - y) \right)^+ - \frac{\bar{a}}{2} \left(\cos(x - y) \right)^-, \end{aligned}$$

so that the solution of PPDE (V.5.1) is given explicitly by $u(t, \omega) = \cos(\omega_t + \bar{\omega}_t)$, which serves as a reference value for the numerical examples. This idea is borrowed from Guo, Zhang and Zhuo [59]. For numerical test, we implemented the finite difference scheme in Section 4.1 and the probabilistic scheme (of Fahim, Touzi and Warin [48]) in Section 4.3. The results are reported in Figure V.1.

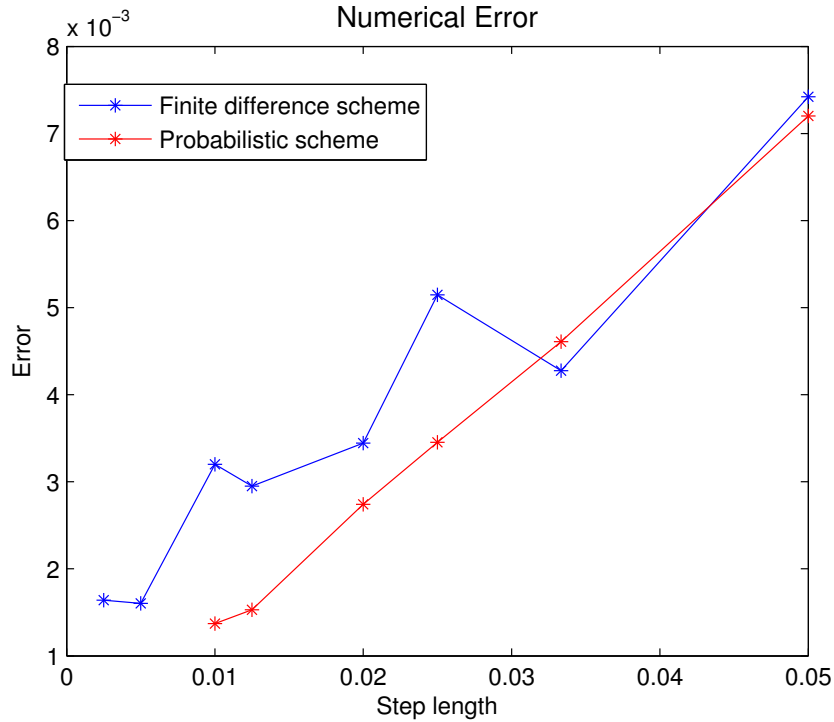


Figure V.1 – For PPDE (V.5.1), we choose $\mu = -0.2$, $\bar{\mu} = 0.2$, $\underline{a} = 0.04$, $\bar{a} = 0.09$, $T = 1$ and $\omega_0 = \bar{\omega}_0 = 0$. Then the reference solution is given by $u(0, 0) = \cos(0) = 1$. We compute the error between the reference solution and the numerical solutions, w.r.t. difference time step length Δt .

A second numerical example The second example of PPDE we considered is given by

$$-\partial_t u - \max_{\underline{a} \leq a \leq \bar{a}} \left(\frac{1}{2} a \partial_{\omega\omega}^2 u - f(t, u, \partial_\omega u, a) \right) = 0, \quad (\text{V.5.2})$$

$$\text{where } f(t, y, z, a) = \frac{1}{2} \left((\sqrt{a}z + b/\sqrt{a})^- \right)^2 - zb - b^2/2a,$$

which is taken from Matoussi, Possamaï and Zhou [85]. The above equation is motivated by solving a robust utility maximization problem using 2BSDE, which can be instead characterized by a PPDE (see e.g. (??)).

We consider the terminal condition

$$u(T, \omega) = K_1 + (\bar{\omega}_T - K_1)^+ - (\bar{\omega}_T - K_2)^+, \quad \bar{\omega}_T := \int_0^T \omega_s ds.$$

Then the solution of PPDE (V.5.2) can also be characterized by the PDE, by adding an associated variable y ,

$$\begin{aligned} -\partial_t v - x\partial_y v - \max_{\underline{a} \leq a \leq \bar{a}} \left(\frac{1}{2} a \partial_{xx}^2 v - f(t, v, \partial_x v, a) \right) &= 0, \\ v(T, x, y) &= K_1 + (y - K_1)^+ - (y - K_2)^+. \end{aligned} \tag{V.5.3}$$

We implemented the finite difference scheme (Section 4.1) and the probabilistic scheme (Section 4.3) for PPDE (V.5.2). For reference, we implemented the classical finite difference scheme of PDE (V.5.3). We also notice that the generator in PPDE (V.5.2) is in fact not Lipschitz but quadratic in z , however, the convergence of the numerical solutions can be still observed, see Figure V.2.

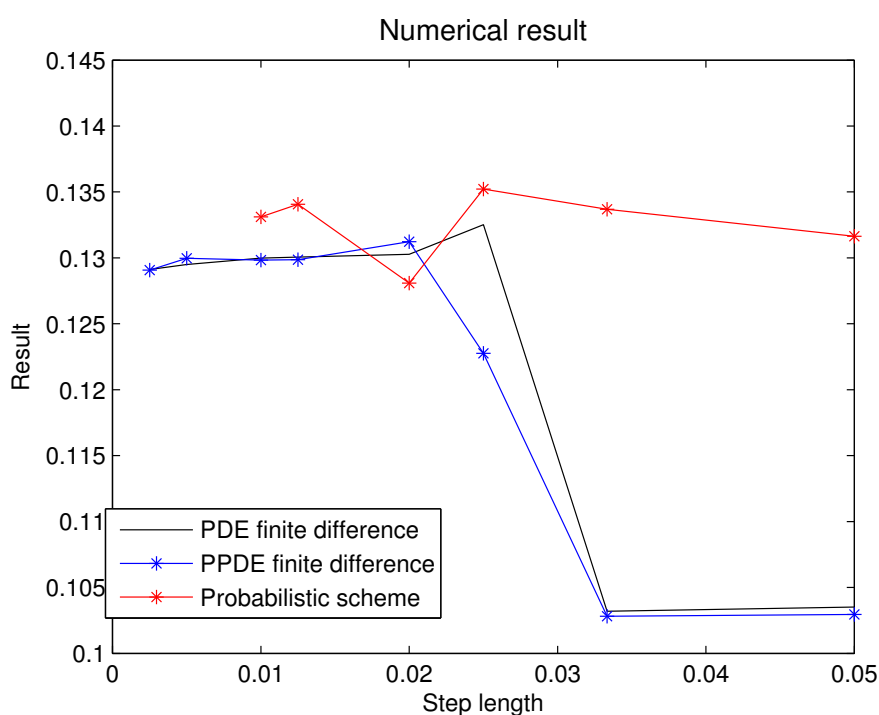


Figure V.2 – For PPDE (V.5.1), we choose $K_1 = -0.2$, $K_2 = 0.2$, $\underline{a} = 0.04$, $\bar{a} = 0.09$, $b = 0.05$ and $T = 1$. We provide all the numerical solutions w.r.t. difference time step length Δt . It seems that the faire value is closed to 0.129. For finite-difference scheme, when Δt is greater than 0.025, we need to use a coarser space-discretization to ensure the monotonicity (similar to the classical CFL condition), which makes a big difference to the numerical solutions for the case $\Delta t < 0.25$. However, the convergence as $\Delta t \rightarrow 0$ is still obvious.

Chapitre VI

Elliptic fully nonlinear path dependent PDEs

1 Preliminary

Let $\Omega := \{\omega \in C(\mathbb{R}^+, \mathbb{R}^d) : \omega_0 = 0\}$ be the set of continuous paths starting from the origin, B be the canonical process, \mathbb{F} be the filtration generated by B , and \mathbb{P}_0 be the Wiener measure. We introduce the following notations :

- \mathbb{S}^d denotes the set of $d \times d$ symmetric matrices and $\gamma : \eta = \text{Tr}[\gamma\eta]$ for all $\gamma, \eta \in \mathbb{S}^d$;
- \mathcal{R} denotes the set of all open, bounded and convex subsets of \mathbb{R}^d containing 0;
- $O_L := \{x \in \mathbb{R}^d : |x| < L\}$, \overline{O}_L denotes the closure of O , $[aI_d, bI_d] := \{\beta \in \mathbb{S}^d : aI_d \leq \beta \leq bI_d\}$;
- $\mathbb{H}^0(E)$ denotes the set of all \mathbb{F} -progressively measurable processes with values in E , and $\mathbb{H}_L^0 := \mathbb{H}^0\left([\sqrt{2/L}I_d, \sqrt{2L}I_d]\right)$ for $L > 0$;
- \mathcal{T}^t denote the set of all stopping times larger than t ; in particular, $\mathcal{T} := \mathcal{T}^0$;
- $\|\omega\|_t := \sup_{s \leq t} |\omega_s|$, $\|\omega\|_s^t := \sup_{s \leq u \leq t} |\omega_u|$ for $\omega \in \Omega$ and $s, t \in \mathbb{R}^+$;
- $(\omega \otimes_t \omega')(s) := \omega_s 1_{[0,t)}(s) + (\omega_t + \omega'_{s-t}) 1_{[t,\infty)}(s)$ for $\omega, \omega' \in \Omega$ and $s, t \in \mathbb{R}^+$;
- given $\varphi : \Omega \rightarrow \mathbb{R}^d$, we define $\varphi^{t,\omega}(\omega') := \varphi(\omega \otimes_t \omega')$.

In this chapter, we focus on a subset of Ω denoted as Ω^e which will be considered as the solution space of elliptic path-dependent PDEs.

- $\Omega^e := \{\omega \in \Omega : \omega = \omega_{t\wedge \cdot} \text{ for some } t \geq 0\}$ denotes the set of all paths with flat tails;
- $\bar{t}(\omega) := \inf \{t : \omega = \omega_{t\wedge \cdot}\}$ for all $\omega \in \Omega^e$;
- given $\varphi : \Omega \rightarrow \mathbb{R}^d$, we define the process $\varphi_t(\omega) := \varphi(\omega_{t\wedge \cdot})$.

Elliptic equations are devoted to model time-invariant phenomena, and in the path space the time-invariance property can be formulated mathematically as follows.

Definition VI.1.1. *Define the distance on Ω^e :*

$$d^e(\omega, \omega') := \inf_{t \in \mathbb{R}^+} \sup_{t \in \mathbb{R}^+} |\omega_{l(t)} - \omega'_t|, \text{ for } \omega, \omega' \in \Omega^e,$$

where $\bar{\omega}$ is the set of all increasing bijections from \mathbb{R}^+ to \mathbb{R}^+ . We say ω is equivalent to ω' , if $d^e(\omega, \omega') = 0$. A function u on Ω^e is time-invariant, if u is well defined on the equivalent class :

$$u(\omega) = u(\omega') \text{ whenever } d^e(\omega, \omega') = 0.$$

Further, $C(\Omega^e)$ denotes the set of all random variables on Ω^e continuous with respect to $d^e(\cdot, \cdot)$. We also use the notations $C(\Omega^e; \mathbb{R}^d)$, $C(\Omega^e; \mathbb{S}^d)$ when we need to emphasize the space where the functions take values. Finally, we say $u \in \text{BUC}(\Omega^e)$ if $u : \Omega^e \rightarrow \mathbb{R}$ is bounded and uniformly continuous with respect to $d^e(\cdot, \cdot)$, i.e. there exists a modulus of continuity ρ such that

$$|u(\omega^1) - u(\omega^2)| \leq \rho(d^e(\omega^1, \omega^2)) \quad \text{for all } \omega^1, \omega^2 \in \Omega^e. \quad (\text{VI.1.1})$$

In this chapter, we assume ρ to be convex.

Example VI.1.2. We show some examples of time-invariant functions :

- Markovian case : Assume that there exists $\bar{u} : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $u(\omega) = \bar{u}(\omega_{\bar{t}(\omega)})$. Since $|\omega_{\bar{t}(\omega^1)}^1 - \omega_{\bar{t}(\omega^2)}^2| \leq d^e(\omega^1, \omega^2)$ for all $\omega^1, \omega^2 \in \Omega^e$, u is time-invariant.
- Maximum dependent case : Assume that there exists $\bar{u} : \mathbb{R} \rightarrow \mathbb{R}$ such that $u(\omega) = \bar{u}(\|\omega\|_\infty)$. Note that $\|\omega\|_\infty = d^e(\omega, 0)$ and $d^e(\omega^1, 0) - d^e(\omega^2, 0) \leq d^e(\omega^1, \omega^2)$. Thus, $\|\omega^1\|_\infty = \|\omega^2\|_\infty$ whenever $d^e(\omega^1, \omega^2) = 0$. Consequently, u is time-invariant.
- Let $(t_i, x_i) \in \mathbb{R}^+ \times \mathbb{R}^d$ for each $1 \leq i \leq n$. We denote by

$$\text{Lin}\{(0, 0), (t_1, x_1), \dots, (t_n, x_n)\} \quad (\text{VI.1.2})$$

the linear interpolation of the points with a flat tail extending to $t = \infty$. Then, for the two paths defined as the interpolations as follows :

$$\omega^i := \text{Lin}\{(0, 0), (t_1^i, x_1), \dots, (t_n^i, x_n)\} \quad \text{for } i = 1, 2,$$

the distance between them is 0, i.e. $d^e(\omega^1, \omega^2) = 0$.

In this chapter, we will prove the wellposedness result for time-invariant solutions to elliptic path-dependent PDEs.

Further, for all $D \in \mathcal{R}$ we have the following notations :

- $\mathcal{D} := \{\omega \in \Omega^e : \omega_t \in D \text{ for all } t \geq 0\}$;
- $D^x := D - x := \{y : x + y \in D\}$ for $x \in D$, and $D^\omega := D^{\omega_{\bar{t}(\omega)}}$ for $\omega \in \mathcal{D}$;
- $\mathbb{H}_D := \inf\{t \geq 0 : \omega_t \notin D\}$ and $\mathcal{H} := \{\mathbb{H}_D : D \in \mathcal{R}\}$;

- $\partial\mathcal{D} := \{\omega \in \Omega^e : \bar{t}(\omega) = H_D(\omega)\}$ defines the boundary of \mathcal{D} , and $\text{cl}(\mathcal{D}) := \mathcal{D} \cup \partial\mathcal{D}$ defines the closure of \mathcal{D} .

Also, $C(\mathcal{D})$ denotes the set of all continuous functions defined on \mathcal{D} . For $\omega \in \Omega^e$ and $\omega' \in \Omega$,

- $(\omega \bar{\otimes} \omega')(s) := (\omega \otimes_{\bar{t}(\omega)} \omega')(s)$ defines the concatenation of the paths;
- given $\varphi : \Omega \rightarrow \mathbb{R}^d$, we define $\varphi^\omega(\omega') := \varphi^{\bar{t}(\omega), \omega}(\omega') = \varphi(\omega \bar{\otimes} \omega')$.

Similarly, for $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$, we define that $\phi^x(y) := \phi(x + y)$ for all $x, y \in \mathbb{R}^d$.

We next introduce the smooth functions on the space Ω^e . First, as in [38], for every constant $L > 0$, we denote by \mathcal{P}_L the collection of all continuous semimartingale measures \mathbb{P} on Ω whose drift and diffusion belong to $\mathbb{H}^0(\bar{O}_L)$ and \mathbb{H}_L^0 , respectively. More precisely, let $\tilde{\Omega} := \Omega \times \Omega \times \Omega$ be an enlarged canonical space, $\tilde{B} := (B, A, M)$ be the canonical process. A probability measure $\mathbb{P} \in \mathcal{P}^L$ means that there exists an extension $\mathbb{Q}^{\alpha, \beta}$ of \mathbb{P} on $\tilde{\Omega}$ such that :

$$\begin{aligned} B &= A + M, \quad A \text{ is absolutely continuous, } M \text{ is a martingale,} \\ \|\alpha^\mathbb{P}\|_\infty &\leq L, \quad \beta^\mathbb{P} \in \mathbb{H}_L^0, \quad \text{where } \alpha_t^\mathbb{P} := \frac{dA_t}{dt}, \quad \beta_t^\mathbb{P} := \sqrt{\frac{d\langle M \rangle_t}{dt}}, \end{aligned} \quad \mathbb{Q}^{\alpha, \beta}\text{-a.s.} \quad (\text{VI.1.3})$$

Further, denote $\mathcal{P}^\infty := \cup_{L>0} \mathcal{P}^L$.

Definition VI.1.3 (Smooth time-invariant processes). *Let $D \in \mathcal{R}$. We say that $u \in C^2(\mathcal{D})$, if $u \in C(\mathcal{D})$ and there exist $Z \in C(\mathcal{D}; \mathbb{R}^d)$, $\Gamma \in C(\mathcal{D}; \mathbb{S}^d)$ such that*

$$du_t = Z_t \cdot dB_t + \frac{1}{2} \Gamma_t : \langle B \rangle_t \text{ for } t \leq H_D, \quad \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}^\infty.$$

By a direct localization argument, we see that the above Z and Γ , if they exist, are unique. Denote $\partial_\omega u := Z$ and $\partial_{\omega\omega}^2 u := \Gamma$.

Remark VI.1.4. *In the Markovian case mentioned in Example VI.1.2, if the function $\bar{u} : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies $\bar{u} \in C^2(D)$, then by the Itô's formula it follows that $u \in C^2(\mathcal{D})$.*

Remark VI.1.5. *In the path-dependent case, Dupire [34] defined derivatives, $\partial_t u$ and $\partial_\omega u$, for process $u : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}^d$. In particular, the t -derivative is defined as :*

$$\partial_t u(s, \omega) := \lim_{h \rightarrow 0^+} \frac{u(t+h, \omega_{t \wedge \cdot}) - u(t, \omega)}{h}.$$

Also, Dupire and other authors, for example [17], proved the functional Itô formula for the processes regular in Dupire's sense :

$$du_s = \partial_t u_s ds + \partial_\omega u_s \cdot dB_s + \frac{1}{2} \partial_{\omega\omega}^2 u_s : \langle B \rangle_s, \quad \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}^\infty,$$

Note that in the time-invariant case it always holds that $\partial_t u = 0$. Consequently, the processes with Dupire's derivatives in $C(\mathcal{D})$ are also smooth according to our definition.

We next introduce the notations about nonlinear expectations. For a measurable set $A \in \Omega$, a random variable ξ and a process X , we define :

- $\mathcal{C}^L[A] := \sup_{\mathbb{P} \in \mathcal{P}^L} \mathbb{P}[A]$, $\bar{\mathcal{E}}^L[\xi] := \sup_{\mathbb{P} \in \mathcal{P}^L} \mathbb{E}^{\mathbb{P}}[\xi]$ and $\underline{\mathcal{E}}^L[\xi] := \inf_{\mathbb{P} \in \mathcal{P}^L} \mathbb{E}^{\mathbb{P}}[\xi]$;
- $\bar{\mathcal{E}}_t^L[\xi](\omega) := \sup_{\mathbb{P} \in \mathcal{P}^L} \mathbb{E}^{\mathbb{P}}[\xi^{t,\omega}]$ and $\underline{\mathcal{E}}_t^L[\xi](\omega) := \inf_{\mathbb{P} \in \mathcal{P}^L} \mathbb{E}^{\mathbb{P}}[\xi^{t,\omega}]$;
- $\bar{\mathcal{S}}_t^L[X_{H_D \wedge \cdot}](\omega) := \sup_{\tau \in \mathcal{T}^t} \bar{\mathcal{E}}_t^L[X_{\tau \wedge H_D}](\omega)$ and $\underline{\mathcal{S}}_t^L[X_{H_D \wedge \cdot}](\omega) := \inf_{\tau \in \mathcal{T}^t} \underline{\mathcal{E}}_t^L[X_{\tau \wedge H_D}](\omega)$.

A process X is an $\bar{\mathcal{E}}^L$ -supermartingale on $[0, T]$, if $X_t(\omega) = \bar{\mathcal{S}}_t^L[X](\omega)$ for all $(t, \omega) \in [0, T] \times \Omega$; similarly, we define the $\underline{\mathcal{E}}^L$ -submartingales. The existing literature gives the following results.

Lemma VI.1.6 (Tower property, Nutz and van Handel [88]). *For a bounded measurable process X , we have*

$$\bar{\mathcal{E}}_\sigma^L[X] = \bar{\mathcal{E}}_\sigma^L[\bar{\mathcal{E}}_\tau^L[X]] \text{ for all stopping times } \sigma \leq \tau.$$

Lemma VI.1.7 (Snell envelop characterization, Ekren, Touzi and Zhang [36]). *Let $H_D \in \mathcal{H}$ and $X \in \text{BUC}(\mathcal{D})$. Define the Snell envelope and the corresponding first hitting time of the obstacles :*

$$Y := \bar{\mathcal{S}}^L[X_{H_D \wedge \cdot}], \quad \tau^* := \inf \{t \geq 0 : Y_t = X_t\}.$$

Then $Y_{\tau^} = X_{\tau^*}$. Y is an $\bar{\mathcal{E}}^L$ -supermartingale on $[0, H_D]$ and an $\bar{\mathcal{E}}^L$ -martingale on $[0, \tau^*]$. Consequently, τ^* is an optimal stopping time.*

Some other properties of the nonlinear expectation will be useful.

Proposition VI.1.8. *Let $D \in \mathcal{R}$ and $O \subset D$ also in \mathcal{R} . Define a sequence of stopping times H_n :*

$$H_0 = 0, \quad H_{i+1} := \inf \{s \geq H_i : B_s - B_{H_i} \notin O\}, \quad i \geq 0. \quad (\text{VI.1.4})$$

Then, it holds that

$$\lim_{n \rightarrow \infty} \mathcal{C}^L[H_n < T] = 0 \text{ for all } T \in \mathbb{R}^+; \quad \bar{\mathcal{E}}^L[H_D] < \infty; \quad \text{and} \quad \lim_{n \rightarrow \infty} \sup_{x \in D} \mathcal{C}^L[H_n < H_D^x] = 0.$$

We report the proof in Appendix.

2 Fully nonlinear elliptic path-dependent PDEs

Let $Q \in \mathcal{R}$ and consider \mathcal{Q} as the domain of the path-dependent Dirichlet problem of the equation :

$$\mathcal{L}u(\omega) := -G(\omega, u, \partial_\omega u, \partial_{\omega\omega}^2 u) = 0 \text{ for } \omega \in \mathcal{Q}, \quad u = \xi \text{ on } \partial\mathcal{Q}, \quad (\text{VI.2.1})$$

with nonlinearity G and boundary condition by ξ .

Assumption VI.2.1. *The nonlinearity $G : \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ satisfies :*

- (i) *For fixed (y, z, γ) , $|G(\cdot, 0, 0, 0)| \leq C_0$;*
- (ii) *G is uniformly elliptic, i.e., there exists $L_0 > 0$ such that for all (ω, y, z)*

$$G(\omega, y, z, \gamma_1) - G(\omega, y, z, \gamma_2) \geq \frac{1}{L_0} I_d : (\gamma_1 - \gamma_2) \text{ for all } \gamma_1 \geq \gamma_2.$$

- (iii) *G is uniformly continuous on Ω^e with respect to $d^e(\cdot, \cdot)$, and is uniformly Lipschitz continuous in (y, z, γ) with a Lipschitz constant L_0 ;*
- (iv) *G is uniformly decreasing in y , i.e. there exists a function $\lambda : \mathbb{R} \rightarrow \mathbb{R}$ strictly increasing and continuous, $\lambda(0) = 0$, and*

$$G(\omega, y_1, z, \gamma) - G(\omega, y_2, z, \gamma) \geq \lambda(y_2 - y_1), \text{ for all } y_2 \geq y_1, (\omega, z, \gamma) \in \Omega^e \times \mathbb{R}^d \times \mathbb{S}^d.$$

For any time-invariant function u on Ω^e , $\omega \in \mathcal{Q}$ and $L > 0$, we define the set of test functions :

$$\begin{aligned} \underline{\mathcal{A}}^L u(\omega) &:= \left\{ \varphi : \varphi \in C^2(\mathcal{O}_\epsilon) \text{ and } (\varphi - u^\omega)_0 = \underline{\mathfrak{S}}_0^L \left[(\varphi - u^\omega)_{\mathbb{H}_{\mathcal{O}_\epsilon}^\omega} \wedge \cdot \right] \text{ for some } \epsilon > 0 \right\}, \\ \overline{\mathcal{A}}^L u(\omega) &:= \left\{ \varphi : \varphi \in C^2(\mathcal{O}_\epsilon) \text{ and } (\varphi - u^\omega)_0 = \overline{\mathfrak{S}}_0^L \left[(\varphi - u^\omega)_{\mathbb{H}_{\mathcal{O}_\epsilon}^\omega} \wedge \cdot \right] \text{ for some } \epsilon > 0 \right\}. \end{aligned}$$

We call $\mathbb{H}_{\mathcal{O}_\epsilon}$ a localization of test function φ . Note that the stopping time $\mathbb{H}_{\mathcal{O}_\epsilon}$ can take the value of ∞ , while u is only defined on Ω^e . However, since $\mathbb{H}_{\mathcal{O}_\epsilon} < \infty$ \mathcal{P}^L -q.s., it is not essential. If necessary, we can define complementarily $u := 0$ on $\Omega \setminus \Omega^e$. Now, we define the viscosity solution to the elliptic path-dependent PDEs (VI.2.1).

Definition VI.2.2. *Let $L > 0$ and $\{u_t\}_{t \in \mathbb{R}^+}$ be a time-invariant progressively measurable process.*

- (i) *u is an L -viscosity subsolution (resp. L -supersolution) of PPDE (VI.2.1) if for $\omega \in \mathcal{Q}$ and*

any $\varphi \in \underline{\mathcal{A}}^L u(\omega)$ (resp. $\varphi \in \overline{\mathcal{A}}^L u(\omega)$) :

$$-G(\omega, u(\omega), \partial_\omega \varphi_0, \partial_{\omega\omega}^2 \varphi_0) \leq \text{(resp. } \geq) 0.$$

(ii) u is a viscosity subsolution (resp. supersolution) of PPDE (VI.2.1) if u is an L -viscosity subsolution (resp. L -supersolution) of PPDE (VI.2.1) for some $L > 0$.

(iii) u is a viscosity solution of PPDE (VI.2.1) if it is both a viscosity subsolution and a viscosity supersolution.

By very similar arguments as in the proof of Theorem 3.16 and Theorem 5.1 in [37], we may easily prove that :

Theorem VI.2.3 (Consistency with classical solution). *Let Assumption VI.2.1 hold. Given a function $u \in C^2(\mathcal{Q})$, then u is a viscosity supersolution (resp. subsolution, solution) to PPDE (VI.2.1) if and only if u is a classical supersolution (resp. subsolution, solution).*

Theorem VI.2.4 (Stability). *Let $L > 0$, G satisfies Assumption VI.2.1, and $u \in \text{BUC}$. Assume*

(i) *for any $\epsilon > 0$, there exist G^ϵ and $u^\epsilon \in \text{BUC}$ such that G^ϵ satisfies Assumption VI.2.1 and u^ϵ is a L -viscosity subsolution (resp. supersolution) of PPDE (VI.2.1) with generator G^ϵ ;*

(ii) *as $\epsilon \rightarrow 0$, (G^ϵ, u^ϵ) converge to (G, u) locally uniformly in the following sense : for any $(\omega, y, z, \gamma) \in \Omega^e \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$, there exists $\delta > 0$ such that*

$$\lim_{\epsilon \rightarrow 0} \sup_{(\tilde{\omega}, \tilde{y}, \tilde{z}, \tilde{\gamma}) \in O_\delta(\omega, y, z, \gamma)} \left[|(G^\epsilon - G)^\omega(\tilde{\omega}, \tilde{y}, \tilde{z}, \tilde{\gamma})| + |(u^\epsilon - u)^\omega(\tilde{\omega})| \right] = 0,$$

where we abuse the notation O_δ to denote a ball in the corresponding space. Then, u is a L -viscosity solution (resp. supersolution) of PPDE (VI.2.1) with generator G .

Following Ekren, Touzi and Zhang [38], we introduce the path-frozen PDEs :

$$(E)_\epsilon^\omega \quad \mathbf{L}^\omega v := -G(\omega, v, Dv, D^2v) = 0 \text{ on } O_\epsilon(\omega) := O_\epsilon \cap Q^\omega. \quad (\text{VI.2.2})$$

Note that ω is a parameter instead of a variable in the above PDE. Similar to [38], our wellposedness result relies on the following condition on the PDE $(E)_\epsilon^\omega$.

Assumption VI.2.5. *For $\epsilon > 0$, $\omega \in \mathcal{Q}$ and $h \in C(\partial O_\epsilon(\omega))$, we have $\bar{v} = \underline{v}$, where*

$$\begin{aligned} \bar{v}(x) &:= \inf \{w(x) : w \in C_0^2(O_\epsilon(\omega)), \mathbf{L}^\omega w \geq 0 \text{ on } O_\epsilon(\omega), w \geq h \text{ on } \partial O_\epsilon(\omega)\}, \\ \underline{v}(x) &:= \sup \{w(x) : w \in C_0^2(O_\epsilon(\omega)), \mathbf{L}^\omega w \leq 0 \text{ on } O_\epsilon(\omega), w \leq h \text{ on } \partial O_\epsilon(\omega)\}, \end{aligned}$$

and $C_0^2(O_\epsilon(\omega)) := C^2(O_\epsilon(\omega)) \cap C(\text{cl}(O_\epsilon(\omega)))$.

We shall use PDEs as tools to study PPDEs. Note that by viscosity solutions of PDEs, we mean the classical definition in the PDE literature, for example in [19].

Remark VI.2.6. *If Equation $(E)_t^\omega$ has the classical solution, then Assumption VI.2.5 holds. For example, let function $G(\omega, \cdot) : \mathbb{S}^d \rightarrow \mathbb{R}$ be convex, and assume that the elliptic PDE :*

$$-G(\omega, D^2v) = 0 \text{ on } O, \quad v = h \text{ on } \partial O$$

has a unique viscosity solution. Then, according to Caffarelli and Cabre [15], the viscosity solution has the interior C^2 -regularity.

The rest of the chapter will be devoted to prove the following two main results.

Theorem VI.2.7 (Comparison result). *Let Assumptions VI.2.1 and VI.2.5 hold true, and let $\xi \in \text{BUC}(\partial\mathcal{Q})$. Let u be a BUC viscosity subsolution and v be a BUC viscosity supersolution to the path-dependent PDE (VI.2.1). Then, if $u_{\mathbb{H}_Q} \leq \xi \leq v_{\mathbb{H}_Q}$, we have $u \leq v$ on \mathcal{Q} .*

Theorem VI.2.8 (Wellposedness). *Let Assumptions VI.2.1 and VI.2.5 hold, and let $\xi \in \text{BUC}(\partial\mathcal{Q})$. Then, the path-dependent PDE (VI.2.1) has a unique viscosity solution in $\text{BUC}(\mathcal{Q})$.*

3 Comparison result

3.1 Partial comparison

Similar to [38], we introduce the class of piecewise smooth processes in our time-invariant context.

Definition VI.3.1. *Let $u : \mathcal{Q} \rightarrow \mathbb{R}$. We say that $u \in \overline{C}^2(\mathcal{Q})$, if u is bounded, process $\{u_t\}_{t \in \mathbb{R}^+}$ is pathwise continuous, and there exists an increasing sequence of \mathbb{F} -stopping times $\{\mathbb{H}_n; n \geq 1\}$ such that*

- (1) *for each i and ω , $\text{ID}_{\mathbb{H}_i, \omega} := \mathbb{H}_{i+1}^{\mathbb{H}_i(\omega), \omega} - \mathbb{H}_i(\omega) \in \mathcal{H}$ whenever $\mathbb{H}_i(\omega) < \mathbb{H}_Q(\omega) < \infty$, i.e. there is $O_{i, \omega} \in \mathcal{R}$ such that $\text{ID}_{\mathbb{H}_i, \omega} = \inf\{t : \omega_t \notin O_{i, \omega}\}$;*
- (2) *$\{i : \mathbb{H}_i(\omega) < \mathbb{H}_Q(\omega)\}$ is finite \mathcal{P}^∞ -q.s. and $\lim_{i \rightarrow \infty} \mathbb{C}_0^L[\mathbb{H}_i^\omega < \mathbb{H}_Q^\omega] = 0$ for all $\omega \in \mathcal{Q}$ and $L > 0$;*
- (3) *for each i , $u^{\omega_{\mathbb{H}_i} \wedge \cdot} \in \text{BUC}(\mathcal{O}_{i, \omega})$, and there exist $\partial_\omega u^i, \partial_{\omega\omega}^2 u^i$ such that for all ω , $(\partial_\omega u^i)^{\omega_{\mathbb{H}_i} \wedge \cdot}$ and $(\partial_{\omega\omega}^2 u^i)^{\omega_{\mathbb{H}_i} \wedge \cdot}$ are both continuous on $\mathcal{O}_{i, \omega}$ and*

$$u_t^{\omega_{\mathbb{H}_i} \wedge \cdot} - u_0^{\omega_{\mathbb{H}_i} \wedge \cdot} = \int_0^t (\partial_\omega u^i)_s^{\omega_{\mathbb{H}_i} \wedge \cdot} \cdot dB_s + \frac{1}{2} \int_0^t (\partial_{\omega\omega}^2 u^i)_s^{\omega_{\mathbb{H}_i} \wedge \cdot} : d\langle B \rangle_s \text{ for all } t \leq \text{ID}_{\mathbb{H}_i, \omega}, \mathcal{P}^\infty\text{-q.s.}$$

The rest of the chapter is devoted to the proof of the following partial comparison result.

Proposition VI.3.2. *Let Assumption VI.2.1 hold. Let $u^2 \in \text{BUC}(\mathcal{Q})$ be a viscosity supersolution of PPDE (VI.2.1) and let $u^1 \in \overline{C}^2(\mathcal{Q})$ satisfies $\mathcal{L}u^1(\omega) \leq 0$ for all $\omega \in \mathcal{Q}$. If $u^1 \leq u^2$ on $\partial\mathcal{Q}$, then $u^1 \leq u^2$ in $\text{cl}(\mathcal{Q})$. A similar result holds if we exchange the roles of u^1 and u^2 .*

In preparation to the proof of Proposition VI.3.2, we prove the following lemma.

Lemma VI.3.3. *Let $D \in \mathcal{R}$ and $X \in \text{BUC}(\mathcal{D})$ and non-negative. Assume that $X_0 > \overline{\mathcal{E}}^L[X_{\mathbb{H}_D}]$, then there exists $\omega^* \in \mathcal{D}$ and $t^* := \bar{t}(\omega^*)$ such that*

$$X_{t^*}(\omega^*) = \overline{\mathcal{S}}_{t^*}^L[X_{\mathbb{H}_D \wedge \cdot}](\omega^*) \quad \text{and} \quad X_{t^*}(\omega^*) > 0.$$

Proof Denote Y as the Snell envelop of $X_{\mathbb{H}_D \wedge \cdot}$, i.e. $Y_t := \overline{\mathcal{S}}_t^L[X_{\mathbb{H}_D \wedge \cdot}]$. By Lemma VI.1.7, we know that $\tau^* := \inf\{t : X_t = Y_t\}$ defines an optimal stopping rule. So, we have

$$\overline{\mathcal{E}}^L[X_{\tau^*}] = Y_0 \geq X_0 > \overline{\mathcal{E}}^L[X_{\mathbb{H}_D}].$$

Hence $\{\tau^* < \mathbb{H}_D\} \neq \emptyset$. Suppose that $X_{\tau^*} = 0$ on $\{\tau^* < \mathbb{H}_D\}$. Then,

$$0 = X_{\tau^*}1_{\{\tau^* < \mathbb{H}_D\}} = Y_{\tau^*}1_{\{\tau^* < \mathbb{H}_D\}} \geq \overline{\mathcal{E}}_{\tau^*}^L[X_{\mathbb{H}_D}]1_{\{\tau^* < \mathbb{H}_D\}}.$$

Since X is non-negative, we obtain that $X_{\mathbb{H}_D}1_{\{\tau^* < \mathbb{H}_D\}} = 0$. So, $X_{\tau^*} = X_{\mathbb{H}_D}$ on $\{\tau^* < \mathbb{H}_D\}$. Thus, we conclude that

$$X_0 \leq Y_0 = \overline{\mathcal{E}}^L[X_{\tau^*}] = \overline{\mathcal{E}}^L[X_{\mathbb{H}_D}] < X_0.$$

This contradiction implies that $\{\tau^* < \mathbb{H}_D, X_{\tau^*} > 0\} \neq \emptyset$. Finally, take $\omega' \in \{\tau^* < \mathbb{H}_D, X_{\tau^*} > 0\}$, and then $\omega^* := \omega'_{\tau^*}(\omega')_{\wedge \cdot}$ is the desired path. \square

Proof of Proposition VI.3.2 Recall the notation \mathbb{H}_n and O_n^ω in Definition VI.3.1. We decompose the proof in two steps.

Step 1. We first show that for all $i \geq 0$ and $\omega \in \mathcal{Q}$

$$(u^1 - u^2)_{\mathbb{H}_i}^+(\omega) \leq \overline{\mathcal{E}}^L \left[\left((u^1_{\mathbb{H}_{i+1}})_{\mathbb{H}_i, \omega} - (u^2_{\mathbb{H}_{i+1}})_{\mathbb{H}_i, \omega} \right)^+ \right].$$

Clearly it suffices to consider $i = 0$. Assume the contrary, i.e.

$$c := (u^1 - u^2)_{\mathbb{H}_1}^+(\mathbf{0}) - \overline{\mathcal{E}}^L \left[(u^1 - u^2)_{\mathbb{H}_1}^+ \right] > 0.$$

Denote $X := (u^1 - u^2)^+$. Then, by Lemma VI.3.3, there exists $\omega^* \in \mathcal{O}_0^0$ and $t^* := \bar{t}(\omega^*)$ such that

$$X_{t^*}(\omega^*) = \bar{\mathfrak{S}}_{t^*}^L[X_{\mathbb{H}_D \wedge \cdot}] \text{ and } X_{t^*}(\omega^*) > 0.$$

Since u^1 is smooth on \mathcal{O}_0^0 , we have $\varphi := (u_{\mathbb{H}_{\mathcal{O}_0^0 \wedge \cdot}^1})^{\omega^*} \in \bar{\mathcal{A}}^L u^2(\omega^*)$. By the L -viscosity supersolution property of u^2 and the assumption on the function G , this implies that

$$\begin{aligned} 0 &\leq -G(\cdot, u^2, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi)(\omega_{t^* \wedge \cdot}^*) \\ &\leq -G(\cdot, u^1, \partial_\omega u^1, \partial_{\omega\omega}^2 u^1)(\omega_{t^* \wedge \cdot}^*) - \lambda \left((u^1 - u^2)_{t^*}(\omega^*) \right) \\ &< -G(\cdot, u^1, \partial_\omega u^1, \partial_{\omega\omega}^2 u^1)(\omega_{t^* \wedge \cdot}^*). \end{aligned}$$

This is in contradiction with the classical subsolution property of u^1 .

Step 2. By the result of Step 1 and the tower property of $\bar{\mathcal{E}}^L$ stated in Lemma VI.1.6, we have

$$(u^1 - u^2)^+(\mathbf{0}) \leq \bar{\mathcal{E}}^L \left[(u^1 - u^2)_{\mathbb{H}_i}^+ \right] \leq \bar{\mathcal{E}}^L \left[(u^1 - u^2)_{\mathbb{H}_Q}^+ \right] + \bar{\mathcal{E}}^L \left[(u^1 - u^2)_{\mathbb{H}_i}^+ - (u^1 - u^2)_{\mathbb{H}_Q}^+ \right] \text{ for all } i \geq 1.$$

By Proposition VI.1.8, we have $\mathcal{C}^L[\mathbb{H}_i < \mathbb{H}_Q] \rightarrow 0$ as $i \rightarrow \infty$. Therefore,

$$(u^1 - u^2)^+(\mathbf{0}) \leq \bar{\mathcal{E}}^L \left[(u^1 - u^2)_{\mathbb{H}_Q}^+ \right] = 0.$$

□

3.2 The Perron type construction

Define the following two functions :

$$\bar{u}(\omega) := \inf \left\{ \psi(\omega) : \psi \in \bar{\mathcal{D}}_Q^\xi(\omega) \right\}, \quad \underline{u}(\omega) := \sup \left\{ \psi(\omega) : \psi \in \underline{\mathcal{D}}_Q^\xi(\omega) \right\}, \quad (\text{VI.3.1})$$

where

$$\bar{\mathcal{D}}_Q^\xi(\omega) := \left\{ \psi \in \bar{\mathcal{C}}^2(\mathcal{Q}^\omega) : \mathcal{L}^\omega \psi \geq 0 \text{ on } \mathcal{Q}, \psi \geq \xi^\omega \text{ on } \partial \mathcal{Q} \right\},$$

$$\underline{\mathcal{D}}_Q^\xi(\omega) := \left\{ \psi \in \bar{\mathcal{C}}^2(\mathcal{Q}^\omega) : \mathcal{L}^\omega \psi \leq 0 \text{ on } \mathcal{Q}, \psi \leq \xi^\omega \text{ on } \partial \mathcal{Q} \right\}.$$

As a direct corollary of Proposition VI.3.2, we have :

Corollary VI.3.4. *Let Assumption VI.2.1 hold. For all BUC viscosity supersolutions (resp. subsolution) u such that $u \geq \xi$ (resp. $u \leq \xi$) on $\partial\Omega$, it holds that $u \geq \underline{u}$ (resp. $u \leq \bar{u}$) on Ω .*

In order to prove the comparison result of Theorem VI.2.7, it remains to show the following result.

Proposition VI.3.5. *Let $\xi \in \text{BUC}(\partial\Omega)$. Under Assumptions VI.2.1 and VI.2.5, we have $\bar{u} = \underline{u}$.*

The proof of this proposition is reported in Subsection 3.4, and requires the preparations in Subsection 3.3.

3.3 Hamilton-Jacobi-Belleman equations

In this subsection, we will recall the relation between HJB equations and stochastic control problems. Recall the constants L_0 and C_0 in Assumption VI.2.1 and consider two functions :

$$\begin{aligned} \bar{g}(y, z, \gamma) &:= C_0 + L_0 |z| + L_0 y^- + \sup_{\beta \in [\sqrt{2/L_0}I_d, \sqrt{2L_0}I_d]} \frac{1}{2}\beta^2 : \gamma, \\ \underline{g}(y, z, \gamma) &:= -C_0 - L_0 |z| - L_0 y^+ + \inf_{\beta \in [\sqrt{2/L_0}I_d, \sqrt{2L_0}I_d]} \frac{1}{2}\beta^2 : \gamma, \end{aligned} \quad (\text{VI.3.2})$$

Then for all nonlinearities G satisfying Assumption VI.2.1, it holds $\underline{g} \leq G \leq \bar{g}$. Consider the HJB equations :

$$\bar{\mathbf{L}}u := -\bar{g}(u, Du, D^2u) = 0 \quad \text{and} \quad \underline{\mathbf{L}}u := -\underline{g}(u, Du, D^2u) = 0.$$

In the rest of this subsection, we show that the explicit solutions of the Dirichlet problems of the above equations on a set $D \in \mathcal{R}$ are given in terms of boundary condition h_D , i.e.

$$\begin{aligned} \bar{w}(x) &:= \sup_{b \in \mathbb{H}^0([0, L_0])} \bar{\mathcal{E}}^{L_0} \left[h_D(B_{\mathbb{H}_D^x}) e^{-\int_0^{\mathbb{H}_D^x} b_r dr} + C_0 \int_0^{\mathbb{H}_D^x} e^{-\int_0^t b_r dr} dt \right], \\ \underline{w}(x) &:= \inf_{b \in \mathbb{H}^0([0, L_0])} \underline{\mathcal{E}}^{L_0} \left[h_D(B_{\mathbb{H}_D^x}) e^{-\int_0^{\mathbb{H}_D^x} b_r dr} + C_0 \int_0^{\mathbb{H}_D^x} e^{-\int_0^t b_r dr} dt \right]. \end{aligned}$$

Lemma VI.3.6. *Let $h_D(x) := \bar{\mathcal{E}}^{L_0} [v(x, B_{\mathbb{H}_D^x}^\wedge)]$ for some $v \in \text{BUC}(\mathbb{R}^d \times \Omega^e)$. Then \bar{w} and \underline{w} are the unique viscosity solutions in $\text{BUC}(\text{cl}(D))$ of the equations $\bar{\mathbf{L}}u = 0$ and $\underline{\mathbf{L}}u = 0$, respectively, with boundary condition $u = h_D$ on ∂D .*

Proof According to Proposition VI.6.1 in Appendix, there exists a modulus of continuity ρ such that

$$\bar{\mathcal{E}}^{L_0} [|\mathbb{H}_D^{x_1} - \mathbb{H}_D^{x_2}|] \leq \rho(|x_1 - x_2|). \quad (\text{VI.3.3})$$

Since $v \in \text{BUC}(\mathbb{R} \times \Omega^e)$, we obtain that

$$|h_D(x_1) - h_D(x_2)| \leq \bar{\mathcal{E}}^{L_0} [|v(x_1, B_{\mathbb{H}_D^{x_1}}) - v(x_2, B_{\mathbb{H}_D^{x_2}})|] \leq \rho \left(|x_1 - x_2| + \bar{\mathcal{E}}^{L_0} [|B_{\mathbb{H}_D^{x_1}} - B_{\mathbb{H}_D^{x_2}}|] \right), \quad (\text{VI.3.4})$$

where we used the convexity of ρ and the Jensen's inequality. We next estimate :

$$\bar{\mathcal{E}}^{L_0} [|B_{\mathbb{H}_D^{x_1}} - B_{\mathbb{H}_D^{x_2}}|] \leq \left(\bar{\mathcal{E}}^{L_0} [|B_{\mathbb{H}_D^{x_1}} - B_{\mathbb{H}_D^{x_2}}|^2] \right)^{\frac{1}{2}} \leq \left(2L_0 \bar{\mathcal{E}}^{L_0} [|\mathbb{H}_D^{x_1} - \mathbb{H}_D^{x_2}|] \right)^{\frac{1}{2}}. \quad (\text{VI.3.5})$$

In view of (VI.3.3), we conclude that h_D is bounded and uniformly continuous. Further, since h_D is bounded and b only takes non-negative values, we can easily obtain that for $x_1, x_2 \in D$,

$$|\bar{w}(x_1) - \bar{w}(x_2)| \leq \bar{\mathcal{E}}^{L_0} [|h_D(B_{\mathbb{H}_D^{x_1}}) - h_D(B_{\mathbb{H}_D^{x_2}})|] + C \bar{\mathcal{E}}^{L_0} [|\mathbb{H}_D^{x_1} - \mathbb{H}_D^{x_2}|].$$

Since $h_D \in \text{BUC}(\mathbb{R}^d)$, by the same arguments in (VI.3.4) and (VI.3.5), we conclude that $\bar{w} \in \text{BUC}(\text{cl}(D))$. Then, by standard argument, one can easily verify that \bar{w} is the viscosity solution to $\bar{\mathbf{L}}u = 0$ with the boundary condition h_D on ∂D . Similarly, we may prove the corresponding result for \underline{w} . \square

3.4 Proof of comparison result

Recall the two functions \bar{u}, \underline{u} defined in (VI.3.1). In the next lemma, we will use the path-frozen PDEs to construct the functions θ_n^ϵ , which will be needed to construct the approximations of \bar{u} and \underline{u} defined in (VI.3.1). Recall the notation of linear interpolation in (VI.1.2). Then

- let $(x_i; 1 \leq i \leq n) \in (\bar{B}_\epsilon^d)^n$ and denote $\pi_n := \text{Lin}\{(0, 0), (1, x_1), \dots, (n, x_n)\}$; in particular, note that $\pi_n \in \Omega^e$;
- denote $\pi_n^x := \text{Lin}\{\pi_n, (n+1, x)\}$ for all $x \in \bar{B}_\epsilon^d$; clearly, we have $\pi_n^x \in \Omega^e$;
- define a sequence of stopping times : for $i \geq 1$

$$\mathbb{H}_0^{\pi_n^x} := 0, \quad \mathbb{H}_1^{\pi_n^x} := \inf \{t \geq \mathbb{H}_i : x + B_t \notin O_\epsilon\} \wedge \mathbb{H}_Q^{\pi_n^x}, \quad \mathbb{H}_{i+1}^{\pi_n^x} := \inf \{t \geq \mathbb{H}_i^{\pi_n^x} : B_t - B_{\mathbb{H}_i} \notin O_\epsilon\} \wedge \mathbb{H}_Q^{\pi_n^x}$$

— given $\omega \in \Omega$, we define for $m > 0$:

$$\begin{aligned} \pi_n^1(x, \omega) &:= \text{Lin} \left\{ \pi_n, (n+1, x + \omega_{H_1^{\pi_n^x}}) \right\} \quad \text{and} \\ \pi_n^m(x, \omega) &:= \text{Lin} \left\{ \pi_n, (n+1, x + \omega_{H_1^{\pi_n^x}}), (n+j, \omega_{H_j^{\pi_n^x}} - \omega_{H_{j-1}^{\pi_n^x}})_{2 \leq j \leq m} \right\}, \quad \text{for all } m > 1. \end{aligned}$$

Lemma VI.3.7. *Let Assumption VI.2.1 hold, and assume that $|\xi| \leq C_0$. Let $\omega \in \Omega$, $|x_i| = \epsilon$ for all $i \geq 1$, $\pi_n := \{x_i\}_{1 \leq i \leq n}$, and $\omega \bar{\otimes} \pi_n^x \in \Omega$. Then*

(i) *there exists a sequence of continuous functions $(\pi_n, x) \mapsto \theta_n^{\omega, \epsilon}(\pi_n, x)$, bounded uniformly in (ϵ, n) , such that*

$$\theta_n^{\omega, \epsilon}(\pi_n; \cdot) \text{ is a viscosity solution of } (E)_\epsilon^{\omega \bar{\otimes} \pi_n},$$

with boundary conditions :

$$\begin{cases} \theta_n^{\omega, \epsilon}(\pi_n; x) = \xi(\omega \bar{\otimes} \pi_n^x), & |x| < \epsilon \text{ and } x \in \partial Q^{\omega \bar{\otimes} \pi_n}, \\ \theta_n^{\omega, \epsilon}(\pi_n; x) = \theta_{n+1}^{\omega, \epsilon}(\pi_n^x; 0), & |x| = \epsilon. \end{cases}$$

(ii) *Moreover, there is a modulus of continuity ρ_ϵ such that for any $\omega^1, \omega^2 \in \Omega$,*

$$\left| \theta_0^{\omega^1, \epsilon}(0; 0) - \theta_0^{\omega^2, \epsilon}(0; 0) \right| \leq \rho_\epsilon \left(d^e(\omega^1, \omega^2) \right). \quad (\text{VI.3.6})$$

For the domain $O_\epsilon(\pi_n)$ defined in (VI.2.2), a part of the boundary belongs to ∂Q^{π_n} , while the other belongs to ∂O_ϵ . On ∂Q^{π_n} , we should set the solution to be equal to the boundary condition of the path-dependent PDE. Otherwise, on ∂O_ϵ , the value of the solution should be consistent with that of the next piece of the path-frozen PDEs. The proof of Lemma VI.3.7 is similar to that of Lemma 6.2 in [38]. However, the stochastic representations and the estimates that we will use are all in the context of the elliptic equations. So it is necessary to explain the proof in detail.

In preparation of the proof of Lemma VI.3.7, we give the following estimate on the viscosity solutions to the path-frozen PDEs. The proof is reported in Appendix, Subsection 6.

Lemma VI.3.8. *Fix $D \in \mathcal{R}$. Let $h^i : \partial D \rightarrow \mathbb{R}$ be continuous ($i = 1, 2$), G satisfy Assumption VI.2.1, and v^i be the viscosity solutions to the following PDEs :*

$$G(\omega^i, v^i, Dv^i, D^2v^i) = 0 \text{ on } D, \quad v^i = h^i \text{ on } \partial D.$$

Then, we have

$$(v^1 - v^2)(x) \leq \bar{\mathcal{E}}^{L_0} \left[\left((h^1 - h^2)_{\mathbb{H}_D}^+ \right)^x \right] + C \|G(\omega^1, \cdot) - G(\omega^2, \cdot)\|_\infty.$$

In particular, if $\omega^1 = \omega^2$, then we have $(v^1 - v^2)(x) \leq \bar{\mathcal{E}}^{L_0} \left[\left((h^1 - h^2)_{\mathbb{H}_D}^+ \right)^x \right]$.

Proof of Lemma VI.3.7 Since ϵ is fixed, to simplify the notation, we omit ϵ in the superscript in the proof. We decompose the proof in five steps.

Step 1. We first prove (i) in the case of $G := \bar{g}$, where \bar{g} is defined in (VI.3.2). For any N , denote

$$\bar{\theta}_{N,N}^\omega(\pi_N; 0) := \bar{\mathcal{E}}^{L_0} \left[(\xi_{\mathbb{H}_Q})^{\omega \bar{\otimes} \pi_N} \right].$$

Thanks to Lemma VI.3.6, we may define $\bar{\theta}_{N,n}^\omega(\pi_n; \cdot)$ as the viscosity solution of the following PDE :

$$-\bar{g}(\theta, D\theta, D^2\theta) = 0 \text{ on } O_\epsilon(\omega \bar{\otimes} \pi_n), \quad \theta(x) = \bar{\theta}_{N,n+1}^\omega(\pi_n^x; 0) \text{ on } \partial O_\epsilon(\omega \bar{\otimes} \pi_n), \quad \text{for all } n \leq N \text{ (VI.3.7)}$$

and we know

$$\begin{aligned} \bar{\theta}_{N,n}^\omega(\pi_n; x) = \\ \sup_{b \in \mathbb{H}^0([0, L_0])} \bar{\mathcal{E}}^{L_0} \left[e^{-\int_0^{\mathbb{H}^{\omega \bar{\otimes} \pi_n^x}} b_r dr} \xi \left(\omega \bar{\otimes} \pi_n^{N-n}(x, B) \bar{\otimes} (B_{\mathbb{H}_Q^{\omega \bar{\otimes} \pi_n^x \wedge})}^{\mathbb{H}_{N-n}^{\omega \bar{\otimes} \pi_n^x}} \right) + C_0 \int_0^{\mathbb{H}_{N-n}^{\omega \bar{\otimes} \pi_n^x}} e^{-\int_0^s b_r dr} ds \right]. \end{aligned} \quad (\text{VI.3.8})$$

Lemma VI.3.6 also implies that $\bar{\theta}_{N,n}^\epsilon(\pi_n; x)$ is continuous in both variables (π_n, x) , and clearly, they are uniformly bounded. We next define

$$\begin{aligned} \bar{\theta}_n^\omega(\pi_n; x) := \\ \sup_{b \in \mathbb{H}^0([0, L_0])} \bar{\mathcal{E}}^{L_0} \left[e^{-\int_0^{\mathbb{H}_Q^{\omega \bar{\otimes} \pi_n^x}} b_r dr} \limsup_{N \rightarrow \infty} \xi \left(\omega \bar{\otimes} \pi_n^{N-n}(x, B) \bar{\otimes} (B_{\mathbb{H}_Q^{\omega \bar{\otimes} \pi_n^x \wedge})}^{\mathbb{H}_{N-n}^{\omega \bar{\otimes} \pi_n^x}} \right) + C_0 \int_0^{\mathbb{H}_Q^{\omega \bar{\otimes} \pi_n^x}} e^{-\int_0^s b_r dr} ds \right]. \end{aligned} \quad (\text{VI.3.9})$$

Then, it is easy to estimate that

$$|\bar{\theta}_n^\omega(\pi_n; x) - \bar{\theta}_{N,n}^\omega(\pi_n; x)| \leq C \mathbb{C}^{L_0} \left[\mathbb{H}_{N-n}^{\omega \bar{\otimes} \pi_n^x} < \mathbb{H}_Q^{\omega \bar{\otimes} \pi_n^x} \right] \rightarrow 0, \quad N \rightarrow \infty.$$

By Proposition VI.1.8, the convergence is uniform in (π_n, x) . This implies that $\bar{\theta}_n^\omega(\pi_n; x)$ is uniformly bounded and continuous in (π_n, x) . Moreover, by the stability of viscosity solutions we see that $\bar{\theta}_n^\omega(\pi_n; \cdot)$ is the viscosity solution of PDE (VI.3.7) in $O_\epsilon(\omega \bar{\otimes} \pi_n)$, with the boundary condition :

$$\begin{cases} \bar{\theta}_n^\omega(\pi_n; x) = \xi(\omega \bar{\otimes} \pi_n^x), & |x| < \epsilon \text{ and } x \in \partial Q^{\omega \bar{\otimes} \pi_n}, \\ \bar{\theta}_n^\omega(\pi_n; x) = \bar{\theta}_{n+1}^\omega(\pi_n^x; 0), & |x| = \epsilon. \end{cases}$$

Hence, we have showed the desired result in the case $G = \bar{g}$. Similarly, we may show that $\underline{\theta}_n^\omega$ defined below is the viscosity solution to the path-frozen PDE when the nonlinearity is \underline{g} :

$$\underline{\theta}_n^\omega(\pi_n; x) := \inf_{b \in \mathbb{H}^0([0, L_0])} \underline{\xi}^{L_0} \left[e^{-\int_0^{\omega \bar{\otimes} \pi_n^x} b_r dr} \limsup_{N \rightarrow \infty} \xi \left(\omega \bar{\otimes} \pi_n^{N-n}(x, B) \bar{\otimes} (B_{\mathbb{H}_Q^{\omega \bar{\otimes} \pi_n^x \wedge}})^{\mathbb{H}_{N-n}^{\omega \bar{\otimes} \pi_n^x}} \right) + C_0 \int_0^{\omega \bar{\otimes} \pi_n^x} e^{-\int_0^s b_r dr} ds \right].$$

Step 2. We next prove (ii) in the case of $G = \bar{g}$. Considering $\pi_n^x \in \mathcal{Q}^{\omega^1} \cap \mathcal{Q}^{\omega^2}$, we have the following estimate :

$$\begin{aligned} \left| \bar{\theta}_{N,n}^{\omega^1}(\pi_n; x) - \bar{\theta}_{N,n}^{\omega^2}(\pi_n; x) \right| &\leq C \bar{\mathcal{E}}^{L_0} \left[\left| \mathbb{H}_{N-n}^{\omega^1 \bar{\otimes} \pi_n^x} - \mathbb{H}_{N-n}^{\omega^2 \bar{\otimes} \pi_n^x} \right| \right] \\ &+ C \bar{\mathcal{E}}^{L_0} \left[\left| \xi \left(\omega^1 \bar{\otimes} \pi_n^{N-n}(x, B) \bar{\otimes} (B_{\mathbb{H}_Q^{\omega^1 \bar{\otimes} \pi_n^x \wedge}})^{\mathbb{H}_{N-n}^{\omega^1 \bar{\otimes} \pi_n^x}} \right) - \xi \left(\omega^2 \bar{\otimes} \pi_n^{N-n}(x, B) \bar{\otimes} (B_{\mathbb{H}_Q^{\omega^2 \bar{\otimes} \pi_n^x \wedge}})^{\mathbb{H}_{N-n}^{\omega^2 \bar{\otimes} \pi_n^x}} \right) \right| \right], \end{aligned}$$

where $\bar{\theta}_{N,n}^{\omega^i}(\pi_n; x)$, $i = 1, 2$, are defined in (VI.3.8). Note that $\left| \mathbb{H}_{N-n}^{\omega^1 \bar{\otimes} \pi_n^x} - \mathbb{H}_{N-n}^{\omega^2 \bar{\otimes} \pi_n^x} \right| \leq \left| \mathbb{H}_Q^{\omega^1 \bar{\otimes} \pi_n^x} - \mathbb{H}_Q^{\omega^2 \bar{\otimes} \pi_n^x} \right|$. As in Lemma VI.3.6, one may easily show that

$$\left| \bar{\theta}_{N,n}^{\omega^1} - \bar{\theta}_{N,n}^{\omega^2} \right| \leq \rho_\epsilon(d^e(\omega^1, \omega^2)),$$

where ρ_ϵ is independent of N . Considering $\bar{\theta}_n^{\omega^i}$ defined in (VI.3.9), we obtain by sending $N \rightarrow \infty$ that

$$\left| \bar{\theta}_n^{\omega^1} - \bar{\theta}_n^{\omega^2} \right| \leq \rho_\epsilon(d^e(\omega^1, \omega^2)).$$

A similar argument provides the same estimate for $\underline{\theta}_n^\omega$:

$$\left| \underline{\theta}_n^{\omega^1} - \underline{\theta}_n^{\omega^2} \right| \leq \rho_\epsilon(d^e(\omega^1, \omega^2)). \quad (\text{VI.3.10})$$

Step 3. We now prove (i) for general G . Given the construction of Step 1, define :

$$\bar{\theta}_m^{\omega,m}(\pi_m; x) := \bar{\theta}_m^{\omega}(\pi_m; x), \quad \underline{\theta}_m^{\omega,m}(\pi_m; x) := \underline{\theta}_m^{\omega}(\pi_m; x); \quad m \geq 1.$$

For $n \leq m-1$, we may define $\bar{\theta}_n^{\omega,m}$ and $\underline{\theta}_n^{\omega,m}$ as the unique viscosity solution of the path-frozen PDE $(E)_\epsilon^{\omega \bar{\otimes} \pi_n}$ with boundary conditions

$$\bar{\theta}_n^{\omega,m}(\pi_n; x) = \bar{\theta}_{n+1}^{\omega,m}(\pi_n^x; 0), \quad \underline{\theta}_n^{\omega,m}(\pi_n; x) = \underline{\theta}_{n+1}^{\omega,m}(\pi_n^x; 0) \quad \text{for } x \in \partial O_\epsilon(\omega \bar{\otimes} \pi_n).$$

Since $g \leq G \leq \bar{g}$, it is easy to deduce that $\bar{\theta}_m^{\epsilon,m}$ and $\underline{\theta}_m^{\epsilon,m}$ are respectively viscosity supersolution and subsolution to the path-frozen PDE $(E)_\epsilon^{\pi_m}$. By the comparison result for viscosity solutions of PDEs, we obtain that

$$\bar{\theta}_m^{\omega,m}(\pi_m; \cdot) \geq \bar{\theta}_m^{\omega,m+1}(\pi_m; \cdot) \geq \underline{\theta}_m^{\omega,m+1}(\pi_m; \cdot) \geq \underline{\theta}_m^{\omega,m}(\pi_m; \cdot) \quad \text{on } O_\epsilon(\omega \bar{\otimes} \pi_m),$$

Using the comparison argument again, we obtain

$$\bar{\theta}_n^{\omega,m}(\pi_n; \cdot) \geq \bar{\theta}_n^{\omega,m+1}(\pi_n; \cdot) \geq \underline{\theta}_n^{\omega,m+1}(\pi_n; \cdot) \geq \underline{\theta}_n^{\omega,m}(\pi_n; \cdot) \quad \text{on } O_\epsilon(\omega \bar{\otimes} \pi_n) \quad \text{for all } n \leq m. \quad (\text{VI.3.11})$$

Denote $\delta\theta_n^{\omega,m} := \bar{\theta}_n^{\omega,m} - \underline{\theta}_n^{\omega,m}$. Applying Lemma VI.3.8 repeatedly and using the tower property of $\bar{\mathcal{E}}^{L_0}$ stated in Lemma VI.1.6, we obtain that

$$|\delta\theta_n^{\omega,m}(\pi_n; x)| \leq \bar{\mathcal{E}}^{L_0} \left[\left| \delta\theta_m^{\omega,m}(\pi_n^{m-n}(x, B); 0) \right| \right].$$

Note that $\delta\theta_m^{\omega,m}(\pi_n^{m-n}(x, B); 0) = 0$ as $\pi_n^{m-n}(x, B) \in \partial Q^\omega$. Then, by Proposition VI.1.8, we have

$$|\delta\theta_n^{\omega,m}(\pi_n; x)| \leq C \mathcal{E}^{L_0} \left[\mathbb{H}_{m-n}^{\omega \bar{\otimes} \pi_n^x} < \mathbb{H}_Q^{\omega \bar{\otimes} \pi_n^x} \right] \rightarrow 0, \quad \text{as } m \rightarrow \infty.$$

Together with (VI.3.11), this implies the existence of θ_n^ω such that

$$\bar{\theta}_n^{\omega,m} \downarrow \theta_n^\omega, \quad \underline{\theta}_n^{\omega,m} \uparrow \theta_n^\omega, \quad \text{as } m \rightarrow \infty. \quad (\text{VI.3.12})$$

Clearly θ_n^ω is uniformly bounded and continuous. Finally, it follows from the stability of viscosity solutions that θ_n^ω satisfies the statement of (i).

Step 4. We next prove (ii) for a general nonlinearity G . For the simplicity of notation, we denote the stopping times :

$$\mathbb{H}^i := \mathbb{H}_Q^{\omega^i \bar{\otimes} \pi_n^x} \text{ for } i = 1, 2, \quad \mathbb{H}^{1,2} := \mathbb{H}^1 \wedge \mathbb{H}^2,$$

$$\mathbb{H}_0 = 0, \quad \mathbb{H}_1 := \inf\{t \geq 0 : x + B_t \notin O_\epsilon\}, \quad \mathbb{H}_{i+1} := \inf\{t \geq \mathbb{H}_i : B_t - B_{\mathbb{H}_i} \notin O_\epsilon\} \text{ for } i \geq 1.$$

First, considering $\bar{\theta}_n^{\omega^1, m}$ defined in Step 3, we claim that for $\pi_n^x \in \mathcal{Q}^{\omega^1} \cap \mathcal{Q}^{\omega^2}$

$$\begin{aligned} (\bar{\theta}_n^{\omega^1, m} - \underline{\theta}_n^{\omega^2, m})(\pi_n; x) &\leq \bar{\mathcal{E}}^{L_0} \left[(\bar{\theta}_m^{\omega^1} - \underline{\theta}_m^{\omega^2}) \left(\pi_n^{m-n}(x, B); 0 \right) 1_{\{\mathbb{H}_{m-n} \leq \mathbb{H}^{1,2}\}} \right] \\ &\quad + I_1 + I_2 + C(m-n)\rho(d^e(\omega^1, \omega^2)), \end{aligned} \quad (\text{VI.3.13})$$

where

$$I_1 := \sum_{k=0}^{m-n-1} \bar{\mathcal{E}}^{L_0} \left[\left(\bar{\theta}_{n+k+1}^{\omega^1, m}(\pi_n^{k+1}(x, B); 0) - \underline{\theta}_{n+k}^{\omega^2, m}(\pi_n^k(x, B); B_{\mathbb{H}^1} - B_{\mathbb{H}_k}) \right) 1_{\{\mathbb{H}^1 < \mathbb{H}_{k+1} \leq \mathbb{H}^2\}} \right],$$

$$I_2 := \sum_{k=0}^{m-n-1} \bar{\mathcal{E}}^{L_0} \left[\left(\bar{\theta}_{n+k}^{\omega^1, m}(\pi_n^k(x, B); B_{\mathbb{H}^2} - B_{\mathbb{H}_k}) - \underline{\theta}_{n+k+1}^{\omega^2, m}(\pi_n^{k+1}(x, B); 0) \right) 1_{\{\mathbb{H}^2 < \mathbb{H}_{k+1} \leq \mathbb{H}^1\}} \right]$$

This claim will be proved in Step 5. We next focus on the term in I_1 :

$$\left(\bar{\theta}_{n+k+1}^{\omega^1, m}(\pi_n^{k+1}(x, B); 0) - \underline{\theta}_{n+k}^{\omega^2, m}(\pi_n^k(x, B); B_{\mathbb{H}^1} - B_{\mathbb{H}_k}) \right) 1_{\{\mathbb{H}^1 < \mathbb{H}_{k+1} \leq \mathbb{H}^2\}}.$$

Note that as $\mathbb{H}^1 < \mathbb{H}_{k+1} \leq \mathbb{H}^2$, we have $\pi_n^{k+1} = \pi_n^k \bar{\otimes} \text{Lin}\left\{((0, 0), (1, B_{\mathbb{H}^1} - B_{\mathbb{H}_k}))\right\}$, and thus

$$\bar{\theta}_{n+k+1}^{\omega^1, n+k+1}(\pi_n^{k+1}(x, B); 0) = \underline{\theta}_{n+k+1}^{\omega^1, n+k+1}(\pi_n^{k+1}(x, B); 0) = \underline{\theta}_{n+k}^{\omega^1, n+k}(\pi_n^k(x, B); B_{\mathbb{H}^1} - B_{\mathbb{H}_k}).$$

So, by using (VI.3.10), we obtain

$$\begin{aligned} &\left| \bar{\theta}_{n+k+1}^{\omega^1, n+k+1}(\pi_n^{k+1}(x, B); 0) - \underline{\theta}_{n+k}^{\omega^2, n+k}(\pi_n^k(x, B); B_{\mathbb{H}^1} - B_{\mathbb{H}_k}) \right| \\ &\leq \left| \underline{\theta}_{n+k}^{\omega^1, n+k}(\pi_n^k(x, B); B_{\mathbb{H}^1} - B_{\mathbb{H}_k}) - \underline{\theta}_{n+k}^{\omega^2, n+k}(\pi_n^k(x, B); B_{\mathbb{H}^1} - B_{\mathbb{H}_k}) \right| \leq \rho_\epsilon(d^e(\omega^1, \omega^2)). \end{aligned} \quad (\text{VI.3.14})$$

Further, as in Step 3, the comparison result implies that

$$\begin{aligned}\underline{\theta}_{n+k}^{\omega^2, m}(\pi_n^k(x, B); B_{\mathbb{H}^1} - B_{\mathbb{H}_k}) &\geq \underline{\theta}_{n+k}^{\omega^2, n+k}(\pi_n^k(x, B); B_{\mathbb{H}^1} - B_{\mathbb{H}_k}); \\ \bar{\theta}_{n+k+1}^{\omega^1, m}(\pi_n^{k+1}(x, B); 0) &\leq \bar{\theta}_{n+k+1}^{\omega^1, n+k+1}(\pi_n^{k+1}(x, B); 0).\end{aligned}$$

Therefore, (VI.3.14) implies that

$$\bar{\theta}_{n+k+1}^{\omega^1, m}(\pi_n^{k+1}(x, B); 0) - \underline{\theta}_{n+k}^{\omega^2, m}(\pi_n^k(x, B); B_{\mathbb{H}^1} - B_{\mathbb{H}_k}) \leq \rho_\epsilon(d^e(\omega^1, \omega^2)).$$

Similarly, we may prove the same estimate for the term in I_2 . Then, by (VI.3.13) we conclude that

$$(\bar{\theta}_n^{\omega^1, m} - \underline{\theta}_n^{\omega^2, m})(\pi_n; x) \leq C\mathcal{C}^L [\mathbb{H}_{m-n} < \mathbb{H}^{1,2}] + C(m-n+1)\rho_\epsilon(d^e(\omega^1, \omega^2)).$$

Recalling (VI.3.12), we obtain that

$$(\theta_n^{\omega^1} - \theta_n^{\omega^2})(\pi_n; x) \leq \rho_\epsilon(d^e(\omega^1, \omega^2)).$$

By exchanging the roles of ω^1 and ω^2 , we have $|(\theta_n^{\omega^1} - \theta_n^{\omega^2})(\pi_n; x)| \leq \rho_\epsilon(d^e(\omega^1, \omega^2))$.

Step 5. We prove Claim (VI.3.13). Suppose that $m \geq n+1$. It suffices to show that

$$\begin{aligned}(\bar{\theta}_n^{\omega^1, m} - \underline{\theta}_n^{\omega^2, m})(\pi_n; x) &\leq \bar{\mathcal{E}}^{L_0} \left[(\bar{\theta}_{n+1}^{\omega^1, m} - \underline{\theta}_{n+1}^{\omega^2, m})(\pi_n^1(x, B); 0) 1_{\{\mathbb{H}_1 \leq \mathbb{H}^{1,2}\}} \right] \\ &\quad + \bar{\mathcal{E}}^{L_0} \left[(\bar{\theta}_{n+1}^{\omega^1, m}(\pi_n^1(x, B); 0) - \underline{\theta}_n^{\omega^2, m}(\pi_n; x + B_{\mathbb{H}^1})) 1_{\{\mathbb{H}^1 < \mathbb{H}_1 \leq \mathbb{H}^2\}} \right] \\ &\quad + \bar{\mathcal{E}}^{L_0} \left[(\bar{\theta}_n^{\omega^1, m}(\pi_n; x + B_{\mathbb{H}^2}) - \underline{\theta}_{n+1}^{\omega^2, m}(\pi_n^1(x, B); 0)) 1_{\{\mathbb{H}^2 < \mathbb{H}_1 \leq \mathbb{H}^1\}} \right] \\ &\quad + C\rho_\epsilon(d^e(\omega^1, \omega^2)).\end{aligned}$$

Then the claim can be easily proved by induction. Recall that $\bar{\theta}_n^{\omega^1, m}$ (resp. $\underline{\theta}_n^{\omega^2, m}$) is a solution to the PDE with generator $G(\omega^1, \cdot)$ (resp. $G(\omega^2, \cdot)$). Now we study those two PDEs on the domain :

$$O_\epsilon \cap Q^{\omega^1} \cap Q^{\omega^2}.$$

The boundary of this set can be divided into three parts which belong to ∂O_ϵ , ∂Q^{ω^1} and ∂Q^{ω^2} respectively. We denote them by Bd_1 , Bd_2 and Bd_3 .

(i) On Bd_1 , we have $\mathbf{H}_1 \leq \mathbf{H}^{1,2}$, and thus

$$\bar{\theta}_n^{\omega^1, m}(\pi_n, x) = \bar{\theta}_{n+1}^{\omega^1, m}(\pi_n^x, 0) \quad \text{and} \quad \underline{\theta}_n^{\omega^2, m}(\pi_n, x) = \underline{\theta}_{n+1}^{\omega^2, m}(\pi_n^x, 0).$$

(ii) On Bd_2 , we have $\mathbf{H}^1 < \mathbf{H}_1 \leq \mathbf{H}^2$, so we only have $\bar{\theta}_n^{\omega^1, m}(\pi_n, x) = \bar{\theta}_{n+1}^{\omega^1, m}(\pi_n^x, 0)$.

(iii) On Bd_3 , we have $\mathbf{H}^2 < \mathbf{H}_1 \leq \mathbf{H}^1$, so we only have $\underline{\theta}_n^{\omega^2, m}(\pi_n, x) = \underline{\theta}_{n+1}^{\omega^2, m}(\pi_n^x, 0)$.

Finally, Lemma VI.3.8 completes the proof. \square

The previous lemma shows the existence of the viscosity solution to the path-frozen PDEs. We now use Assumption VI.2.5 to construct smooth super- and sub-solutions to the PPDE from the solution to the path-frozen PDEs.

Denote

$$\theta_n^\epsilon := \theta_n^{0, \epsilon}, \quad \mathbf{H}_n := \mathbf{H}_n^0 \quad \text{and} \quad \hat{\pi}_n := \text{Lin}\left\{(\mathbf{H}_i, \omega_{\mathbf{H}_i}); 0 \leq i \leq n\right\}.$$

Lemma VI.3.9. *There exists $\psi^\epsilon \in \bar{C}^2(\mathcal{Q})$ such that*

$$\psi^\epsilon(0) = \theta_0^\epsilon(0) + \epsilon, \quad \psi^\epsilon \geq h \quad \text{on} \quad \mathcal{Q} \quad \text{and} \quad \mathcal{L}^{\hat{\pi}_n} \psi^\epsilon \geq 0 \quad \text{on} \quad \mathcal{O}_\epsilon(\hat{\pi}_n) \quad \text{for all } n \in \mathbb{N}.$$

Proof For simplicity, in the proof we omit the superscript ϵ . Set $\delta_n := 2^{-n-2}\epsilon$. First, since PDE $(E)_\epsilon^0$ satisfies Assumption VI.2.5 and $G(\omega, y, z, \gamma)$ is decreasing in y , there exists a function $v_0 \in C_0^2(O_\epsilon(0))$ such that

$$v_0(0) = \theta_0(0) + \frac{\epsilon}{2}, \quad \mathbf{L}^0 v_0 \geq 0 \quad \text{on} \quad O_\epsilon(0) \quad \text{and} \quad v_0 \geq \theta_0 \quad \text{on} \quad \partial O_\epsilon(0).$$

Define

$$\psi(\omega) := v_0(\omega_{\bar{t}}) + \sum_{i \geq 0} \delta_i \quad \text{on} \quad \text{cl}(O_\epsilon(0)).$$

By the monotonicity of G , it is clear that

$$\psi(0) - \theta_0(0) = \frac{\epsilon}{2} + \sum_{i \geq 0} \delta_i = \epsilon, \quad \psi \in C^2(O_\epsilon(0)) \quad \text{and} \quad \mathcal{L}^0 \psi \geq \mathbf{L}^0 v_0 \geq 0 \quad \text{on} \quad O_\epsilon(0).$$

Next, applying again Assumption VI.2.5 to PDE $(E)_\epsilon^{\hat{\pi}_1}$, we can find a function $v_1(\hat{\pi}_1; \cdot) \in C_0^2(O_\epsilon(\hat{\pi}_1))$ such that

$$v_1(\hat{\pi}_1; 0) = v_0(x_1) + \delta_0, \quad \mathbf{L}^{\hat{\pi}_1} v_1 \geq 0 \quad \text{on} \quad O_\epsilon(\hat{\pi}_1), \quad v_1(\hat{\pi}_1; \cdot) \geq \theta_1(\hat{\pi}_1; \cdot) \quad \text{on} \quad \partial O_\epsilon(\hat{\pi}_1).$$

Then, define

$$\psi(\omega) := v_1(\hat{\pi}_1; \omega_{\bar{t}} - \omega_{H_1}) + \sum_{i \geq 1} \delta_i, \quad \text{for } \omega \in \mathcal{O}_\epsilon(\hat{\pi}_1)$$

It is clear that the updated ψ is in \bar{C}^2 and $\hat{\pi}_1 \varphi \geq 0$ on $\mathcal{O}_\epsilon(\hat{\pi}_n)$. Repeating the procedure, we may find a sequence of functions v_n and thus construct the desired $\psi \in \bar{C}^2(\mathcal{Q})$. \square

Finally, we have done all the necessary constructions and are ready to show the main result of the section.

Proof of Proposition VI.3.5 For any $\epsilon > 0$, let ψ^ϵ be as in Lemma VI.3.9, and $\bar{\psi}^\epsilon := \psi^\epsilon + \rho(2\epsilon) + \lambda^{-1}(\rho(2\epsilon))$, where ρ is the modulus of continuity of ξ and G and λ^{-1} is the inverse of the function in Assumption VI.2.1. Then clearly $\bar{\psi}^\epsilon \in \bar{C}^2(\mathcal{Q})$ and bounded. Also,

$$\bar{\psi}^\epsilon(\omega) - \xi(\omega) \geq \psi^\epsilon(\omega) + \rho(2\epsilon) - \xi(\omega) \geq \xi(\omega^\epsilon) - \xi(\omega) + \rho(2\epsilon) \geq 0 \text{ on } \partial\mathcal{Q}.$$

Moreover, when $\bar{t}(\omega) \in [H_n(\omega), H_{n+1}(\omega))$, we have that

$$\begin{aligned} \mathcal{L}\bar{\psi}^\epsilon(\omega) &= -G(\omega, \bar{\psi}^\epsilon, \partial_\omega \psi^\epsilon, \partial_{\omega\omega}^2 \psi^\epsilon) \\ &\geq -G(\hat{\pi}_n, \psi^\epsilon + \lambda^{-1}(\rho(2\epsilon)), \partial_\omega \psi^\epsilon, \partial_{\omega\omega}^2 \psi^\epsilon) - \rho(2\epsilon) \\ &\geq -G(\hat{\pi}_n, \psi^\epsilon, \partial_\omega \psi^\epsilon, \partial_{\omega\omega}^2 \psi^\epsilon) \geq 0. \end{aligned}$$

Then by the definition of \bar{u} we see that

$$\bar{u}(0) \leq \bar{\psi}^\epsilon(0) = \psi^\epsilon + \rho(2\epsilon) + \lambda^{-1}(\rho(2\epsilon)) \leq \theta_0^\epsilon(0) + \epsilon + \rho(2\epsilon) + \lambda^{-1}(\rho(2\epsilon)). \quad (\text{VI.3.15})$$

Similarly, $\underline{u}(0) \geq \theta_0^\epsilon(0) - \epsilon - \rho(2\epsilon) - \lambda^{-1}(\rho(2\epsilon))$. That implies that

$$\bar{u}(0) - \underline{u}(0) \leq 2\epsilon + 2\rho(2\epsilon) + 2\lambda^{-1}(\rho(2\epsilon)).$$

Since ϵ is arbitrary, this shows that $\bar{u}(0) = \underline{u}(0)$. Similarly, we can show that $\bar{u}(\omega) = \underline{u}(\omega)$ for all $\omega \in \mathcal{Q}$. \square

4 Existence

In this section, we verify that

$$u := \bar{u} = \underline{u} \tag{VI.4.1}$$

is the unique BUC viscosity solution to the path-dependent PDE (VI.2.1). We will prove that u is BUC in Subsection 4.1 and u satisfies the viscosity property in Subsection 4.2.

4.1 Regularity

The non-continuity of the hitting time $H_Q(\omega)$ brings difficulty to the proof of the regularity of u . One cannot adapt the method used in [38]. In our approach, we make use of the uniform continuity of the solution of the path-frozen PDEs proved in (ii) of Lemma VI.3.7.

First, it is easy to show that u is bounded.

Proposition VI.4.1. *Let Assumption VI.2.1 hold and $\xi \in \text{BUC}(\partial\mathcal{Q})$. Then \bar{u} is bounded from above and \underline{u} is bounded from below.*

Proof Assume that $|\xi| \leq C_0$. Define :

$$\psi := \lambda^{-1}(C_0) + C_0.$$

Note that $\psi \in \bar{C}^2$. Observe that $\psi_T \geq C_0 \geq \xi$. Also,

$$\mathcal{L}^\omega \psi_s = -G^\omega(\cdot, \psi_s, 0, 0) \geq C_0 - G^\omega(\cdot, 0, 0, 0) \geq 0.$$

It follows that $\psi \in \bar{\mathcal{D}}_Q^\xi(\omega)$, and thus $\bar{u}(\omega) \leq \psi(0) = \lambda^{-1}(C_0) + C_0$. Similarly, one can show that $\underline{u}(\omega) \geq -\lambda^{-1}(C_0) - C_0$. \square

The rest of the subsection is devoted to prove the uniform continuity of u .

Proposition VI.4.2. *The function u defined in (VI.4.1) is uniformly continuous in \mathcal{Q} .*

Proof Recall (VI.3.15), i.e. for $\omega^1, \omega^2 \in \mathcal{Q}$, it holds that

$$\bar{u}(\omega^1) \leq \theta_0^{\omega^1}(0) + \epsilon + \rho(2\epsilon) \quad \text{and} \quad \underline{u}(\omega^2) \geq \theta_0^{\omega^2}(0) - \epsilon - \rho(2\epsilon).$$

Hence, it follows from Lemma VI.3.7 that

$$u(\omega^1) - u(\omega^2) = \bar{u}(\omega^1) - \underline{u}(\omega^2) \leq \theta_0^{\omega^1}(0) - \theta_0^{\omega^2}(0) + 2(\epsilon + \rho(2\epsilon)) \leq \rho_\epsilon(d^e(\omega^1, \omega^2)) + \rho(2\epsilon).$$

By exchanging the roles of ω^1 and ω^2 , we obtain that u is uniformly continuous. \square

4.2 Viscosity property

After having shown that u is uniformly continuous, we need to verify that it indeed satisfies the viscosity property. The following proof is similar to that of Proposition 4.3 in [38].

Proposition VI.4.3. *u is the viscosity solution to PPDE (VI.2.1).*

Proof Without loss of generality, we prove only that \bar{u} is a L_0 -viscosity supersolution at 0. Assume the contrary, i.e. there exists $\varphi \in \bar{\mathcal{A}}^{L_0}\bar{u}(0)$ such that $-c := \mathcal{L}\varphi(0) < 0$. For any $\psi \in \bar{\mathcal{D}}_Q^\xi(0)$ and $\omega \in \mathcal{Q}$ it is clear that $\psi^\omega \in \bar{\mathcal{D}}_Q^\xi(\omega)$ and $\psi(\omega) \geq \bar{u}(\omega)$. Now by the definition of \bar{u} , there exists $\psi^n \in \bar{C}^2(\mathcal{Q})$ such that

$$\delta_n := \psi^n(0) - \bar{u}(0) \downarrow 0 \text{ as } n \rightarrow \infty, \quad \mathcal{L}\psi^n(\omega) \geq 0, \quad \omega \in \mathcal{Q}. \quad (\text{VI.4.2})$$

Let H_{O_ϵ} be a localization of test function φ . Since $\varphi \in C^2(\mathcal{O}_\epsilon)$ and $\bar{u} \in \text{BUC}(\mathcal{Q})$, without loss of generality we may assume that

$$\mathcal{L}\varphi(\omega_{t\wedge\cdot}) \leq -\frac{c}{2} \text{ and } |\varphi_t - \varphi_0| + |\bar{u}_t - \bar{u}_0| \leq \frac{c}{6L_0} \text{ for all } t \leq \mathsf{H}_{O_\epsilon}.$$

Since $\varphi \in \bar{\mathcal{A}}^{L_0}\bar{u}(0)$, this implies for all $\mathbb{P} \in \mathcal{P}^{L_0}$ that :

$$0 \geq \mathbb{E}^\mathbb{P} [(\varphi - \bar{u})_{\mathsf{H}_{O_\epsilon}}] \geq \mathbb{E}^\mathbb{P} [(\varphi - \psi^n)_{\mathsf{H}_{O_\epsilon}}]. \quad (\text{VI.4.3})$$

Denote $\mathcal{G}^\mathbb{P}\phi := \alpha^\mathbb{P} \cdot \partial_\omega \phi + \frac{1}{2}(\beta^\mathbb{P})^2 : \partial_{\omega\omega}^2 \phi$. Then, since $\varphi \in C^2(\mathcal{O}_\epsilon)$ and $\psi^n \in \bar{C}^2(\mathcal{Q})$, it follows from (VI.4.2) that :

$$\begin{aligned} \delta_n &\geq \mathbb{E}^\mathbb{P} [(\varphi - \psi^n)_{\mathsf{H}_{O_\epsilon}} - (\varphi - \psi^n)_0] = \mathbb{E}^\mathbb{P} \left[\int_0^{\mathsf{H}_{O_\epsilon}} \mathcal{G}^\mathbb{P}(\varphi - \psi^n) ds \right] \\ &\geq \mathbb{E}^\mathbb{P} \left[\int_0^{\mathsf{H}_{O_\epsilon}} \left(\frac{c}{2} - G(\omega_{s\wedge\cdot}, \varphi, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi) + G(\omega_{s\wedge\cdot}, \psi^n, \partial_\omega \psi^n, \partial_{\omega\omega}^2 \psi^n) + \mathcal{G}^\mathbb{P}(\varphi - \psi^n) \right) ds \right] \\ &\geq \mathbb{E}^\mathbb{P} \left[\int_0^{\mathsf{H}_{O_\epsilon}} \left(\frac{c}{2} - G(\omega_{s\wedge\cdot}, \varphi, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi) + G(\omega_{s\wedge\cdot}, \bar{u}, \partial_\omega \psi^n, \partial_{\omega\omega}^2 \psi^n) + \mathcal{G}^\mathbb{P}(\varphi - \psi^n) \right) ds \right], \end{aligned}$$

where the last inequality is due to the monotonicity in y of G . Since $\varphi_0 = \bar{u}_0$, we get

$$\begin{aligned} \delta_n \geq \mathbb{E}^{\mathbb{P}} \left[\int_0^{\mathbb{H}_{O_\epsilon}} \left(\frac{c}{3} - G(\omega_{s\wedge\cdot}, \bar{u}_0, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi) \right. \right. \\ \left. \left. + G(\omega_{s\wedge\cdot}, \bar{u}_0, \partial_\omega \psi^n, \partial_{\omega\omega}^2 \psi^n) + \mathcal{G}^{\mathbb{P}}(\varphi - \psi^n) \right) ds \right]. \end{aligned}$$

We next let $\eta > 0$, and for each n , define $\tau_0^n := 0$ and

$$\begin{aligned} \tau_{j+1}^n : = \mathbb{H}_{O_\epsilon} \wedge \inf \{ t \geq \tau_j^n : \rho(d^e(\omega_{t\wedge\cdot}, \omega_{\tau_j^n\wedge\cdot})) + |\partial_\omega \varphi(\omega_{t\wedge\cdot}) - \partial_\omega \varphi(\omega_{\tau_j^n\wedge\cdot})| \\ + |\partial_{\omega\omega}^2 \varphi(\omega_{t\wedge\cdot}) - \partial_{\omega\omega}^2 \varphi(\omega_{\tau_j^n\wedge\cdot})| + |\partial_\omega \psi^n(\omega_{t\wedge\cdot}) - \partial_\omega \psi^n(\omega_{\tau_j^n\wedge\cdot})| \\ + |\partial_{\omega\omega}^2 \psi^n(\omega_{t\wedge\cdot}) - \partial_{\omega\omega}^2 \psi^n(\omega_{\tau_j^n\wedge\cdot})| \geq \eta \}. \end{aligned}$$

Since $\varphi \in C^2(\mathcal{O}_\epsilon)$ and $\psi^n \in \bar{C}^2(\mathcal{Q})$, one can easily check that $\tau_j^n \uparrow \mathbb{H}_D$ \mathcal{P}^{L_0} -q.s. as $j \rightarrow \infty$. Thus,

$$\begin{aligned} \delta_n \geq \left(\frac{c}{3} - C\eta \right) \mathbb{E}^{\mathbb{P}}[\mathbb{H}_{O_\epsilon}] + \sum_{j \geq 0} \mathbb{E}^{\mathbb{P}} \int_{\tau_j^n}^{\tau_{j+1}^n} \left(-G(\cdot, \bar{u}_0, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi) \right. \\ \left. + G(\cdot, \bar{u}_0, \partial_\omega \psi^n, \partial_{\omega\omega}^2 \psi^n) + \mathcal{G}^{\mathbb{P}}(\varphi - \psi^n) \right)_{\tau_j^n} ds \\ = \left(\frac{c}{3} - C\eta \right) \mathbb{E}^{\mathbb{P}}[\mathbb{H}_{O_\epsilon}] + \sum_{j \geq 0} \mathbb{E}^{\mathbb{P}} \int_{\tau_j^n}^{\tau_{j+1}^n} \left(\alpha_j^n \cdot \partial_\omega(\psi^n - \varphi) \right. \\ \left. + \frac{1}{2}(\beta_j^n)^2 : \partial_{\omega\omega}^2(\psi^n - \varphi) + \mathcal{G}^{\mathbb{P}}(\varphi - \psi^n) \right)_{\tau_j^n} ds, \end{aligned}$$

for some α_j^n, β_j^n such that $|\alpha_j^n| \leq L$ and $\beta_j^n \in \mathbb{H}_L^0$. Note that α_j^n and β_j^n are both $\mathcal{F}_{\tau_j^n}$ -measurable. Take $\mathbb{P}_n \in \mathcal{P}^{L_0}$ such that $\alpha_t^{\mathbb{P}_n} = \alpha_j^n, \beta_t^{\mathbb{P}_n} = \beta_j^n$ for $t \in [\tau_j^n, \tau_{j+1}^n)$. Then

$$\delta_n \geq \left(\frac{c}{3} - C\eta \right) \mathbb{E}^{\mathbb{P}_n}[\mathbb{H}_{O_\epsilon}].$$

Let $\eta := \frac{c}{6C}$, and it follows $\underline{\mathcal{E}}^{L_0}[\mathbb{H}_{O_\epsilon}] \leq \mathbb{E}^{\mathbb{P}_n}[\mathbb{H}_{O_\epsilon}] \leq \delta_n$. By putting $n \rightarrow \infty$, we get $\underline{\mathcal{E}}^{L_0}[\mathbb{H}_{O_\epsilon}] = 0$, contradiction. \square

5 Path-dependent time-invariant stochastic control

In this section, we present an application of fully nonlinear elliptic PPDE. An important question which is most relevant since the recent financial crisis is the risk of model misspecification. The uncertain volatility model (see Avellaneda, Levy and Paras [1], Lyons [83] or

Nutz [87]) provides a conservative answer to this problem.

In the present application, the canonical process B represents the price process of some primitive asset, and our objective is the hedging of the derivative security defined by the payoff $\xi(B)$ at some maturity H_Q defined as the exiting time from some domain Q .

In contrast with the standard Black-Scholes modeling, we assume that the probability space (Ω, \mathcal{F}) is endowed with a family of probability measures \mathcal{P}^{UVM} . In the uncertain volatility model, the quadratic variation of the canonical process is assumed to lie between two given bounds,

$$\underline{\sigma}^2 dt \leq d\langle B \rangle_t \leq \bar{\sigma}^2 dt, \quad \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}^{\text{UVM}}.$$

Then, by the possible frictionless trading of the underlying asset, it is well known that the non-arbitrage condition is characterized by the existence of an equivalent martingale measure. Consequently, we take

$$\mathcal{P}^{\text{UVM}} := \{ \mathbb{P} \in \mathcal{P}^\infty : B \text{ is } \mathbb{P}\text{-martingale and } \frac{d\langle B \rangle_t}{dt} \in [\underline{\sigma}^2, \bar{\sigma}^2], \mathbb{P}\text{-a.s.} \}.$$

The superhedging problem under model uncertainty was initially formulated by Denis&Martini [29] and Neufeld&Nutz [86], and involves delicate quasi-sure analysis. Their main result expresses the cost of robust superhedging as

$$u_0 := \bar{\mathcal{E}}^{\text{UVM}} \left[e^{-rH_Q} \xi(B_{H_Q \wedge \cdot}) \right] := \sup_{\mathbb{P} \in \mathcal{P}^{\text{UVM}}} \mathbb{E}^{\mathbb{P}} \left[e^{-rH_Q} \xi(B_{H_Q \wedge \cdot}) \right],$$

where r is the discount rate. Further, define u on Ω^e as :

$$u(\omega) := \bar{\mathcal{E}}^{\text{UVM}} \left[e^{-rH_Q^\omega} \xi((B_{H_Q \wedge \cdot})^\omega) \right], \quad \text{for all } \omega \in \Omega^e. \quad (\text{VI.5.1})$$

We are interested in characterizing u as the viscosity solution of the corresponding fully nonlinear elliptic PPDE.

Assumption VI.5.1. *Assume that*

$$\xi \in \text{BUC}(\Omega^e), \quad \underline{\sigma} > 0, \quad \text{and the discount rate } r > 0.$$

Proposition VI.5.2. *Under Assumption VI.5.1, the function u defined in (VI.5.1) is BUC and is a viscosity solution to the elliptic path-dependent HJB equation :*

$$ru - L|\partial_\omega u| - \sup_{\beta \in [\underline{\sigma}, \bar{\sigma}]} \frac{1}{2} \beta^2 \partial_{\omega\omega}^2 u = 0 \text{ on } \Omega, \quad \text{and } u = \xi \text{ on } \partial\Omega$$

In preparation to the proof of the proposition, we first show two lemmas.

Lemma VI.5.3. *The function u defined in (VI.5.1) is BUC.*

Proof By (VI.3.3) and the fact that $\xi \in \text{BUC}(\partial\Omega)$, one may easily obtain the desired result. \square

Lemma VI.5.4. *Let $\xi \in \text{BUC}(\partial\Omega)$. There exists a probability measure $\mathbb{P}^* \in \mathcal{P}^{\text{UVM}}$ such that $e^{-r \cdot} u(B_{t \wedge \cdot})$ is a \mathbb{P}^* -martingale.*

Proof Denote $t_i^n := \frac{i}{2^n}$ for $i \leq n2^n$ and $n \in \mathbb{N}$. Define process $\tilde{u}_t := e^{-rt} u_t$. Note that the tower property implies that $\tilde{u}_0 = \bar{\mathcal{E}}^{\text{UVM}}[\tilde{u}_{t_1^n}]$. Since \mathcal{P}^{UVM} is weakly compact and u is BUC, there exists a probability measure \mathbb{P}_0^n such that

$$\tilde{u}_0 = \bar{\mathcal{E}}^{\text{UVM}}[\tilde{u}_{t_1^n}] = \mathbb{E}^{\mathbb{P}_0^n}[\tilde{u}_{t_1^n}].$$

Since the space Ω is separable, we may find a countable $\mathbb{F}_{t_1^n}$ -measurable partition $\{E_i\}_{i \in \mathbb{N}}$ such that $\|\omega - \omega'\|_{t_1^n} < \epsilon$ for all $\omega, \omega' \in E_i$. Fix an $\omega^i \in E_i$. As before, there exists probability measures $\mathbb{P}_{1,i}^{n,\epsilon}$ such that

$$\tilde{u}_{t_1^n}(\omega^i) = \mathbb{E}^{\mathbb{P}_{1,i}^{n,\epsilon}}[(\tilde{u}_{t_2^n})^{t_1^n, \omega^i}].$$

For $\omega \in E_i$, we have $|u_{t_1^n}(\omega) - u_{t_1^n}(\omega^i)| \leq \rho(\epsilon)$ and $\|(u_{t_2^n})^{t_1^n, \omega} - (u_{t_2^n})^{t_1^n, \omega^i}\| \leq \rho(\epsilon)$, where ρ is the modulus of continuity of u . Thus, we obtain that

$$\tilde{u}_{t_1^n}(\omega) \leq \mathbb{E}^{\mathbb{P}_{1,i}^{n,\epsilon}}[(\tilde{u}_{t_2^n})^{t_1^n, \omega}] + 2\rho(\epsilon). \quad (\text{VI.5.2})$$

Define $\mathbb{P}_1^{n,\epsilon}(A) := \mathbb{E}^{\mathbb{P}_0^n}[\sum_i \mathbb{P}_{1,i}^{n,\epsilon}(A^{t_1^n, B}) 1_{\{B_{t_1^n \wedge \cdot} \in E_i\}}]$. Clearly, still $\mathbb{P}_1^{n,\epsilon} \in \mathcal{P}^{\text{UVM}}$. Note that (VI.5.2) implies that $\tilde{u}_{t_1^n} \leq \mathbb{E}^{\mathbb{P}_1^{n,\epsilon}}[\tilde{u}_{t_2^n} | \mathcal{F}_{t_1^n}] + 2\rho(\epsilon)$, \mathbb{P}_1^n -a.s. Again, since \mathcal{P}^{UVM} is weakly compact and u is BUC, there exists $\mathbb{P}_1^n \in \mathcal{P}^{\text{UVM}}$ such that $\tilde{u}_{t_1^n} \leq \mathbb{E}^{\mathbb{P}_1^n}[\tilde{u}_{t_2^n} | \mathcal{F}_{t_1^n}]$, \mathbb{P}_1^n -a.s.. On the other hand, by Theorem 2.3 in [88], it holds that $\tilde{u}_{t_1^n} \geq \mathbb{E}^{\mathbb{P}_1^n}[\tilde{u}_{t_2^n} | \mathcal{F}_{t_1^n}]$, \mathbb{P}_1^n -a.s. It follows that

$$\tilde{u}_{t_1^n} = \mathbb{E}^{\mathbb{P}_1^n}[\tilde{u}_{t_2^n} | \mathcal{F}_{t_1^n}], \quad \mathbb{P}_1^n\text{-a.s.}$$

Note that by the definition of $\mathbb{P}_1^{n,\epsilon}$, we know that $\mathbb{P}_1^n = \mathbb{P}_0^n$ on $\mathcal{F}_{t_1^n}$. So, it also holds that $\tilde{u}_0 = \mathbb{E}^{\mathbb{P}_0^n}[\tilde{u}_{t_1^n}]$. Repeating the construction, we may find a sequence of probability measures $\mathbb{P}_0^n, \dots, \mathbb{P}_{n2^n}^n$. Denote $\mathbb{P}^n := \mathbb{P}_{n2^n}^n$. It holds that for all $m \leq n$

$$\tilde{u}_{t_i^m} = \mathbb{E}^{\mathbb{P}^n}[\tilde{u}_{t_j^m} | \mathcal{F}_{t_i^m}], \quad \mathbb{P}^n\text{-a.s. for } i \leq j \leq m2^m.$$

Finally, since \mathcal{P}^{UVM} is weakly compact and u is BUC, there exists $\mathbb{P}^* \in \mathcal{P}^{\text{UVM}}$ such that \tilde{u} is a \mathbb{P}^* -martingale. \square

Proof of Proposition VI.5.2 *Step 1.* We first verify the viscosity supersolution property. Take $L > 0$ such that $L \geq \bar{\sigma}$ and $\frac{1}{L} \leq \underline{\sigma}$. Without loss of generality, we only verify it at the point 0. Let $\varphi \in \bar{\mathcal{A}}^L u(0)$ i.e. $(\varphi - u)_0 = \max_{\tau} \bar{\mathcal{E}}^L [(\varphi - u)_{\text{HO}_\epsilon \wedge \tau}]$. Then, for all $\mathbb{P} \in \mathcal{P}^L$, we obtain that for all $h > 0$

$$\begin{aligned} 0 &\geq \mathbb{E}^{\mathbb{P}} \left[\varphi_{\text{HO}_\epsilon \wedge h} - \varphi_0 - u_{\text{HO}_\epsilon \wedge h} + u_0 \right] \\ &\geq \mathbb{E}^{\mathbb{P}} \left[\int_0^{\text{HO}_\epsilon \wedge h} \left(\frac{1}{2} \partial_{\omega\omega}^2 \varphi_s d\langle B \rangle_s + \partial_\omega \varphi_s dB_s \right) \right] \\ &\quad + \mathbb{E}^{\mathbb{P}} \left[(e^{-r(\text{HO}_\epsilon \wedge h)} - 1) u_{\text{HO}_\epsilon \wedge h} \right] - \mathbb{E}^{\mathbb{P}} \left[e^{-r(\text{HO}_\epsilon \wedge h)} u_{\text{HO}_\epsilon \wedge h} \right] + u_0 \\ &\geq \mathbb{E}^{\mathbb{P}} \left[\int_0^{\text{HO}_\epsilon \wedge h} \left(\frac{1}{2} \partial_{\omega\omega}^2 \varphi_s d\langle B \rangle_s + \partial_\omega \varphi_s dB_s \right) \right] + \mathbb{E}^{\mathbb{P}} \left[(e^{-r(\text{HO}_\epsilon \wedge h)} - 1) u_{\text{HO}_\epsilon \wedge h} \right]. \end{aligned}$$

Now, we take $\mathbb{P}_{\lambda, \beta} \in \mathcal{P}^{\text{UVM}}$ such that there exists a $\mathbb{P}_{\lambda, \beta}$ -Brownian motion W such that $B_t = \lambda t + \beta W_t$, $\mathbb{P}_{\lambda, \beta}$ -a.s. It follows that

$$0 \geq \frac{1}{h} \mathbb{E}^{\mathbb{P}_{\lambda, \beta}} \left[\int_0^{\text{HO}_\epsilon \wedge h} \left(\frac{1}{2} \beta^2 \partial_{\omega\omega}^2 \varphi_s + \lambda \partial_\omega \varphi_s \right) ds + \left(e^{-r(\text{HO}_\epsilon \wedge h)} - 1 \right) u_{\text{HO}_\epsilon \wedge h} \right].$$

Let $h \rightarrow 0$, we obtain that $0 \geq -ru_0 + \frac{1}{2} \beta^2 \partial_{\omega\omega}^2 \varphi_0 + \lambda \partial_\omega \varphi_0$. Since $\lambda \in [-L, L]$, $\beta \in [\underline{\sigma}, \bar{\sigma}]$ can be arbitrary, we finally have

$$ru_0 - L |\partial_\omega \varphi_0| - \sup_{\beta \in [\underline{\sigma}, \bar{\sigma}]} \frac{1}{2} \beta^2 \partial_{\omega\omega}^2 \varphi_0 \geq 0.$$

Step 2. Now we verify the viscosity subsolution property. Without loss of generality, we only verify it at the point 0. Let $\varphi \in \underline{\mathcal{A}}^L$, i.e. $(\varphi - u)_0 = \min_{\tau} \underline{\mathcal{E}}^L [(\varphi - u)_{\text{HO}_\epsilon \wedge \tau}]$. Take the probability measure $\mathbb{P}^* \in \mathcal{P}^{\text{UVM}}$ in Lemma VI.5.4, so that u is a \mathbb{P}^* -martingale. Then it holds that for all

$h > 0$

$$\begin{aligned}
 0 &\leq \mathbb{E}^{\mathbb{P}^*} [\varphi_{\mathbb{H}_{O_\epsilon \wedge h}} - \varphi_0 - u_{\mathbb{H}_{O_\epsilon \wedge h}} + u_0] \\
 &\leq \mathbb{E}^{\mathbb{P}^*} \left[\int_0^{\mathbb{H}_{O_\epsilon \wedge h}} \left(\frac{1}{2} \partial_{\omega\omega}^2 \varphi_s d\langle B \rangle_s + \partial_\omega \varphi_s dB_s \right) \right] \\
 &\quad + \mathbb{E}^{\mathbb{P}^*} \left[(e^{-r(\mathbb{H}_{O_\epsilon \wedge h})} - 1) u_{\mathbb{H}_{O_\epsilon \wedge h}} \right] - \mathbb{E}^{\mathbb{P}^*} \left[e^{-r(\mathbb{H}_{O_\epsilon \wedge h})} u_{\mathbb{H}_{O_\epsilon \wedge h}} \right] + u_0 \\
 &\leq \mathbb{E}^{\mathbb{P}^*} \left[\int_0^{\mathbb{H}_{O_\epsilon \wedge h}} \left(L |\partial_\omega \varphi_s| + \sup_{\beta \in [\underline{\sigma}, \bar{\sigma}]} \frac{1}{2} \beta^2 \partial_{\omega\omega}^2 \varphi_s \right) ds + (e^{-r(\mathbb{H}_{O_\epsilon \wedge h})} - 1) u_{\mathbb{H}_{O_\epsilon \wedge h}} \right].
 \end{aligned}$$

Divide the right side by h , and let $h \rightarrow 0$. Finally, we get

$$ru_0 - L |\partial_\omega \varphi_0| - \sup_{\beta \in [\underline{\sigma}, \bar{\sigma}]} \frac{1}{2} \beta^2 \partial_{\omega\omega}^2 \varphi_0 \leq 0.$$

□

6 Appendix

We complete some proofs in this section.

Proof of Proposition VI.1.8 The first result is easy, and we omit its proof. We decompose the proof in two steps.

Step 1. We first prove that $\bar{\mathcal{E}}^L[\mathbb{H}_D] < \infty$. Without loss of generality, we may assume that $D = O_r$. Further, since

$$\mathbb{H}_{O_r} \leq \mathbb{H}_r^1 := \inf\{t \geq 0 : |B_t^1| \geq r\},$$

without loss of generality, we may assume that the dimension $d = 1$.

We first consider the following Dirichlet problem of ODE :

$$-L |\partial_x u| - \frac{1}{L} \partial_{xx}^2 u - 1 = 0, \quad u(r) = u(-r) = 0. \quad (\text{VI.6.1})$$

It is easy to verify that Equation (VI.6.1) has a classical solution :

$$u(x) = \frac{1}{L^3} \left(e^{L^2 r} - e^{L^2 x} \right) - \frac{1}{L} (R - x) \text{ for } 0 \leq x \leq r, \quad \text{and } u(x) = u(-x) \text{ for } -r \leq x \leq 0.$$

Further, it is clear that u is concave, so u is also a classical solution to the equation :

$$-L|\partial_x u| - \frac{1}{2} \sup_{\frac{2}{L} \leq \beta \leq 2L} \beta \partial_{xx}^2 u - 1 = 0, \quad u(r) = u(-r) = 0. \quad (\text{VI.6.2})$$

Then by Itô's formula, we obtain

$$0 = u(B_{H_{O_r}}) = u_0 + \int_0^{H_{O_r}} \partial_x u(B_t) dB_t + \frac{1}{2} \int_0^{H_{O_r}} \partial_{xx}^2 u(B_t) d\langle B \rangle_t.$$

Taking the nonlinear expectation on both sides and recalling the definition of $\mathbb{Q}^{\alpha, \beta}$ in (VI.1.3), we have

$$0 = u_0 + \mathbb{E}^{\mathbb{Q}^{\alpha, \beta}} \left[\int_0^{H_{O_r}} \left(\alpha_t \partial_x u(B_t) + \frac{1}{2} \beta_t^2 \partial_{xx}^2 u(B_t) \right) dt \right] \quad \text{for all } \|\alpha\| \leq L, \frac{2}{L} \leq \beta \leq 2L. \quad (\text{VI.6.3})$$

Since u is a solution of Equation (VI.6.2), we have

$$\mathbb{E}^{\mathbb{Q}^{\alpha, \beta}} \left[\int_0^{H_{O_r}} \left(\alpha_t \partial_x u(B_t) + \frac{1}{2} \beta_t^2 \partial_{xx}^2 u(B_t) \right) dt \right] \leq -\mathbb{E}^{\mathbb{Q}^{\alpha, \beta}} [H_{O_r}]$$

Hence $u_0 \geq \bar{\mathcal{E}}^L [H_{O_r}]$. On the other hand, taking $\alpha^* := L \text{sgn}(\partial_x u(B_t))$ and $\beta^* := \sqrt{\frac{2}{L}}$, we obtain from (VI.6.2) and (VI.6.3) that

$$u_0 = \mathbb{E}^{\mathbb{Q}^{\alpha^*, \beta^*}} [H_{O_r}].$$

So, we have proved that $u_0 = \bar{\mathcal{E}}^L [H_{O_r}]$. Consequently, $\bar{\mathcal{E}}^L [H_{O_r}] < \infty$.

Step 2. Note that

$$\mathcal{C}^L [H_D \geq T] \leq \frac{\bar{\mathcal{E}}^L [H_D]}{T}.$$

By the result of Step 1, we have $\mathcal{C}^L [H_D \geq T] \leq \frac{C}{T}$, and then $\lim_{T \rightarrow \infty} \mathcal{C}^L [H_D \geq T] = 0$. Further,

$$\begin{aligned} \mathcal{C}^L [H_n < H_D] &\leq \mathcal{C}^L [H_n < H_D; H_D \leq T] + \mathcal{C}^L [H_n < H_D; H_D > T] \\ &\leq \mathcal{C}^L [H_n < T] + \mathcal{C}^L [H_D > T]. \end{aligned}$$

We conclude that $\lim_{n \rightarrow \infty} \mathcal{C}^L [H_n < H_D] = 0$

Denote $D^x := \{y : \exists z \in D, y = z - x\}$. Then, define $\hat{D} := \cup_{x \in D} D^x$. Note that

$$H_D^x \leq H_{\hat{D}} \text{ for all } x \in D.$$

Hence we have

$$\sup_{x \in D} \mathcal{C}^L [\mathbb{H}_D^x \geq T] \leq \mathcal{C}^L [\mathbb{H}_{\hat{D}} \geq T] \rightarrow 0.$$

□

Proof of Lemma VI.3.8 Denote $\delta h := h^1 - h^2$. By standard argument, one can easily verify that function $w(x) := \bar{\mathcal{E}}^{L_0} \left[\left((\delta h)_{\mathbb{H}_D}^+ \right)^x + \int_0^{\mathbb{H}_D^x} c_0 dt \right]$ is a viscosity solution of the nonlinear PDE :

$$-c_0 - L_0 |Dw| - \frac{1}{2} \sup_{\sqrt{\frac{2}{L_0}} I_d \leq \sigma \leq \sqrt{2L_0} I_d} \sigma^2 : D^2 w = 0 \text{ on } D, \quad \text{and } w = (\delta h)^+ \text{ on } \partial D.$$

Let K be a smooth nonnegative kernel with unit total mass. For all $\eta > 0$, we define the mollification $w^\eta := w * K^\eta$ of w . Then w^η is smooth, and it follows from a convexity argument as in [75] that w^η is a classic supersolution of

$$-c_0 - L_0 |Dw^\eta| - \frac{1}{2} \sup_{\sqrt{\frac{2}{L_0}} I_d \leq \sigma \leq \sqrt{2L_0} I_d} \sigma^2 : D^2 w^\eta \geq 0 \text{ on } D, \quad \text{and } w^\eta = (\delta h)^+ * K^\eta \text{ on } \partial D. \quad (\text{VI.6.4})$$

We claim that

$$\bar{w}^\eta + v^2 \text{ is a supersolution to the PDE with generator } g^1,$$

where $\bar{w}^\eta := w^\eta + \|w^\eta - (\delta h)^+\|_{L^\infty}$. Then we note that $\bar{w}^\eta + v^2 \geq w^\eta + h^2 + \|w^\eta - (\delta h)^+\|_{L^\infty} \geq h^1 = v^1$ on ∂D . By comparison principle in PDEs, we have $\bar{w}^\eta + v^2 \geq v^1$ on $\text{cl}(D)$. Setting $\eta \rightarrow 0$, we obtain that $v^1 - v^2 \leq w$. The desired result follows.

It remains to prove that $\bar{w} + v^2$ is a supersolution of the PDE with generator g^1 . Let $x_0 \in D$, $\phi \in C^2(D)$ be such that $0 = (\phi - \bar{w} - v^2)(x_0) = \max(\phi - \bar{w} - v^2)$. Then, it follows from the viscosity supersolution property of v^2 that $\mathbf{L}^2(\phi - \bar{w})(x_0) \geq 0$. Hence, at the point x_0 , by

(VI.6.4) we have

$$\begin{aligned}
 \mathbf{L}^1\phi &\geq \mathbf{L}^1\phi - \mathbf{L}^2(\phi - \bar{w}^\eta) \\
 &= -g^1(\phi, D\phi, D^2\phi) + g^2(\phi - \bar{w}^\eta, D(\phi - \bar{w}^\eta), D^2(\phi - \bar{w}^\eta)) \\
 &\geq -g^1(\phi, D\phi, D^2\phi) + g^2(\phi, D(\phi - \bar{w}^\eta), D^2(\phi - \bar{w}^\eta)) \\
 &\geq c_0 + L_0|Dw^\eta| + \frac{1}{2} \sup_{\sqrt{\frac{2}{L_0}}I_d \leq \sigma \leq \sqrt{2L_0}I_d} \sigma^2 : D^2w^\eta - c_0 - \alpha \cdot Dw^\eta - \frac{1}{2}\gamma : D^2w^\eta \\
 &\geq 0,
 \end{aligned}$$

where $|\alpha| \leq L_0$ and $\frac{2}{L_0}I_d \leq \gamma \leq 2L_0I_d$. □

Proposition VI.6.1. *For all $n \geq 1$, there exists a modulus of continuity ρ such that*

$$\bar{\mathcal{E}}^L \left[|\mathbb{H}_Q^{\omega^1} - \mathbb{H}_Q^{\omega^2}| \right] \leq \rho \left(d^e(\omega^1, \omega^2) \right). \quad (\text{VI.6.5})$$

Proof By the tower property, we have

$$\begin{aligned}
 \bar{\mathcal{E}}^L \left[|\mathbb{H}_Q^{\omega^1} - \mathbb{H}_Q^{\omega^2}| \right] &\leq \bar{\mathcal{E}}^L \left[|\mathbb{H}_Q^{\omega^1} - \mathbb{H}_Q^{\omega^2}| 1_{\{\mathbb{H}_Q^{\omega^1} \leq \mathbb{H}_Q^{\omega^2}\}} \right] + \bar{\mathcal{E}}^L \left[|\mathbb{H}_Q^{\omega^1} - \mathbb{H}_Q^{\omega^2}| 1_{\{\mathbb{H}_Q^{\omega^1} > \mathbb{H}_Q^{\omega^2}\}} \right] \\
 &\leq \bar{\mathcal{E}}^L \left[\bar{\mathcal{E}}^L \left[\mathbb{H}_Q^{\omega^2 \otimes B_{\mathbb{H}_Q^{\omega^1} \wedge}} \right] 1_{\{\mathbb{H}_Q^{\omega^1} \leq \mathbb{H}_Q^{\omega^2}\}} \right] + \bar{\mathcal{E}}^L \left[\bar{\mathcal{E}}^L \left[\mathbb{H}_Q^{\omega^1 \otimes B_{\mathbb{H}_Q^{\omega^2} \wedge}} \right] 1_{\{\mathbb{H}_Q^{\omega^1} > \mathbb{H}_Q^{\omega^2}\}} \right].
 \end{aligned}$$

So, it suffices to show that there exists a modulus of continuity ρ such that

$$\bar{\mathcal{E}}^L \left[\mathbb{H}_Q^{\omega^2 \otimes \omega'_{\mathbb{H}_Q^{\omega^1} \wedge}} \right] \leq \rho \left(d^e(\omega^1, \omega^2) \right), \text{ for all } \omega' \text{ such that } \mathbb{H}_Q^{\omega^1}(\omega') \leq \mathbb{H}_Q^{\omega^2}(\omega').$$

Further, without loss of generality, we may assume that the dimension $d = 1$ and $Q = (a, a+h)$.

Denote $x_i := \omega_{t_i}^i$ and $y_i := x_i + \omega'_{\mathbb{H}_Q^{\omega^1}}$ for $i = 1, 2$. Note that

$$|y_1 - y_2| = |x_1 - x_2|, \quad y_1 \in \partial Q, \quad y_2 \in Q, \quad \text{and} \quad \mathbb{H}_Q^{\omega^2 \otimes \omega'_{\mathbb{H}_Q^{\omega^1} \wedge}} = \mathbb{H}_Q^{y_2}.$$

In particular, $d(y_2, \partial Q) = |x_1 - x_2|$. We next consider the Dirichlet problem of ODE :

$$-L|\partial_x u| - \frac{1}{2} \sup_{\frac{2}{L} \leq \beta \leq 2L} \beta \partial_{xx}^2 u - 1 = 0 \quad \text{and} \quad u\left(-\frac{h}{2}\right) = u\left(\frac{h}{2}\right) = 0 \quad (\text{VI.6.6})$$

Then, as in the proof of Proposition VI.1.8 in Section 6, we can prove that Equation (VI.6.6) has a classical solution and

$$\bar{\mathcal{E}}^L[\mathbb{H}_{Q^{y_2}}] = u\left(\frac{h}{2} - |x_1 - x_2|\right) \leq \rho(|x_1 - x_2|),$$

where ρ is the modulus of continuity of u . The proof is complete.

Chapitre VII

Large deviation for non-Markovian diffusion

1 Problem formulation and main results

Let $\Omega_d := \{\omega \in C^0([0, T], \mathbb{R}^d) : \omega_0 = 0\}$ be the canonical space of continuous paths starting from the origin, B the canonical process defined by $B_t(\omega) := \omega_t$, $t \in [0, 1]$, and $\mathbb{F} := \{\mathcal{F}_t, t \in [0, T]\}$ the corresponding filtration. We shall use the following notation for the supremum norm :

$$\|\omega\|_t := \sup_{s \in [0, t]} |\omega_s| \quad \text{and} \quad \|\omega\| := \|\omega\|_T \quad \text{for all } t \in [0, T], \omega \in \Omega_d.$$

Let \mathbb{P}_0 be the Wiener measure on Ω_d . For all $\varepsilon \geq 0$, we denote by $\mathbb{P}^\varepsilon := \mathbb{P}_0 \circ (\sqrt{\varepsilon}B)^{-1}$ the probability measure such that

$$\left\{ W_t^\varepsilon := \frac{1}{\sqrt{\varepsilon}} B_t, 0 \leq t \leq T \right\} \quad \text{is a } \mathbb{P}^\varepsilon \text{ - Brownian motion.}$$

Our main interest in this paper is on the solution of the path-dependent stochastic differential equation :

$$dX_t = b_t(B, X)dt + \sigma_t(B, X)dB_t, \quad X_0 = x_0, \quad \mathbb{P}^\varepsilon\text{-a.s.} \quad (\text{VII.1.1})$$

where the process X takes values in \mathbb{R}^n for some integer $n \geq 1$, and its paths are in $\Omega_n := C^0([0, T], \mathbb{R}^n)$.

The supremum norm on Ω_n is also denoted $\|\cdot\|_t$, without reference to the dimension of the underlying space. The coefficients $b : [0, T] \times \Omega_d \times \Omega_n \longrightarrow \mathbb{R}^n$ and $\sigma : [0, T] \times \Omega_d \times \Omega_n \longrightarrow \mathbb{R}^{n \times d}$ are assumed to satisfy the following conditions which guarantee existence and uniqueness of a strong solution for all $\varepsilon > 0$.

Assumption VII.1.1. *The coefficients $f \in \{b, \sigma\}$ are :*

- *non-anticipative, i.e. $f_t(\omega, x) = f_t((\omega_s)_{s \leq t}, (x_s)_{s \leq t})$,*

- L -Lipschitz-continuous in (ω, x) , uniformly in t , for some $L > 0$:

$$\left| f_t(\omega, x) - f_t(\omega', x') \right| \leq L(\|\omega - \omega'\|_t + \|x - x'\|_t); \quad t \in [0, T], (\omega, x), (\omega', x') \in \Omega_d \times \Omega_n,$$

Under \mathbb{P}^ε , the stochastic differential equation (VII.1.1) is driven by a small noise, and our objective is to provide some large deviation asymptotics in the present path-dependent case, which extend the corresponding results of Freidlin & Wentzell [55] in the Markovian case. We shall adapt to our path-dependent case the PDE approach to large deviations of stochastic differential equation as initiated by Fleming [51] and Evans & Ishii [45], see also Fleming & Soner [52], Chapter VII.

1.1 Laplace transform near infinity

As a first example, we consider the Laplace transform of some path-dependent random variable $\xi((\omega_s)_{s \leq T}, (x_s)_{s \leq T})$ for some final horizon $T > 0$:

$$L_0^\varepsilon := -\varepsilon \ln \mathbb{E}^{\mathbb{P}^\varepsilon} \left[e^{-\frac{1}{\varepsilon} \xi(B, X)} \right]. \quad (\text{VII.1.2})$$

In the following statement \mathbb{L}_d^2 denotes the collection of measurable functions $\alpha : [0, T] \rightarrow \mathbb{R}^d$ such that $\int_0^T |\alpha_t|^2 dt < \infty$. Our first main result is :

Theorem VII.1.2. *Let ξ be a bounded uniformly continuous \mathcal{F}_T -measurable r.v. Then, under Assumption VII.1.1, we have :*

$$L_0^\varepsilon \longrightarrow L_0 := \inf_{\alpha \in \mathbb{L}_d^2} \ell_0^\alpha \quad \text{as } \varepsilon \rightarrow 0, \quad \text{where } \ell_0^\alpha := \xi(\omega^\alpha, x^\alpha) + \frac{1}{2} \int_0^T |\alpha_t|^2 dt,$$

and $(\omega^\alpha, x^\alpha)$ are defined by the controlled ordinary differential equations :

$$\omega_t^\alpha = \int_0^t \alpha_s ds, \quad x_t^\alpha = X_0 + \int_0^t b_s(\omega^\alpha, x^\alpha) ds + \int_0^t \sigma_s(\omega^\alpha, x^\alpha) d\omega_s^\alpha, \quad t \in [0, T].$$

The proof of this result is reported in Section 3.

Remark VII.1.3. Theorem VII.1.2 is still valid in the context where the coefficient b depends also on the parameter ε , so that the process X is replaced by X^ε defined by :

$$dX_t^\varepsilon = b_t^\varepsilon(B, X^\varepsilon) dt + \sigma_t(B, X^\varepsilon) dB_t, \quad X_0^\varepsilon = x_0, \quad \mathbb{P}^\varepsilon\text{-a.s.}$$

Since this extension will be needed for our application in Section 2, we provide a precise formulation. Let Assumption VII.1.1 hold uniformly in $\varepsilon \in [0, 1)$, and assume further that $\varepsilon \mapsto b^\varepsilon$ is uniformly Lipschitz on $[0, 1)$. Then the statement of Theorem VII.1.2 holds with x^α defined by :

$$x_t^\alpha = X_0 + \int_0^t b_s^0(\omega^\alpha, x^\alpha) ds + \int_0^t \sigma_s(\omega^\alpha, x^\alpha) d\omega_s^\alpha, \quad t \in [0, T].$$

This slight extension does not induce any additional technical difficulty in the proof. We shall therefore provide the proof in the context of Theorem VII.1.2.

1.2 Exiting from a given domain before some maturity

As a second example, we consider the asymptotic behavior of the probability of exiting from some given subset of \mathbb{R}^n before the maturity T :

$$Q_0^\varepsilon := -\varepsilon \ln \mathbb{P}^\varepsilon[H < T], \quad \text{where } H := \inf\{t > 0 : X_t \notin O\}, \quad (\text{VII.1.3})$$

and O is a bounded open set in \mathbb{R}^n . We also introduce the corresponding subset of paths in Ω_n :

$$\mathcal{O} := \left\{ \omega \in \Omega_n : \omega_t \in O \text{ for all } t \leq T \right\}. \quad (\text{VII.1.4})$$

The analysis of this problem requires additional conditions.

Assumption VII.1.4. *The coefficients b and σ are uniformly bounded, and σ is uniformly elliptic, i.e. $a := \sigma\sigma^T$ is invertible with bounded inverse a^{-1} .*

The present example exhibits a singularity on the boundary ∂O because Q_0^ε vanishes whenever the path ω is started on the boundary ∂O . Our second main result is the following.

Theorem VII.1.5. *Let O be a bounded open set in \mathbb{R}^n with C^3 boundary. Then, under Assumptions VII.1.1 and VII.1.4, we have :*

$$Q_0^\varepsilon \longrightarrow Q_0 := \inf \left\{ q_0^\alpha : \alpha \in \mathbb{L}_d^2, x_{T \wedge \cdot}^\alpha \notin \mathcal{O} \right\}, \quad \text{where } q_0^\alpha := \frac{1}{2} \int_0^T |\alpha_s|^2 ds,$$

and x^α is defined as in Theorem VII.1.2.

The proof of this result is reported in Section 4.

Remark VII.1.6. (i) A similar result of Theorem VII.1.5 can be found in Gao-Liu [56]. However, our proof has a completely different flavor, and follows the lines of the simpler and more direct PDE argument.

(ii) The condition on the boundary ∂O can be slightly weakened. Examining the proof of Lemma VII.4.2, where this condition is used, we see that it is sufficient to assume that O can be approximated from outside by open bounded sets with C^3 boundary.

Remark VII.1.7. The result of Theorem VII.1.5 is still valid in the context of Remark VII.1.3. This can be immediately verified by examining the proof of Theorem VII.1.5.

1.3 Path-dependent Eikonal equation

We next provide a characterization of our asymptotics in terms of partial differential equations. We refer to Evans & Ishii [45], Fleming & Souganidis [53], Evans-Souganidis [46], Evans, Souganidis, Fournier & Willem [47], Fleming & Soner [52], for the corresponding PDE literature with a derivation by means of the powerful theory of viscosity solutions.

Due to the path dependence in the dynamics of our state process X , and the corresponding limiting system x^α , our framework is clearly not covered by any of these existing works. Therefore, we shall adapt the notion of viscosity solutions introduced in Lukoyanov [82].

Denote $\hat{\Omega} := \Omega_d \times \Omega_n$ and $\hat{\omega} = (\omega, x)$ a generic element of $\hat{\Omega}$, $\Theta := [0, T] \times \hat{\Omega}$, and $\Theta^0 := [0, T] \times \hat{\Omega}$. Consider the truncated Eikonal equation :

$$\left\{ -\partial_t u - F_{K_0}(\cdot, \partial_\omega u, \partial_x u) \right\}(t, \hat{\omega}) = 0 \quad \text{for } (t, \hat{\omega}) \in \Theta^0, \quad (\text{VII.1.5})$$

where K_0 is a fixed parameter, and the nonlinearity F_{K_0} is given by :

$$F_{K_0}(t, \hat{\omega}, p_\omega, p_x) := b_t(\hat{\omega}) \cdot p_x + \inf_{\alpha \in \mathbb{R}^d, |\alpha| \leq K_0} \left\{ \frac{1}{2} |\alpha|^2 + a \cdot (p_\omega + \sigma_t(\hat{\omega})^\top p_x) \right\}, \quad (\text{VII.1.6})$$

for all $(t, \hat{\omega}) \in \Theta$, $p_\omega \in \mathbb{R}^d$ and $p_x \in \mathbb{R}^n$. Notice that

$$F_{K_0}(t, \hat{\omega}, p_\omega, p_x) \longrightarrow b_t(\hat{\omega}) \cdot p_x - \frac{1}{2} |p_\omega + \sigma_t(\hat{\omega})^\top p_x|^2 \quad \text{as } K_0 \rightarrow \infty,$$

the equation (VII.1.5) thus leads to a path-dependent Eikonal equation. We note that the truncated feature of the equation (VII.1.5) is induced by the fact that the corresponding solution will be shown to be Lipschitz under our assumptions.

1.3.1 Classical derivatives

The set Θ is endowed with the pseudo-distance

$$d(\theta, \theta') := |t - t'| + \|\hat{\omega}_{t\wedge} - \hat{\omega}'_{t'\wedge}\| \quad \text{for all } \theta = (t, \hat{\omega}), \theta' = (t', \hat{\omega}') \in \Theta.$$

For any integer $k > 0$, we denote by $C^0(\Theta, \mathbb{R}^k)$ the collection of all continuous functions $u : \Theta \rightarrow \mathbb{R}^k$. Notice, in particular, that any $u \in C^0(\Theta, \mathbb{R}^k)$ is non-anticipative, i.e. $u(t, \hat{\omega}) = u(t, (\hat{\omega}_s)_{s \leq t})$ for all $(t, \hat{\omega}) \in \Theta$.

We denote $\hat{\Omega}_K$ as the set of all K -Lipschitz paths. For $\theta = (t, \hat{\omega}) \in \Theta^0$, we denote $\Theta(\theta) := \cup_{K \geq 0} \Theta_K(\theta)$, where :

$$\Theta_K(\theta) := \left\{ (t', \hat{\omega}') \in \Theta : t' \geq t, \hat{\omega}'_{t\wedge} = \hat{\omega}_{t\wedge}, \text{ and } \hat{\omega}'|_{[t, T]} \text{ is } K\text{-Lipschitz} \right\}.$$

Definition VII.1.8. A function $\varphi : \Theta \rightarrow \mathbb{R}$ is said to be $C^{1,1}(\Theta)$ if $\varphi \in C^0(\Theta, \mathbb{R})$, and we may find $\partial_t \varphi \in C^0(\Theta, \mathbb{R})$, $\partial_{\hat{\omega}} \varphi \in C^0(\Theta, \mathbb{R}^{d+n})$, such that for all $\theta = (t, \hat{\omega}) \in \Theta$:

$$\varphi(\theta') = \varphi(\theta) + \partial_t \varphi(\theta)(t' - t) + \partial_{\hat{\omega}} \varphi(\theta)(\hat{\omega}'_{t'} - \hat{\omega}_t) + o_{\hat{\omega}'}(t' - t) \quad \text{for all } \theta' \in \Theta(\theta),$$

where $o_{\hat{\omega}'}(h)/h \rightarrow 0$ as $h \searrow 0$. The derivatives ∂_{ω} and ∂_x are defined by the natural decomposition $\partial_{\hat{\omega}} \varphi = (\partial_{\omega} \varphi, \partial_x \varphi)^T$.

The last collection of smooth functions will be used for our subsequent definition of viscosity solutions.

1.3.2 Viscosity solutions of the path-dependent Eikonal equation

Let $\Theta_K^0 := [0, T] \times \hat{\Omega}_K$. The set of test functions is defined for all $K > 0$ and $\theta \in \Theta_K^0$ by :

$$\underline{\mathcal{A}}^K u(\theta) := \left\{ \varphi \in C^{1,1}(\Theta) : (\varphi - u)(\theta) = \min_{\theta' \in \Theta_K} (\varphi - u)(\theta') \right\}, \quad (\text{VII.1.7})$$

$$\overline{\mathcal{A}}^K u(\theta) := \left\{ \varphi \in C^{1,1}(\Theta) : (\varphi - u)(\theta) = \max_{\theta' \in \Theta_K} (\varphi - u)(\theta') \right\}. \quad (\text{VII.1.8})$$

Definition VII.1.9. Let $u : \Theta \rightarrow \mathbb{R}$ be a continuous function.

(i) u is a K -viscosity subsolution of (VII.1.5), if for all $\theta \in \Theta_K^0$, we have

$$\left\{ -\partial_t \varphi - F_{K_0}(\cdot, \partial_{\hat{\omega}} \varphi) \right\}(\theta) \leq 0 \quad \text{for all } \varphi \in \underline{\mathcal{A}}^K u(\theta).$$

(ii) u is a K -viscosity supersolution of (VII.1.5), if for all $\theta \in \Theta_K^0$, we have

$$\left\{ -\partial_t \varphi - F_{K_0}(\cdot, \partial_{\hat{\omega}} \varphi) \right\}(\theta) \geq 0 \quad \text{for all } \varphi \in \overline{\mathcal{A}}^K u(\theta).$$

(iii) u is a K -viscosity solution of (VII.1.5) if it is both K -viscosity subsolution and supersolution.

1.3.3 Wellposedness of the path-dependent Eikonal equation

We only focus on the asymptotics of Laplace transform. For simplicity, we adopt the following strengthened version of Assumption VII.1.1.

Assumption VII.1.10. *The coefficients b and σ are both bounded and satisfy Assumption VII.1.1.*

A natural candidate solution of equation (VII.1.5), with the terminal condition $u = \xi$, is the dynamic version of the limit L^0 introduced in Theorem VII.1.2 :

$$u(t, \hat{\omega}) := \inf_{\alpha \in \mathbb{L}_d^2([t, T])} \left\{ \xi^{t, \hat{\omega}}(\hat{\omega}^{\alpha, t, \hat{\omega}}) + \frac{1}{2} \int_t^T |\alpha_s|^2 ds \right\}, \quad (t, \hat{\omega}) \in \Theta, \quad (\text{VII.1.9})$$

where $\hat{\omega}^{\alpha, t, \hat{\omega}} := (\omega^{\alpha, t, \hat{\omega}}, x^{\alpha, t, \hat{\omega}})$ is defined by :

$$\omega_s^{\alpha, t, \hat{\omega}} = \int_0^s \alpha_{t+r} dr, \quad x_s^{\alpha, t, \hat{\omega}} = \int_0^s b_{t+r}(\hat{\omega} \otimes_t \hat{\omega}^{\alpha, t, \hat{\omega}}) dr + \int_0^s \sigma_{t+r}(\hat{\omega} \otimes_t \hat{\omega}^{\alpha, t, \hat{\omega}}) d\omega_r^{\alpha, t, \hat{\omega}},$$

with the notation $(\hat{\omega} \otimes_t \hat{\omega}')_s := \mathbf{1}_{\{s \leq t\}} \hat{\omega}_s + \mathbf{1}_{\{s > t\}} (\hat{\omega}_t + \hat{\omega}'_{s-t})$, and

$$\xi^{t, \hat{\omega}}(\hat{\omega}') := \xi\left((\hat{\omega} \otimes_t \hat{\omega}')_{T \wedge \cdot}\right) \quad \text{for all } \hat{\omega}, \hat{\omega}' \in \hat{\Omega}.$$

Theorem VII.1.11. *Let Assumption VII.1.10 hold true, and let ξ be a bounded Lipschitz function on $\hat{\Omega}$. Then, for K_0 sufficiently large and $K \geq (\|b\|_\infty \vee \|\sigma\|_\infty)(1 + K_0)$, the function u defined in (VII.1.9) is the unique bounded K -viscosity solution of the path-dependent PDE (VII.1.5).*

The proof of this result is reported in Section 5.

2 Application to implied volatility asymptotics

2.1 Implied volatility surface

The Black-Scholes formula $\text{BS}(K, \sigma^2 T)$ expresses the price of a European call option with time to maturity T and strike K in the context of a geometric Brownian motion model for the underlying stock, with volatility parameter $\sigma \geq 0$:

$$\widehat{\text{BS}}(k, v) := \frac{\text{BS}(K, v)}{S_0} := \begin{cases} (1 - e^k)^+ & \text{for } v = 0, \\ \mathbf{N}(d_+(k, v)) - e^k \mathbf{N}(d_-(k, v)), & \text{for } v > 0, \end{cases}$$

where S_0 denotes the spot price of the underlying asset, $v := \sigma^2 T$ is the total variance, $k := \ln(K/S_0)$ is the log-moneyness of the call option, $\mathbf{N}(x) := (2\pi)^{-1/2} \int_{-\infty}^x e^{-y^2/2} dy$,

$$d_{\pm}(k, v) := \frac{-k}{\sqrt{v}} \pm \frac{\sqrt{v}}{2},$$

and the interest rate is reduced to zero.

We assume that the underlying asset price process is defined by the following dynamics under the risk-neutral measure \mathbb{P}_0 :

$$dS_t = S_t \sigma_t(B, S) dB_t, \quad \mathbb{P}_0 - \text{a.s.}$$

so that the price of the T -maturity European call option with strike K is given by $\mathbb{E}^{\mathbb{P}_0}[(S_T - K)^+]$. The implied volatility surface $(T, k) \mapsto \Sigma(T, k)$ is then defined as the unique non-negative solution of the equation

$$\mathbf{N}(d_+(k, \Sigma^2 T)) - e^k \mathbf{N}(d_-(k, \Sigma^2 T)) = \hat{C}(T, k) := \mathbb{E}^{\mathbb{P}_0}[(e^{X_T} - e^k)^+],$$

where $X_t := \ln(S_t/S_0)$, $t \geq 0$.

Our interest in this section is on the short maturity asymptotics $T \searrow 0$ of the implied volatility surface $\Sigma(T, k)$ for $k > 0$. This is a relevant practical problem which is widely used by derivatives traders, and has induced an extensive literature initiated by Berestycki, Busca & Florent [9, 10]. See e.g. Henry-Labordère [65], Hagan, Lesniewski, & Woodward [63], Ford and Jacquier [54], Gatheral, Hsu, Laurence, Ouyang & Wang [57], Deuschel, Friz, Jacquier & Violante [30, 31], and De Marco & Friz [27].

Our starting point is the following limiting result which follows from standard calculus :

$$\lim_{v \rightarrow 0} v \ln \widehat{\text{BS}}(k, v) = -\frac{k^2}{2}, \quad \text{for all } k > 0.$$

We also compute directly that, for $k > 0$, we have $\hat{C}(T, k) \rightarrow 0$ as $T \searrow 0$. Then $T\Sigma(T, k)^2 \rightarrow 0$ as $T \searrow 0$, and it follows from the previous limiting result that

$$\lim_{T \rightarrow 0} T\Sigma(T, k)^2 \ln \hat{C}(T, k) = -\frac{k^2}{2}, \quad \text{for all } k > 0. \quad (\text{VII.2.1})$$

Consequently, in order to study the asymptotic behavior of the implied volatility surface $\Sigma(T, k)$ for small maturity T , we are reduced to the asymptotics of $T \ln \hat{C}(T, k)$ for small T , which will be shown in the next subsection to be closely related to the large deviation problem of Subsection 1.2. Hence, our path-dependent large deviation results enable us to obtain the short maturity asymptotics of the implied volatility surface in the context where the underlying asset is a non-Markovian martingale under the risk-neutral measure.

2.2 Short maturity asymptotics

Recall the process $X_t := \ln(S_t/S_0)$. By Itô's formula, we deduce the dynamic for the process X :

$$dX_t = -\frac{1}{2}\sigma_t^X(B, X)^2 d\langle B \rangle_t + \sigma_t^X(B, X) dB_t, \quad (\text{VII.2.2})$$

where $\sigma^X(\omega, x) := \sigma(\omega, S_0 e^x)$. For the purpose of the application in this section, we need to convert the short maturity asymptotics into a small noise problem, so as to apply the main results from the previous section. In the present path-dependent case, this requires to impose a special structure on the coefficients of the stochastic differential equation (VII.2.2).

For a random variable Y and a probability measure \mathbb{P} , we denote by $\mathcal{L}^{\mathbb{P}}(Y)$ the \mathbb{P} -distribution of Y .

In this section, we shall adopt the simplest

Assumption VII.2.1. *The diffusion coefficient $\sigma^X : [0, T] \times \Omega_d \times \Omega_n \rightarrow \mathbb{R}$ is non-anticipative, Lipschitz-continuous, takes values in $[\underline{\sigma}, \bar{\sigma}]$ for some $\bar{\sigma} \geq \underline{\sigma} > 0$, and satisfies the following small-maturity small-noise correspondence :*

$$\mathcal{L}^{\mathbb{P}_0}(X_\varepsilon) = \mathcal{L}^{\mathbb{P}^\varepsilon}(X_1) \quad \text{for all } \varepsilon \in [0, 1).$$

Remark VII.2.2. (i) Assumption VII.2.1 is the simplest sufficient condition which turns the small maturity problem into a small noise one. Clearly, one could weaken it substantially by allowing for some small perturbations. For simplicity, we refrain from any further refinement in this direction.

(ii) Assume that σ is independent of ω and satisfies the following time-indifference property :

$$\sigma_{ct}^X(x) = \sigma_t^X(x^c) \text{ for all } c > 0, \text{ where } x_s^c := x_{cs}, \text{ } s \in [0, T]. \quad (\text{VII.2.3})$$

Then, $\mathcal{L}^{\mathbb{P}^0}((X_s)_{s \leq \varepsilon}) = \mathcal{L}^{\mathbb{P}^\varepsilon}((X_s)_{s \leq 1})$ for all $\varepsilon \in [0, 1)$, which implies that the small-maturity small-noise correspondence holds true.

Notice that Condition (VII.2.3) holds for a large class of path-dependent examples. For instance, given a pair $(t, x) \in [0, T] \times \Omega_n$, define the trace of x as the image $\mathcal{X}_t := \{y \in \mathbb{R}^d : y = x_s \text{ for some } s \in [0, t]\}$, and let

$$\sigma_t^X(x) := \zeta(\mathcal{X}_t), \quad (t, x) \in [0, T] \times \Omega_n,$$

for some function ζ . Then σ^X satisfies Condition (VII.2.3). In particular, this example covers the following three cases :

- the homogeneous Markovian case $\sigma_t^X(x) = \sigma^X(x_t)$,
- the running maximum dependence $\sigma_t^X(x) = \sigma^X(x_t, \max_{s \leq t} |x_s|)$,
- the running max/min dependence $\sigma_t^X(x) = \sigma^X(x_t, \max_{s \leq t} \{a \cdot x_s\}, \min_{s \leq t} \{a \cdot x_s\})$, for some $a \in \mathbb{R}^n$.

In view of (VII.2.1) and the small-maturity small-noise correspondence of Assumption VII.2.1, we are reduced to the asymptotics of

$$\varepsilon \ln \mathbb{E}^{\mathbb{P}^\varepsilon} [(e^{X_1} - e^k)^+] \text{ as } \varepsilon \rightarrow 0.$$

Under \mathbb{P}^ε the dynamics of X is given by the stochastic differential equation :

$$dX_t = -\frac{\varepsilon}{2} \sigma_t^X(B, X)^2 dt + \sigma_t^X(B, X) dB_t, \quad \mathbb{P}^\varepsilon - \text{a.s.}$$

whose coefficients satisfy the conditions given in Remarks VII.1.3 and VII.1.7. Consider the stopping time

$$H_{a,b} := \inf\{t : X_t \notin (a, b)\} \text{ for } -\infty < a < b < +\infty.$$

Then, it follows from Theorem VII.1.5 and Remark VII.1.7 that

$$Q_0^\varepsilon := -\varepsilon \ln \mathbb{P}^\varepsilon [H_{a,b} \leq 1] \longrightarrow Q_0(a, b) \quad \text{as } \varepsilon \searrow 0,$$

where $Q_0(a, b)$ is defined as in Theorem VII.1.5 in terms of the controlled function x^α of Theorem VII.1.2 :

$$Q_0(a, b) := \inf \left\{ \frac{1}{2} \int_0^1 |\alpha_s|^2 ds : \alpha \in \mathbb{L}_d^2, x_{1\wedge \cdot}^\alpha \notin \mathcal{O}_{a,b} \right\},$$

where $\mathcal{O}_{a,b} := \{x : x_t \in (a, b) \text{ for all } t \in [0, 1]\}$. The rest of this section is devoted to the following result.

Proposition VII.2.3. $\lim_{\varepsilon \rightarrow 0} -\varepsilon \ln \mathbb{E}^{\mathbb{P}^\varepsilon} [(e^{X_1} - e^k)^+] = Q_0(k) := \lim_{a \rightarrow -\infty} Q_0(a, k)$.

Proof 1. We first show that

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \ln \mathbb{E}^{\mathbb{P}^\varepsilon} [(e^{X_1} - e^k)^+] \leq -Q_0(k). \quad (\text{VII.2.4})$$

Fix some $p > 1$ and the corresponding conjugate $q > 1$ defined by $\frac{1}{p} + \frac{1}{q} = 1$. By the Hölder inequality, we estimate that

$$\mathbb{E}^{\mathbb{P}^\varepsilon} [(e^{X_1} - e^k)^+] \leq \mathbb{E}^{\mathbb{P}^\varepsilon} \left[e^{X_1} \mathbf{1}_{\{X_1 \geq k\}} \right] \leq \mathbb{E}^{\mathbb{P}^\varepsilon} [e^{qX_1}]^{1/q} \mathbb{P}^\varepsilon [H_{a,k} \leq 1]^{1/p}, \text{ for all } a < k.$$

By standard estimates, we may find a constant C_p such that $\mathbb{E}^{\mathbb{P}^\varepsilon} [e^{qX_1}] \leq C_p$ for all $\varepsilon \in (0, 1)$. Then,

$$\varepsilon \ln \mathbb{E}^{\mathbb{P}^\varepsilon} [(e^{X_1} - e^k)^+] \leq \frac{\varepsilon}{q} \ln C_p + \frac{\varepsilon}{p} \ln \mathbb{P}^\varepsilon [H_{a,k} \leq 1],$$

which provides (VII.2.4) by sending $\varepsilon \rightarrow 0$ and then $p \rightarrow 1$.

2. We next prove the following inequality :

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \ln \mathbb{E}^{\mathbb{P}^\varepsilon} [(e^{X_1} - e^k)^+] \geq -Q_0(k). \quad (\text{VII.2.5})$$

For $n \in \mathbb{N}$, denote $f_n(x) := (e^{-n} - x)^+ + (x - e^k)^+$ for $x \in \mathbb{R}$. Since f_n is convex and e^X is \mathbb{P}^ε -martingale, the process $f(e^X)$ is a non-negative \mathbb{P}^ε -submartingale. For a sufficiently small $\delta > 0$, set $a_{n,\delta} := \ln(e^{-n} - \delta)$ and $k_\delta := \ln(e^k + \delta)$. Then, it follows from the Doob inequality

that

$$\mathbb{P}^\epsilon[H_{a_n, \delta, k_\delta} \leq 1] = \mathbb{P}^\epsilon\left[\max_{t \leq 1} f_n(e^{X_t}) \geq \delta\right] \leq \frac{1}{\delta} \mathbb{E}^{\mathbb{P}^\epsilon}[f_n(e^{X_1})]. \quad (\text{VII.2.6})$$

We shall prove in Step 3 below that

$$\lim_{\epsilon \rightarrow 0} \frac{\mathbb{E}^{\mathbb{P}^\epsilon}[(e^{-n} - e^{X_1})^+]}{\mathbb{E}^{\mathbb{P}^\epsilon}[(e^{X_1} - e^k)^+]} = 0 \quad \text{for large } n. \quad (\text{VII.2.7})$$

Then, it follows from (VII.2.6), by sending $\epsilon \rightarrow 0$, that

$$-Q_0(a_n, \delta, k_\delta) \leq \liminf_{\epsilon \rightarrow 0} \epsilon \ln \mathbb{E}^{\mathbb{P}^\epsilon}[(e^{X_1} - e^k)^+].$$

Further, function $Q_0(a, b)$ is clearly decreasing in a , and thus

$$-Q_0(k_\delta) \leq -Q_0(a_n, \delta, k_\delta) \leq \liminf_{\epsilon \rightarrow 0} \epsilon \ln \mathbb{E}^{\mathbb{P}^\epsilon}[(e^{X_1} - e^k)^+].$$

It remains to prove that

$$\limsup_{\delta \rightarrow 0} Q_0(k_\delta) \leq Q(k) \quad (\text{VII.2.8})$$

It is easy to show that

$$\begin{aligned} Q_0(b) &= \inf_{\alpha \in \mathbb{L}_d^2} \left\{ \frac{1}{2} \int_0^1 |\alpha_s|^2 ds + \infty \cdot 1_{\{\max_{t \leq 1} x_t^\alpha < b\}} \right\} \\ &= \inf_{\alpha \in \mathbb{L}_d^2} \left\{ \frac{1}{2} \int_0^1 |\alpha_s|^2 ds + \infty \cdot 1_{\{\max_{t \leq 1} x_t^\alpha \leq b\}} \right\}. \end{aligned}$$

Consequently, Q_0 is upper semicontinuous, as the infimum of upper semicontinuous functions. This implies (VII.2.8) and thus (VII.2.5).

3. It remains to prove (VII.2.7). By the assumption $\underline{\sigma} \leq \sigma \leq \bar{\sigma}$ and the convexity of $s \mapsto (e^{-n} - s)^+$ and $s \mapsto (s - e^k)^+$, it follows from [41] that

$$\begin{aligned} \mathbb{E}^{\mathbb{P}^\epsilon}[(e^{-n} - e^{X_1})^+] &\leq \mathbb{E}^{\mathbb{P}^\epsilon}[(e^{-n} - e^{-\frac{1}{2}\epsilon\bar{\sigma}^2 + \bar{\sigma}B_1})^+], \\ \text{and } \mathbb{E}^{\mathbb{P}^\epsilon}[(e^{X_1} - e^k)^+] &\geq \mathbb{E}^{\mathbb{P}^\epsilon}[(e^{-\frac{1}{2}\epsilon\underline{\sigma}^2 + \underline{\sigma}B_1} - e^k)^+]. \end{aligned}$$

Thus

$$\frac{\mathbb{E}^{\mathbb{P}^\epsilon}[(e^{-n} - e^{X_1})^+]}{\mathbb{E}^{\mathbb{P}^\epsilon}[(e^{X_1} - e^k)^+]} \leq \frac{\mathbb{E}^{\mathbb{P}^\epsilon}[(e^{-n} - e^{-\frac{1}{2}\epsilon\bar{\sigma}^2 + \bar{\sigma}B_1})^+]}{\mathbb{E}^{\mathbb{P}^\epsilon}[(e^{-\frac{1}{2}\epsilon\sigma^2 + \sigma B_1} - e^k)^+]}$$

Further, we have

$$\mathbb{E}^{\mathbb{P}^\epsilon}[(e^{-n} - e^{-\frac{1}{2}\epsilon\bar{\sigma}^2 + \bar{\sigma}B_1})^+] \leq e^{-n} \mathbf{N}\left(\frac{1}{2}\bar{\sigma}\sqrt{\epsilon} - \frac{n}{\bar{\sigma}\sqrt{\epsilon}}\right),$$

and, by the Chebyshev inequality,

$$\mathbb{E}^{\mathbb{P}^\epsilon}[(e^{-\frac{1}{2}\epsilon\sigma^2 + \sigma B_1} - e^k)^+] \geq \lambda \mathbb{P}^\epsilon[e^{-\frac{1}{2}\epsilon\sigma^2 + \sigma B_1} \geq e^k + \lambda] = \lambda \mathbf{N}\left(-\frac{1}{2}\sigma\sqrt{\epsilon} - \frac{\ln(e^k + \lambda)}{\sigma\sqrt{\epsilon}}\right).$$

Using the estimate $\mathbf{N}(-x) \sim \frac{1}{\sqrt{2\pi}}x^{-1}e^{-\frac{x^2}{2}}$, we obtain that

$$\limsup_{\epsilon \rightarrow 0} \frac{\mathbb{E}^{\mathbb{P}^\epsilon}[(e^{-n} - e^{X_1})^+]}{\mathbb{E}^{\mathbb{P}^\epsilon}[(e^{X_1} - e^k)^+]} \leq C \exp\left\{-\lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \left(\frac{n^2}{\bar{\sigma}^2} - \frac{(\ln(e^k + \lambda))^2}{\sigma^2}\right)\right\} = 0,$$

whenever $n^2 > \frac{\bar{\sigma}^2}{\sigma^2}(\ln(e^k + \lambda))^2$. □

3 Asymptotics of Laplace transforms

Our starting point is a characterization of Y_0^ϵ in terms of a quadratic backward stochastic differential equation. Let

$$Y_t^\epsilon := -\epsilon \ln \mathbb{E}_t^{\mathbb{P}^\epsilon} \left[e^{-\frac{1}{\epsilon}\xi(B, X)} \right], \quad t \in [0, T], \quad (\text{VII.3.1})$$

where $\mathbb{E}_t^{\mathbb{P}^\epsilon}$ denotes expectation operator under \mathbb{P}^ϵ , conditional to \mathcal{F}_t .

Proposition VII.3.1. *The processes Y^ϵ is bounded by $\|\xi\|_\infty$, and there exists a process Z^ϵ such that the pair (Y^ϵ, Z^ϵ) is the unique solution of the following “quadratic backward stochastic differential equation” :*

$$Y_t^\epsilon = \xi - \frac{1}{2} \int_t^T |Z_s^\epsilon|^2 ds + \int_t^T Z_s^\epsilon \cdot dB_s, \quad \mathbb{P}^\epsilon - a.s.$$

Moreover, the process Z^ϵ satisfies the "BMO estimate" :

$$\|Z\|_{\mathbb{H}_{\text{BMO}}^2(\mathbb{P}^\epsilon)} := \sup_{t \in [0, T]} \left\| \mathbb{E}_t^{\mathbb{P}^\epsilon} \int_t^T |Z_s^\epsilon|^2 ds \right\|_{\mathbb{L}^\infty(\mathbb{P}^\epsilon)} \leq 4\|\xi\|_\infty. \quad (\text{VII.3.2})$$

Proof Since ξ is bounded, we see immediately that $Y_t^\epsilon \leq -\epsilon \ln \left(e^{-\frac{1}{\epsilon}\|\xi\|_\infty} \right) = \|\xi\|_\infty$ and, similarly $Y_t^\epsilon \geq -\|\xi\|_\infty$. Consequently, the process

$$p^\epsilon := e^{-\frac{1}{\epsilon}Y^\epsilon} = \mathbb{E}_t^{\mathbb{P}^\epsilon} \left[e^{-\frac{1}{\epsilon}\xi(B, X)} \right]$$

is a bounded martingale. By martingale representation, there exists a process q^ϵ , with $\mathbb{E}^{\mathbb{P}^\epsilon} \left[\int_0^T |q_t^\epsilon|^2 dt \right] < \infty$, such that $p_t^\epsilon = p_0^\epsilon + \int_0^t q_s^\epsilon \cdot dB_s$, for all $t \in [0, T]$. Then, Y^ϵ solves the quadratic backward SDE by Itô's formula. The estimate $\|Z\|_{\mathbb{H}_{\text{BMO}}^2(\mathbb{P}^\epsilon)}$ follows immediately by taking expectations in the quadratic backward SDE, and using the boundedness of Y^ϵ by $\|\xi\|_\infty$. \square

We note that the norm $\|\cdot\|_{\mathbb{H}_{\text{BMO}}^2(\mathbb{P}^\epsilon)}$ defined in (VII.3.2) is known as the "BMO" norm (we refer to [22] for more details on the BMO theory). We next provide a stochastic control representation for the process Y^ϵ . For all $\alpha \in \mathbb{H}_{\text{BMO}}^2$, we introduce

$$M_T^{\epsilon, \alpha} := e^{\frac{1}{\sqrt{\epsilon}} \int_0^T \alpha_t \cdot dB_t - \frac{1}{2\epsilon} \int_0^T |\alpha_t|^2 dt}.$$

Then $\mathbb{E}^{\mathbb{P}^\epsilon} [M_T^{\epsilon, \alpha}] = 1$, and we may introduce an equivalent probability measure $\mathbb{P}^{\epsilon, \alpha}$ by the density $d\mathbb{P}^{\epsilon, \alpha} := M_T^{\epsilon, \alpha} d\mathbb{P}^\epsilon$. Define :

$$Y_t^{\epsilon, \alpha} = \mathbb{E}_t^{\mathbb{P}^{\epsilon, \alpha}} \left[\xi + \frac{1}{2} \int_t^T |\alpha_s|^2 ds \right], \quad \mathbb{P}^\epsilon - \text{a.s.}$$

Lemma VII.3.2. *We have*

$$Y_0^\epsilon = Y_0^{\epsilon, Z^\epsilon} = \inf_{\alpha \in \mathbb{H}_{\text{BMO}}^2(\mathbb{P}^\epsilon)} Y_0^{\epsilon, \alpha}.$$

Proof By the martingale representation theorem, there is a process $Z^{\epsilon, \alpha}$ such that the pair $(Y^{\epsilon, \alpha}, Z^{\epsilon, \alpha})$ solves the linear backward SDE

$$dY_t^{\epsilon, \alpha} = -Z_t^{\epsilon, \alpha} \cdot dB_t - \left(Z_t^{\epsilon, \alpha} \cdot \alpha_t - \frac{1}{2} |\alpha_t|^2 \right) dt, \quad \mathbb{P}^\epsilon - \text{a.s.}$$

Since $-\frac{1}{2}z^2 = \inf_{a \in \mathbb{R}^d} \left\{ -a \cdot z + \frac{1}{2}a^2 \right\}$, it follows from the comparison of BSDEs (see for example Section 2.2 of [42]) that $Y^{\epsilon, \alpha} \geq Y^\epsilon$. The required result follows from the observation that the

last supremum is attained by $a^* = z$, and that $Y^{\epsilon, Z^\epsilon} = Y^\epsilon$. \square

Proof of Theorem VII.1.2. First, it is clear that $\mathbb{L}_d^2 \subset \cap_{\epsilon > 0} \mathbb{H}_{\text{BMO}}^2(\mathbb{P}^\epsilon)$. Let $\alpha \in \mathbb{L}_d^2$ and any $\epsilon > 0$ be fixed. Since α is deterministic, it follows from the Girsanov Theorem that

$$B|_{\mathbb{P}^{\epsilon, \alpha}} \stackrel{\mathcal{L}}{=} W^{\epsilon, \alpha}|_{\mathbb{P}_0}, \quad \text{and} \quad X|_{\mathbb{P}^{\epsilon, \alpha}} \stackrel{\mathcal{L}}{=} X^{\epsilon, \alpha}|_{\mathbb{P}_0}, \quad (\text{VII.3.3})$$

where, under \mathbb{P}_0 , for $t \in [0, T]$,

$$\begin{aligned} W_t^{\epsilon, \alpha} &:= \sqrt{\epsilon} B_t + \int_0^t \alpha_s ds, \\ X_t^{\epsilon, \alpha} &= X_0 + \int_0^t b_s(W_s^{\epsilon, \alpha}, X_s^{\epsilon, \alpha}) ds + \int_0^t \sigma_s(W_s^{\epsilon, \alpha}, X_s^{\epsilon, \alpha}) dW_s^{\epsilon, \alpha}. \end{aligned}$$

Therefore, we have the following representation :

$$Y_0^{\epsilon, \alpha} = \mathbb{E}^{\mathbb{P}_0} \left[\xi(W^{\epsilon, \alpha}, X^{\epsilon, \alpha}) + \frac{1}{2} \int_0^T |\alpha_t|^2 dt \right]. \quad (\text{VII.3.4})$$

By the given regularities, it is clear that $\lim_{\epsilon \rightarrow 0} Y_0^{\epsilon, \alpha} = \ell_0^\alpha$. Then it follows from Lemma VII.3.2 that

$$\limsup_{\epsilon \rightarrow 0} Y_0^\epsilon \leq \limsup_{\epsilon \rightarrow 0} Y_0^{\epsilon, \alpha} = \ell_0^\alpha.$$

By the arbitrariness of $\alpha \in \mathbb{L}_d^2$, this shows that $\limsup_{\epsilon \rightarrow 0} Y_0^\epsilon \leq L_0$.

To prove the reverse inequality, we use the minimizer from Lemma VII.3.2. Note that \mathbb{P}^ϵ is equivalent to $\mathbb{P}^{\epsilon, Z^\epsilon}$ and for \mathbb{P}^ϵ -a.e. ω , $\alpha^{\epsilon, \omega} := Z^\epsilon(\omega) \in \mathbb{L}_d^2$. Then we compute that

$$\begin{aligned} Y_0^\epsilon &= Y_0^{\epsilon, Z^\epsilon} = \mathbb{E}^{\mathbb{P}^{\epsilon, Z^\epsilon}} \left[\xi(B, X) + \frac{1}{2} \int_0^T |Z_t^\epsilon|^2 dt \right] \\ &\geq L_0 + \mathbb{E}^{\mathbb{P}^{\epsilon, Z^\epsilon}} \left[\xi(B, X) - \xi(\omega^{Z^\epsilon(\omega)}, x^{Z^\epsilon(\omega)}(\omega)) \right] \\ &\geq L_0 - \mathbb{E}^{\mathbb{P}^{\epsilon, Z^\epsilon}} \left[\rho \left(\|B - \omega^{Z^\epsilon(\omega)}\|_T + \|X - x^{Z^\epsilon(\omega)}(\omega)\|_T \right) \right], \end{aligned}$$

where ρ is the modulus of continuity of ξ . By definition of ω^α , notice that $W^\epsilon := \epsilon^{-1/2}(B - \omega^{Z^\epsilon})$ defines a Brownian motion under $\mathbb{P}^{\epsilon, Z^\epsilon}$. Then it is clear that

$$\limsup_{\epsilon \rightarrow 0} \mathbb{E}^{\mathbb{P}^{\epsilon, Z^\epsilon}} \left[\|B - \omega^{Z^\epsilon}\|_T \right] = \limsup_{\epsilon \rightarrow 0} \mathbb{E}^{\mathbb{P}^{\epsilon, Z^\epsilon}} \left[\sqrt{\epsilon} \|W^\epsilon\|_T \right] = 0.$$

Furthermore, recall that σ and b are Lipschitz-continuous, it follows from the comparison of SDEs that $\underline{\delta}_t \leq X - x^{Z^\epsilon} \leq \bar{\delta}_t$, where $\underline{\delta}_0 = \bar{\delta}_0 = 0$, and

$$\begin{aligned} d\underline{\delta}_t &= \sigma_t(B, X)\sqrt{\epsilon}dW_t^\epsilon - L\left(\sqrt{\epsilon}\|W^\epsilon\|_t + \|\underline{\delta}\|_t\right)(|Z_t^\epsilon| + 1) dt, \\ d\bar{\delta}_t &= \sigma_t(B, X)\sqrt{\epsilon}dW_t^\epsilon + L\left(\sqrt{\epsilon}\|W^\epsilon\|_t + \|\bar{\delta}\|_t\right)(|Z_t^\epsilon| + 1) dt. \end{aligned}$$

We now estimate $\bar{\delta}$. The estimation of $\underline{\delta}$ follows the same line of argument. Denote $K_t := \int_0^t \sigma_s(B, X)dW_s^\epsilon$. By Gronwall's inequality, we obtain

$$\begin{aligned} \epsilon^{-1/2}\|\bar{\delta}_T\| &= L\|W^\epsilon\|_T \int_0^T e^{L \int_0^t (|Z_s^\epsilon|+1)ds} (|Z_t^\epsilon| + 1) dt + \int_0^T e^{L \int_0^t (|Z_s^\epsilon|+1)ds} d\|K\|_t \\ &\leq e^{L \int_0^T (|Z_s^\epsilon|+1)ds} (\|W^\epsilon\|_T + \|K\|_T). \end{aligned}$$

Then,

$$\begin{aligned} \epsilon^{-1/2}e^{-LT}\mathbb{E}^{\mathbb{P}^{\epsilon, Z^\epsilon}}[\|\bar{\delta}_T\|] &\leq \mathbb{E}^{\mathbb{P}^{\epsilon, Z^\epsilon}}\left[e^{L \int_0^T |Z_s^\epsilon|ds}[\|W^\epsilon\|_T + \|K\|_T]\right] \\ &\leq \left(\mathbb{E}^{\mathbb{P}^{\epsilon, Z^\epsilon}}\left[e^{2L \int_0^T |Z_s^\epsilon|ds}\right]\right)^{\frac{1}{2}} \left(\mathbb{E}^{\mathbb{P}^{\epsilon, Z^\epsilon}}\left[\|W^\epsilon\|_T^2 + \|K\|_T^2\right]\right)^{\frac{1}{2}}. \end{aligned}$$

Recall that $\sigma_t(0, x)$ is bounded. One may easily check that, for some constant C independent of ϵ ,

$$\mathbb{E}^{\mathbb{P}^{\epsilon, Z^\epsilon}}\left[\|W^\epsilon\|_T^2 + \|K\|_T^2\right] \leq C.$$

Moreover, note that

$$Y_t^\epsilon = \xi + \frac{1}{2} \int_t^T |Z_s^\epsilon|^2 ds - \sqrt{\epsilon} \int_t^T Z_s^\epsilon dW_s^\epsilon.$$

Then, it follows that $\|Z\|_{\mathbb{H}_{\text{BMO}}^2(\mathbb{P}^{\epsilon, Z^\epsilon})} \leq 4\|\xi\|_\infty$, and $\mathbb{E}^{\mathbb{P}^{\epsilon, Z^\epsilon}}\left[e^{\eta \int_0^T |Z_s^\epsilon|^2 ds}\right] \leq C$ for all $\epsilon > 0$, for some $\eta > 0$ and $C > 0$ independent of ϵ , see for example Lemma 9.6.5 on page 175 of [22]. This implies $\mathbb{E}^{\mathbb{P}^{\epsilon, Z^\epsilon}}\left[e^{2L \int_0^T |Z_s^\epsilon|ds}\right] \leq C$ and thus

$$\mathbb{E}^{\mathbb{P}^{\epsilon, Z^\epsilon}}[\|\bar{\delta}\|_T] \leq C\sqrt{\epsilon}, \quad \text{for all } \epsilon > 0.$$

Similarly, $\mathbb{E}^{\mathbb{P}^{\epsilon, Z^\epsilon}} [\|\delta\|_T] \leq C\sqrt{\epsilon}$, and we may conclude that

$$\mathbb{E}^{\mathbb{P}^{\epsilon, Z^\epsilon}} \left[\rho \left(\|B - \omega^{Z^\epsilon}\|_T + \|X - x^{Z^\epsilon}\|_T \right) \right] \longrightarrow 0, \text{ as } \epsilon \searrow 0,$$

completing the proof. □

4 Asymptotics of the exiting probability

This section is dedicated to the proof of Theorem VII.1.5. As before, we introduce the processes :

$$Y_t^\epsilon := -\epsilon \ln p_t^\epsilon, \quad p_t^\epsilon := \mathbb{P}_t^\epsilon[H < T] \quad \text{for all } t \leq T.$$

Unlike the previous problem, the present example features an additional difficulty due to the singularity of the terminal condition :

$$\lim_{t \rightarrow T} Y_t^\epsilon = \infty \quad \text{on } \{H \geq T\}.$$

We shall first show that $\limsup_{\epsilon \downarrow 0} Y_0^\epsilon \leq Q_0$. In light of the arguments in Fleming & Soner [52, Lemma 10.1, p. 283], we define

$$\delta(x, A) := \inf_{y \in A} |x - y|, \quad \text{for a set } A \subset \mathbb{R}^n.$$

We first need the following regularity result on the distance function δ . We believe that this should be a standard result, but we could not find a reference. Thus we shall provide a proof in Appendix for completeness.

Lemma VII.4.1. *Let O be a bounded open set in \mathbb{R}^n with C^3 boundary. Then the function $\delta(\cdot, \partial O) \in C^2$ on $\{x : \delta(x, \partial O) < \eta\}$ for some $\eta > 0$. □*

The following lemma is crucial.

Lemma VII.4.2. *There exists a constant K such that for any $\epsilon > 0$ we have*

$$Y_t^\epsilon \leq \frac{K\delta(X_t, \partial O)}{T - t} \quad \text{for all } t < T \text{ and } t \leq H, \quad \mathbb{P}^\epsilon\text{-a.e.}$$

Proof First, fix $T_1 < T$. For $x \in \mathbb{R}^d$, we denote by x^1 its first component. Since O is bounded,

there exists a constant μ such that $x^1 + \mu > 0$ for all $x \in O$. Define a function :

$$g^\epsilon(t, x) := \exp\left(-\frac{\lambda(x^1 + \mu)}{\epsilon(T_1 - t)}\right), \quad \text{for } t < T_1, x \in \text{cl}(O),$$

where λ is some constant to be chosen later and $\text{cl}(O)$ denotes the closure of O . Recall that $a = (a^{i,j})_{i,j} := \sigma\sigma^T$. By Itô's formula, we have \mathbb{P}^ϵ -a.s.,

$$dg^\epsilon(t, X_t) = \frac{g^\epsilon(t, X_t)}{\epsilon(T_1 - t)^2} \left[\frac{1}{2} a_t^{1,1}(B, X) \lambda^2 - \lambda(X_t^1 + \mu) - (T_1 - t) \lambda b_t^1(B, X) \right] dt + dM_t,$$

for some \mathbb{P}^ϵ -martingale M . Since $a^{1,1}$ is uniformly bounded away from zero and b^1 is uniformly bounded, the dt -term of the above expression is positive for a sufficiently large $\lambda = \lambda^*$. Hence, $g^\epsilon(t, X_t)$ is a submartingale on $[0, T_1 \wedge H]$. Also, note that $g^\epsilon(T_1, X_{T_1}) = 0 \leq p_{T_1}^\epsilon$ and $g^\epsilon(H, X_H) \leq 1 = p_H^\epsilon$. Since p^ϵ is a martingale, we conclude that

$$g^\epsilon(t, X_t) \leq p_t^\epsilon \quad \text{for all } t \leq T_1 \wedge H, \mathbb{P}^\epsilon\text{-a.s.}$$

Denote $\delta(x) := \delta(x, \partial O)$. Since ∂O is C^3 , it follows from Lemma VII.4.1 that there exists a constant η such that on $\{x \in O : \delta(x) < \eta\}$, the function d is C^2 . Now, define

$$\tilde{g}^\epsilon(t, x) := \exp\left(-\frac{K\delta(x)}{\epsilon(T_1 - t)}\right), \quad \text{for } t < T_1, x \in \text{cl}(O),$$

for some $K \geq \frac{\lambda^*(C+\mu)}{\eta}$. Clearly, for $t \leq T_1 \wedge H$ and $\delta(X_t) \geq \eta$, we have

$$\tilde{g}^\epsilon(t, X_t) \leq g^\epsilon(t, X_t) \leq p_t^\epsilon, \quad \mathbb{P}^\epsilon - \text{a.s.}$$

In the remaining case $t \leq T_1 \wedge H$ and $\delta(X_t) < \eta$, we will now verify that

$$\left\{ \tilde{g}^\epsilon(s, X_s) 1_{\{\delta(X_t) < \eta\}}, s \in [t, H_\eta \wedge H \wedge T] \right\} \quad \text{is a } \mathbb{P}^\epsilon\text{-submartingale,}$$

where $H_\eta := \inf\{s : \delta(X_s) \geq \eta\}$. By Itô's formula, together with the fact that $|D\delta(x)| = 1$,

$$\begin{aligned} d\tilde{g}^\epsilon(s, X_s) &= \frac{K\tilde{g}^\epsilon(s, X_s)}{\epsilon(T_1 - s)^2} \left[\frac{K}{2} a_s D\delta(X_s) \cdot D\delta(X_s) - \epsilon \frac{T_1 - s}{2} \text{tr}(a_s D^2\delta(X_s)) \right. \\ &\quad \left. - (T_1 - s) b_s \cdot D\delta(X_s) - \delta(X_s) \right] ds + dM_s \\ &\geq \frac{K\tilde{g}^\epsilon(s, X_s)}{\epsilon(T_1 - s)^2} \left(\frac{K}{2} \delta - \epsilon \frac{T_1 - s}{2} |a_s| |D^2\delta(X_s)| - (T_1 - s) \|b_s\| \right) ds + dM_s. \end{aligned}$$

Hence, for sufficiently large $K = K^*$, the dt -term is positive, and $\tilde{g}^\epsilon(s, X_s)1_{\{\delta(X_t) < \eta\}}$ is a sub-martingale for $s \in [t, H_\eta \wedge H \wedge T]$. We also verify directly that

$$\tilde{g}^\epsilon(H_\eta \wedge H \wedge T, X_{H_\eta \wedge H \wedge T})1_{\{\delta(X_t) < \eta\}} \leq p_{H_\eta \wedge H \wedge T}^\epsilon, \quad \mathbb{P}^\epsilon - \text{a.s.}$$

Since p^ϵ is a \mathbb{P}^ϵ -martingale, we deduce that $\tilde{g}^\epsilon(t, X_t) \leq p_t^\epsilon$ for $t \leq T_1 \wedge H$ and $\delta(X_t) < \eta$. Thus, we may conclude that

$$\tilde{g}^\epsilon(t, X_t) \leq p_t^\epsilon \text{ for all } t \leq T_1 \wedge H, \quad \mathbb{P}^\epsilon\text{-a.s.}$$

Let $T_1 \rightarrow T$, we finally get

$$Y_t^\epsilon \leq \frac{K\delta(X_t)}{T-t} \text{ for all } t < T \text{ and } t \leq H, \quad \mathbb{P}^\epsilon\text{-a.s.}$$

□

Proposition VII.4.3. $\limsup_{\epsilon \downarrow 0} Y_0^\epsilon \leq Q_0$.

Proof As in Proposition VII.3.1, we may show that there exists a process Z^ϵ such that for any $T_1 < T$:

$$Y_t^\epsilon = Y_{T_1}^\epsilon - \frac{1}{2} \int_t^{T_1} |Z_s^\epsilon|^2 ds + \int_t^{T_1} Z_s^\epsilon \cdot dB_s, \quad \mathbb{P}^\epsilon - \text{a.s.}$$

Define a sequence of BSDEs :

$$\bar{Y}_t^{\epsilon, T_1} = \frac{K\delta(X_{T_1}, O^c)}{T-T_1} - \frac{1}{2} \int_t^{T_1} |Z_s^{\epsilon, T_1}|^2 ds + \int_t^{T_1} Z_t^{\epsilon, T_1} \cdot dB_s, \quad \mathbb{P}^\epsilon - \text{a.s.}$$

Note that $Y_{T_1 \wedge H}^\epsilon \leq \frac{K\delta(X_{T_1 \wedge H}, O^c)}{T-T_1 \wedge H} \leq \frac{K\delta(X_{T_1}, O^c)}{T-T_1}$. By Lemma VII.4.2 and the comparison result of quadratic BSDE (see Theorem 2.6 of [73]), we deduce that

$$Y_0^\epsilon \leq \bar{Y}_0^{\epsilon, T_1} \text{ for all } T_1 < T.$$

Denote $\xi(x) := \frac{K\delta(x_{T_1}, O^c)}{T-T_1}$, and note that $\bar{Y}_0^{\epsilon, T_1} = -\epsilon \ln \mathbb{E}^{\mathbb{P}^\epsilon} [e^{-\frac{1}{\epsilon}\xi(X)}]$. Since ξ is bounded and uniformly continuous, it follows from Theorem VII.1.2 that

$$\lim_{\epsilon \rightarrow 0} \bar{Y}_0^{\epsilon, T_1} = y_0^{T_1} := \inf_{\alpha \in \mathbb{L}^2} \left\{ \frac{1}{2} \int_0^{T_1} \alpha_t^2 dt + \frac{K\delta(x_{T_1}^\alpha, O^c)}{T-T_1} \right\}.$$

Thus, we have

$$\limsup_{\epsilon \downarrow 0} Y_0^\epsilon \leq \inf_{\alpha \in \mathbb{L}^2} \left\{ \frac{1}{2} \int_0^{T_1} \alpha_t^2 dt + \frac{K\delta(x_{T_1}^\alpha, O^c)}{T-T_1} \right\} \leq \inf_{\alpha \in \mathbb{L}^2, x_{T_1}^\alpha \notin O} \left\{ \frac{1}{2} \int_0^T \alpha_t^2 dt \right\}.$$

Finally, observe that

$$\inf_{\alpha \in \mathbb{L}^2, x_{T_1}^\alpha \notin O} \left\{ \frac{1}{2} \int_0^T \alpha_t^2 dt \right\} = \inf_{\alpha \in \mathbb{L}^2, x_{T_1}^\alpha \wedge \cdot \notin O} \left\{ \frac{1}{2} \int_0^T \alpha_t^2 dt \right\} \longrightarrow Q_0, \quad \text{as } T_1 \rightarrow T.$$

□

To complete the proof of Theorem VII.1.5, we next complement the result of Proposition VII.4.3 by the opposite inequality.

Proposition VII.4.4. $\liminf_{\epsilon \downarrow 0} Y_0^\epsilon \geq Q_0.$

Proof We organize the proof in three steps.

1. Define another sequence of BSDEs :

$$\underline{Y}_t^{\epsilon, T_1, m} = m\delta(X_{T_1}, O^c) \wedge Y_{T_1}^\epsilon - \frac{1}{2} \int_t^{T_1} |Z_s^{\epsilon, T_1, m}|^2 ds + \int_t^{T_1} Z_t^{\epsilon, T_1, m} \cdot dB_s, \quad \mathbb{P}^\epsilon\text{-a.s.}$$

By the comparison result of quadratic BSDEs, we have that $\underline{Y}_t^{\epsilon, T_1, m} \leq Y_t^\epsilon$ for all $t \leq T_1$. Then, by the stability of BSDEs, we know that $Y^{\epsilon, T_1, m}$ converges to the solution of the following BSDE as $T_1 \rightarrow T$:

$$\underline{Y}_t^{\epsilon, m} = m\delta(X_T, O^c) - \frac{1}{2} \int_t^T |Z_s^{\epsilon, m}|^2 ds + \int_t^T Z_t^{\epsilon, m} \cdot dB_s, \quad \mathbb{P}^\epsilon\text{-a.s.}$$

Note that $\underline{Y}_0^{\epsilon, m} = -\epsilon \ln \mathbb{E}^{\mathbb{P}^\epsilon} [e^{-\frac{1}{\epsilon} m\delta(X_T, O^c)}]$. We may apply Theorem VII.1.2 and get that

$$\liminf_{\epsilon \downarrow 0} Y_0^\epsilon \geq \lim_{\epsilon \downarrow 0} \underline{Y}_0^{\epsilon, m} = y_0^m := \inf_{\alpha \in \mathbb{L}^2} \left\{ \frac{1}{2} \int_0^T \alpha_s^2 ds + m\delta(x_T^\alpha, O^c) \right\}. \quad (\text{VII.4.1})$$

2. We now prove that the sequence $(y_0^m)_m$ is bounded. Take $\alpha_t \equiv C \cdot 1$. Then

$$x_T^\alpha = x_0 + \int_0^T (b_t + C\sigma_t \cdot 1) dt.$$

Since b is bounded and σ is positive, when $C = C_0$ is sufficiently large, we will have $x_T^\alpha \notin O$. Hence, $y_0^m \leq \frac{1}{2} C_0^2 T d$.

3. In view of (VII.4.1), we now conclude the proof of the proposition by verifying that $y_0^m \longrightarrow Q_0$, as $m \rightarrow \infty$. Let $\rho > 0$. By the definition of y_0^m , there is a ρ -optimal α^ρ :

$$y_0^m + \rho > \frac{1}{2} \int_0^T |\alpha_t^\rho|^2 dt + m\delta(x_T^\rho, O^c),$$

where we denoted $x^\rho := x^{\alpha^\rho}$. By the boundedness of $(y_0^m)_m$ in Step 2, we have $\delta(x_T^\rho, O^c) \leq \frac{C}{m}$. So, there exists a point $x_0 \in \partial O$ such that $|x_T^\rho - x_0| \leq \frac{C}{m}$. Define :

$$\tilde{\alpha}_t := \alpha_t^\rho + \sigma_t^{-1} \frac{x_0 - x_T^\rho}{T}.$$

Then, $x_T^{\tilde{\alpha}} = x_0 \notin O$. Also, note that $\sigma_t^{-1} \frac{x_0 - x_T^\rho}{T} = o(\frac{1}{m})$ when $m \rightarrow \infty$. Hence,

$$\frac{1}{2} \int_0^T |\alpha_t^\rho|^2 dt = \frac{1}{2} \int_0^T |\tilde{\alpha}_t - \sigma_t^{-1} \frac{x_0 - x_T^\rho}{T}|^2 dt \geq \inf_{\alpha \in \mathbb{L}^2, x_T^\alpha \notin O} \left\{ \frac{1}{2} \int_0^T |\alpha_t|^2 dt \right\} + o\left(\frac{1}{m}\right).$$

Finally, sending $m \rightarrow \infty$, we see that $\lim_{m \rightarrow \infty} y_0^m + \rho \geq Q_0$. Since ρ is arbitrary, the proof is complete. \square

5 Viscosity property of the candidate solution

We first cite the result by Lukoyanov (Theorem 2 in [82]).

Theorem VII.5.1 (Lukoyanov [82]). *Assume that*

- *the generator F and the terminal condition ξ is continuous in all components;*
- *it holds the estimates :*

$$|F(t, \hat{\omega}, 0)| \leq \rho(t, \hat{\omega}), \quad |F(t, \hat{\omega}, p_\omega) - F(t, \hat{\omega}, p'_\omega)| \leq \rho(t, \hat{\omega})(|p_\omega - p'_\omega|),$$

where $\rho(t, \hat{\omega}) := C(1 + \|\hat{\omega}\|_t)$, C is a constant, and $p_\omega := (p_\omega, p_x)$;

- *for any compact set $D \subset \Omega_{d+n}$ there is a constant $\mathbb{L}(D)$ such that*

$$|F(t, \hat{\omega}, p_\omega) - F(t, \hat{\omega}', p_\omega)| \leq \mathbb{L}(D)(1 + |p_\omega|) \sqrt{\mu(t, \hat{\omega}, \hat{\omega}')},$$

where $\mu(t, \hat{\omega}, \hat{\omega}') := |\hat{\omega}_t - \hat{\omega}'_t|^2 + \int_0^t |\hat{\omega}_s - \hat{\omega}'_s|^2 ds$.

Then the Dirichlet problem of the path dependent PDE :

$$-\partial_t u - F(t, \omega, p_\omega) = 0 \quad \text{with} \quad u_T = \xi,$$

has a unique continuous viscosity solution.

Clearly our equation (VII.1.5) satisfies the conditions in the above theorem, so uniqueness holds for (VII.1.5) within the class of continuous functions and, in order to prove Theorem VII.1.11 it remains to verify that u satisfies the viscosity properties.

Lemma VII.5.2. Fix $K \geq 0$. There exists a constant C such that for any $t \in [0, T]$ and $\hat{\omega}^1, \hat{\omega}^2 \in \hat{\Omega}$,

$$\sup_{\alpha: \int_t^T |\alpha|_s^2 ds \leq K} \|\hat{\omega}^{\alpha, t, \hat{\omega}^1} - \hat{\omega}^{\alpha, t, \hat{\omega}^2}\| \leq C \|\hat{\omega}^1 - \hat{\omega}^2\|_t$$

Proof By the definition of $\hat{\omega}^{\alpha, t, \hat{\omega}^i}$ ($i = 1, 2$), we know that the components $\omega^{\alpha, t, \hat{\omega}^i}$ are equal. The difference comes from the component $x^{\alpha, t, \hat{\omega}^i}$. Denote $\delta x_t := \|x^{\alpha, t, \hat{\omega}^1} - x^{\alpha, t, \hat{\omega}^2}\|_t^2$. Then, by the definition of $x^{\alpha, t, \hat{\omega}^i}$ and the Lipschitz continuity of b and σ , we obtain that

$$\begin{aligned} \delta x_s &\leq \int_0^s C(\|\hat{\omega}^1 - \hat{\omega}^2\|_t^2 + \delta x_r) dr + C\left(\int_0^s (\|\hat{\omega}^1 - \hat{\omega}^2\|_t + \delta x_r) |\alpha_r| dr\right)^2 \\ &\leq \int_0^s C(\|\hat{\omega}^1 - \hat{\omega}^2\|_t^2 + \delta x_r) dr + 2KC\left(\int_0^s (\|\hat{\omega}^1 - \hat{\omega}^2\|_t^2 + \delta x_r) dr\right) \end{aligned}$$

Finally, the claim follows from the Gronwall's inequality. \square

Lemma VII.5.3 (Dynamic programming). Let u be the value function defined in (VII.1.9). Then, for all $0 \leq t \leq s \leq T$ and $\hat{\omega} \in \hat{\Omega}$, we have

$$u(t, \hat{\omega}) = \inf_{\alpha \in \mathbb{L}_d^2} \left\{ \frac{1}{2} \int_t^s |\alpha_r|^2 dr + u^{t, \hat{\omega}}(s - t, \hat{\omega}^{\alpha, t, \hat{\omega}}) \right\}, \quad (\text{VII.5.1})$$

where $u^{t, \hat{\omega}}(t', \hat{\omega}') := u(t + t', \hat{\omega} \otimes_t \hat{\omega}')$.

Proof 1. By the definition of infimum and that of u , it holds for all $\alpha, \alpha' \in \mathbb{L}_d^2$:

$$\begin{aligned} \text{r.h.s.} &\leq \frac{1}{2} \int_t^s |\alpha_r|^2 dr + u^{t, \hat{\omega}}(s - t, \hat{\omega}^{\alpha, t, \hat{\omega}}) \\ &\leq \frac{1}{2} \int_t^s |\alpha_r|^2 dr + \frac{1}{2} \int_s^T |\alpha'_r|^2 dr + \xi^{s, \tilde{\omega}}(\hat{\omega}^{\alpha', s, \tilde{\omega}}), \quad \text{with } \tilde{\omega} := \hat{\omega} \otimes_t \hat{\omega}^{\alpha, t, \hat{\omega}}. \end{aligned}$$

Denote $\tilde{\alpha}_r := \alpha_r 1_{[t, s)}(r) + \alpha'_r 1_{[s, T]}(r)$, and then $\tilde{\alpha} \in \mathbb{L}_d^2$. Also note that $\tilde{\omega} \otimes_s \hat{\omega}^{\alpha', s, \tilde{\omega}} = \hat{\omega} \otimes \hat{\omega}^{\tilde{\alpha}, t, \hat{\omega}}$. Further, since α, α' are arbitrary, we obtain that

$$\text{r.h.s.} \leq \inf_{\alpha \in \mathbb{L}_d^2} \left\{ \frac{1}{2} \int_t^T |\alpha_r|^2 dr + \xi^{t, \hat{\omega}}(\hat{\omega}^{\alpha, t, \hat{\omega}}) \right\} = u(t, \hat{\omega}). \quad (\text{VII.5.2})$$

2. Again by the definition of infimum and that of u , for any $\epsilon > 0$ there exists $\alpha, \alpha' \in \mathbb{L}_d^2$ such that

$$\begin{aligned} \text{r.h.s.} &> \frac{1}{2} \int_t^s |\alpha_r|^2 dr + u^{t, \hat{\omega}}(s - t, \hat{\omega}^{\alpha, t, \hat{\omega}}) - \epsilon \\ &> \frac{1}{2} \int_t^s |\alpha_r|^2 dr + \frac{1}{2} \int_s^T |\alpha'_r|^2 dr + \xi^{s, \tilde{\omega}}(\hat{\omega}^{\alpha', s, \tilde{\omega}}) - 2\epsilon, \quad \text{with } \tilde{\omega} := \hat{\omega} \otimes_t \hat{\omega}^{\alpha, t, \hat{\omega}}. \end{aligned}$$

It follows that

$$\text{r.h.s.} > \frac{1}{2} \int_t^T |\tilde{\alpha}_r|^2 dr + \xi^{t, \hat{\omega}}(\hat{\omega}^{\tilde{\alpha}, t, \hat{\omega}}) - 2\epsilon \geq u(t, \hat{\omega}) - 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we obtain that $\text{r.h.s.} \geq u(t, \hat{\omega})$. Combing with (VII.5.2), we have (VII.5.1). \square

Lemma VII.5.4. *The function u defined in (VII.1.9) is bounded and Lipschitz-continuous.*

Proof Clearly, u inherits the bound of ξ . For $t \in [0, T]$, $\hat{\omega}^1, \hat{\omega}^2 \in \hat{\Omega}$, since ξ is bounded, there exists a constant K such that

$$\begin{aligned} u(t, \hat{\omega}^i) &= \inf_{\alpha \in \mathbb{L}_a^2} \left\{ \frac{1}{2} \int_t^T |\alpha_s|^2 ds + \xi^{t, \hat{\omega}^i}(\hat{\omega}^{\alpha, t, \hat{\omega}^i}) \right\} \\ &= \inf_{\alpha: \int_t^T |\alpha_s|^2 ds \leq K} \left\{ \frac{1}{2} \int_t^T |\alpha_s|^2 ds + \xi^{t, \hat{\omega}^i}(\hat{\omega}^{\alpha, t, \hat{\omega}^i}) \right\}. \end{aligned}$$

It follows from Lemma VII.5.2 that :

$$\left| u(t, \hat{\omega}^1) - u(t, \hat{\omega}^2) \right| \leq \sup_{\alpha: \int_t^T |\alpha_s|^2 ds \leq K} \left\{ \left| \xi^{t, \hat{\omega}^1}(\hat{\omega}^\alpha) - \xi^{t, \hat{\omega}^2}(\hat{\omega}^\alpha) \right| \right\} \leq C \left\| \hat{\omega}_{t\wedge\cdot}^1 - \hat{\omega}_{t\wedge\cdot}^2 \right\|. \quad (\text{VII.5.3})$$

On the other hand, fixing $\hat{\omega}$, it follows from the dynamic programming principle that

$$u(t+h, \hat{\omega}_{t\wedge\cdot}) - u(t, \hat{\omega}) = \sup_{\alpha \in \mathbb{L}^2} \left\{ -\frac{1}{2} \int_t^{t+h} \alpha_s^2 ds - u^{t, \hat{\omega}}(h, \hat{\omega}^{\alpha, t, \hat{\omega}}) + u(t+h, \hat{\omega}_{t\wedge\cdot}) \right\} \geq \text{VII.5.4}$$

where the last inequality is induced by the constant control $\alpha = 0$. Moreover, since b and σ are bounded, note that $\|(\hat{\omega} \otimes_t \hat{\omega}^{\alpha, t, \hat{\omega}})_{(t+h)\wedge\cdot} - \hat{\omega}_{t\wedge\cdot}\| \leq C \int_t^{t+h} (1 + |\alpha_s|) ds$. Then, using again the dynamic programming principle together with (VII.5.3), we obtain

$$u(t+h, \hat{\omega}_{t\wedge\cdot}) - u(t, \hat{\omega}) \leq \sup_{\alpha \in \mathbb{L}^2} \left\{ \int_t^{t+h} \left(-\frac{1}{2} \alpha_s^2 + C|\alpha_s| + C \right) ds \right\} \leq \left(\frac{C^2}{2} + C \right) h. \quad (\text{VII.5.5})$$

Combining this with (VII.5.3), we see that

$$\begin{aligned} \left| u(t+h, \hat{\omega}^1) - u(t, \hat{\omega}^2) \right| &\leq \left| u(t+h, \hat{\omega}^1) - u(t+h, \hat{\omega}_{t\wedge\cdot}^1) \right| \\ &\quad + \left| u(t+h, \hat{\omega}_{t\wedge\cdot}^1) - u(t, \hat{\omega}^1) \right| + \left| u(t, \hat{\omega}^1) - u(t, \hat{\omega}^2) \right| \\ &\leq C' (\|\hat{\omega}^1\|_t^{t+h} + h + \|\hat{\omega}_{t\wedge\cdot}^1 - \hat{\omega}_{t\wedge\cdot}^2\|) \\ &\leq 3C' (h + \|\hat{\omega}_{(t+h)\wedge\cdot}^1 - \hat{\omega}_{t\wedge\cdot}^2\|). \end{aligned}$$

□

Now, consider a functional u_K :

$$u_K(t, \hat{\omega}) := \inf_{\|\alpha\|_\infty \leq K} \left[\xi(\hat{\omega} \otimes_t \hat{\omega}^{\alpha, t, \hat{\omega}}) + \frac{1}{2} \int_t^T |\alpha_s|^2 ds \right];$$

Notice that $u_K \geq u_{K-1} \geq u$.

Proposition VII.5.5. *For K sufficiently large, we have $u = u_K$.*

Proof Similar to Lemma VII.5.4, for each K , one may easily see that $u_K(t, \cdot)$ is uniformly Lipschitz in ω with the same Lipschitz constant denoted as L . We first claim that there exists α^K such that

$$u_K(0, 0) = \xi(\hat{\omega}^{\alpha^K}) + \frac{1}{2} \int_0^T |\alpha_t^K|^2 dt. \quad (\text{VII.5.6})$$

Then for any t and h , one can easily show that

$$u_K(t, \hat{\omega}^{\alpha^K}) = u_K(t+h, \hat{\omega}^{\alpha^K}) + \frac{1}{2} \int_t^{t+h} |\alpha_s^K|^2 ds.$$

On the other hand, by the dynamic programming,

$$u_K(t, \hat{\omega}^{\alpha^K}) \leq u_K(t+h, \hat{\omega}_{t \wedge \cdot}^{\alpha^K}).$$

Then

$$\begin{aligned} \frac{1}{2} \int_t^{t+h} |\alpha_s^K|^2 ds &\leq u_K(t+h, \hat{\omega}_{t \wedge \cdot}^{\alpha^K}) - u_K(t+h, \hat{\omega}^{\alpha^K}) \\ &\leq L \|\hat{\omega}^{\alpha^K} - \hat{\omega}_{t \wedge \cdot}^{\alpha^K}\|_{t+h} \leq CL \int_t^{t+h} (1 + |\alpha_s^K|) ds, \end{aligned}$$

where C is a common bound for the coefficients b and σ . Since t and h are arbitrary, we get $\|\alpha^K\|_\infty \leq C'$ for some constant C' independent of K . Then $u_K = u_{C'}$ for any $K \geq C'$, and thus $u = u_{C'}$.

We now prove the existence claim (VII.5.6). Let $\alpha^{K,n}$ be a minimum sequence of controls for $u_K(0, 0)$, namely

$$u_K(0, 0) = \lim_{n \rightarrow \infty} \left[\xi(\hat{\omega}^{\alpha^{K,n}}) + \frac{1}{2} \int_0^T |\alpha_t^{K,n}|^2 dt \right]. \quad (\text{VII.5.7})$$

By compactness of Ω_K , the sequence $\{\omega^{\alpha^{K,n}}, n \geq 1\}$ has a limit $\omega^K \in \Omega_K$, after possibly passing

to a subsequence :

$$\lim_{n \rightarrow \infty} \|\omega^{\alpha^{K,n}} - \omega^K\|_T = 0. \quad (\text{VII.5.8})$$

By (VII.5.7) and since ξ is bounded, it is clear that $\sup_n \int_0^T |\alpha_t^{K,n}|^2 dt < \infty$. Then without loss of generality we may assume that $\{\alpha^{K,n}, n \geq 1\}$ converges to certain α^K weakly in $\mathbb{L}^2([0, T])$. Then for any t and h ,

$$\omega_{t+h}^K - \omega_t^K = \lim_{n \rightarrow \infty} [\omega_{t+h}^{\alpha^{K,n}} - \omega_t^{\alpha^{K,n}}] = \lim_{n \rightarrow \infty} \int_t^{t+h} \alpha_s^{K,n} ds = \int_t^{t+h} \alpha_s^K ds.$$

This implies that $\omega^K = \omega^{\alpha^K}$. Further, by Gronwall's inequality, we obtain that

$$\lim_{n \rightarrow \infty} \|x^{\alpha^{K,n}} - x^{\alpha^K}\|_T = 0. \quad (\text{VII.5.9})$$

Now by Mazur's lemma, there exist convex combinations $\tilde{\alpha}^{K,n} = \sum_i c_i^n \alpha^{K,m_i^n}$, where $m_i^n \geq n$, such that $\{\tilde{\alpha}^{K,n}, n \geq 1\}$ converges to α^K strongly in $\mathbb{L}^2([0, T])$. Then by Jensen's inequality we see that

$$\int_0^T |\alpha_t^K|^2 dt = \lim_{n \rightarrow \infty} \int_0^T |\tilde{\alpha}_t^{K,n}|^2 dt \leq \lim_{n \rightarrow \infty} \sum_i c_i^n \int_0^T |\alpha_t^{K,m_i^n}|^2 dt.$$

On the other hand, by (VII.5.8), (VII.5.9) and since ξ is Lipschitz continuous, we have

$$\xi(\hat{\omega}^{\alpha^K}) = \lim_{n \rightarrow \infty} \sum_i c_i^n \xi(\hat{\omega}^{\alpha^{K,m_i^n}}).$$

Then

$$\xi(\hat{\omega}^{\alpha^K}) + \frac{1}{2} \int_0^T |\alpha_t^K|^2 dt \leq \lim_{n \rightarrow \infty} \sum_i c_i^n \left[\xi(\hat{\omega}^{\alpha^{K,m_i^n}}) + \frac{1}{2} \int_0^T |\alpha_t^{K,m_i^n}|^2 dt \right] = u_K(0, 0),$$

where the last equality follows from (VII.5.7). This proves the claim. \square

Proof of Theorem VII.1.11 Fix K_0 such that $u = u_{K_0}$. Recall that b and σ are bounded by C . Then, define $K := C(1 + K_0)$, so that for all $\|\alpha\|_\infty \leq K_0$ and $\hat{\omega} \in \hat{\Omega}_K$, we have $\hat{\omega}^{\alpha, t, \hat{\omega}} \in \hat{\Omega}_K$.

We first prove the viscosity subsolution property. Let $(t, \hat{\omega}) \in \Theta_K$, and $\varphi \in \underline{\mathcal{A}}^K u(t, \hat{\omega})$. By the dynamic programming principle, we have :

$$u(t, \hat{\omega}) = \inf_{\alpha \in \mathbb{L}^2} \left\{ \frac{1}{2} \int_t^{t+h} \alpha_r^2 dr + u^{t, \hat{\omega}}(h, \hat{\omega}^{\alpha, t, \hat{\omega}}) \right\} \text{ for } h \geq 0.$$

Since $\varphi \in \underline{\mathcal{A}}^K u(t, \hat{\omega})$, we have for all $\|\alpha\|_\infty \leq K_0$:

$$0 \leq \frac{1}{2} \int_t^{t+h} |\alpha|_r^2 dr + u^{t, \hat{\omega}}(h, \hat{\omega}^{\alpha, t, \hat{\omega}}) - u(t, \hat{\omega}) \leq \frac{1}{2} \int_t^{t+h} |\alpha|_r^2 dr + \varphi^{t, \hat{\omega}}(h, \hat{\omega}^{\alpha, t, \hat{\omega}}) - \varphi(t, \hat{\omega}).$$

By the smoothness of φ , this provides :

$$0 \leq \frac{1}{h} \int_0^h \left(\partial_t \varphi + b \partial_x \varphi + \frac{1}{2} |\alpha|^2 + \alpha \cdot (\partial_\omega \varphi + \sigma^T \partial_x \varphi) \right)^{t, \hat{\omega}}(r, \hat{\omega}^{\alpha, t, \hat{\omega}}) dr. \quad (\text{VII.5.10})$$

By sending $h \rightarrow 0$, we obtain

$$- \left(\partial_t \varphi + b \cdot \partial_x \varphi + \inf_{|\alpha| \leq K_0} \left(\frac{1}{2} |\alpha|^2 + \alpha \cdot (\partial_\omega \varphi + \sigma^T \partial_x \varphi) \right) \right) (t, \hat{\omega}) \leq 0.$$

We next prove the viscosity supersubsolution property. Assume not, then there exists $\varphi \in \overline{\mathcal{A}}^K u(t, \hat{\omega})$ such that

$$c := - \left(\partial_t \varphi + b \cdot \partial_x \varphi + \inf_{|\alpha| \leq K_0} \left(\frac{1}{2} |\alpha|^2 + \alpha \cdot (\partial_\omega \varphi + \sigma^T \partial_x \varphi) \right) \right) (t, \hat{\omega}) > 0.$$

Without loss of generality, we may assume that $\varphi(t, \hat{\omega}) = u(t, \hat{\omega})$. Recall that $u = u_{K_0}$. Now for any $h > 0$, by the dynamic programming,

$$\begin{aligned} \varphi(t, \hat{\omega}) = u(t, \hat{\omega}) &= \inf_{\|\alpha\|_\infty \leq K_0} \left[u_h^{t, \hat{\omega}}(\hat{\omega}^{\alpha, t, \hat{\omega}}) + \frac{1}{2} \int_t^{t+h} |\alpha_s|^2 ds \right] \\ &\geq \inf_{\|\alpha\|_\infty \leq K_0} \left[\varphi_h^{t, \hat{\omega}}(\hat{\omega}^{\alpha, t, \hat{\omega}}) + \frac{1}{2} \int_t^{t+h} |\alpha_s|^2 ds \right]. \end{aligned}$$

Then,

$$\begin{aligned} 0 &\geq \inf_{\|\alpha\|_\infty \leq K_0} \left[\varphi_h^{t, \hat{\omega}}(\hat{\omega}^{\alpha, t, \hat{\omega}}) - \varphi(t, \hat{\omega}) + \frac{1}{2} \int_t^{t+h} |\alpha_s|^2 ds \right] \\ &= \inf_{\|\alpha\|_\infty \leq K_0} \int_0^h \left[\partial_t \varphi + b \cdot \partial_x \varphi + \frac{1}{2} |\alpha|^2 + \alpha \cdot (\partial_\omega \varphi + \sigma^T \partial_x \varphi) \right]^{t, \hat{\omega}}(s, \hat{\omega}^{\alpha, t, \hat{\omega}}) ds \\ &\geq \inf_{\|\alpha\|_\infty \leq K_0} \int_0^h \left[c - C \left(|\partial_t \varphi^{t, \hat{\omega}}(s, \hat{\omega}^{\alpha, t, \hat{\omega}}) - \partial_t \varphi(t, \hat{\omega})| + |\partial_{\hat{\omega}} \varphi^{t, \hat{\omega}}(s, \hat{\omega}^{\alpha, t, \hat{\omega}}) - \partial_{\hat{\omega}} \varphi(t, \hat{\omega})| \right) \right] ds \\ &\geq \left[c - \rho(d_\infty((1+K)h)) \right] h, \end{aligned}$$

which leads to a contradiction by choosing h sufficiently small. \square

6 Appendix

Proof of Lemma VII.4.1 Since O is of C^3 boundary, we may consider a ball \mathcal{V}_0 covering a part of the boundary, on which there are a local coordinate and a function $f_1 \in C^3(\mathbb{R}^{n-1}, \mathbb{R})$ such that $\partial O \cap \mathcal{V}_0 = \{f(z) := (z, f_1(z))\}$. Let

$$\eta := \frac{1}{2C_0}, \quad \text{where } C_0 := \sup_{f(z) \in \mathcal{V}_0} \|\nabla^2 f_1(z)\|,$$

where $\|\cdot\|$ denotes the spectral norm. We may find an open subset $\mathcal{V}_1 \subset \mathcal{V}_0 \cap \{x : \delta(x, \partial O) < \eta\}$ such that

$$\delta(x, \partial O) = \min_{f(z) \in \mathcal{V}_0} |x - f(z)| = |x - f(z^*(x))|, \quad \text{for all } x \in \mathcal{V}_1,$$

where z^* satisfies the first order condition :

$$x_i - z_i^* + (x_n - f_1(z^*)) \partial_{z_i} f_1(z^*) = 0, \quad \text{for } 1 \leq i \leq n-1. \quad (\text{VII.6.1})$$

Since $f_1 \in C^2$, we obtain by direct differentiation that

$$\nabla z^* = \left(I_{n-1} + \nabla f_1(z^*) \nabla f_1(z^*)^T - (x_n - f_1(z^*)) \nabla^2 f_1(z^*) \right)^{-1},$$

where the matrix on the right hand side is invertible because $(x_n - f_1(z^*)) \nabla^2 f_1(z^*) \leq \frac{1}{2} I_{n-1}$. Finally, by a standard compactness argument, we may prove that may choose η independent of \mathcal{V}_0 . This shows that $z^* \in C^1$ on a small neighborhood of the boundary.

Since $f_1 \in C^3$, we may also prove similarly that $z^* \in C^2$ on a small neighborhood of the boundary. □

Chapitre VIII

A Dual algorithm for stochastic control problems

1 Duality result for European options

1.1 Notations

We begin by introducing some basic notations. For any $k \in \mathbb{N}$ let

$$\Omega^k := \{\omega : \omega \in C([0, T], \mathbb{R}^k), \omega_0 = 0\}.$$

Let $d, m \in \mathbb{N}$ and $T > 0$. Define $\Omega := \Omega^d$, $\Theta := [0, T] \times \Omega$ and let B denote the canonical process on Ω^m with $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ the filtration generated by B . Finally, denote by \mathbb{P}_0 the Wiener measure.

For $h > 0$, consider a finite partition $\{t_i^h\}_i$ of $[0, T]$ with mesh less than h , i.e. such that $t_{i+1}^h - t_i^h \leq h$ for all i . For some $M > 0$, let A be a compact subset of

$$O_M := \{x \in \mathbb{R}^k : |x| \leq M\}, \quad \text{for some } k \in \mathbb{N},$$

and N^h be a finite h -net of A , i.e. for all $a, b \in N^h \subset A$, we have $|a - b| \leq h$. We define sets :

- $\mathcal{A} := \left\{ \varphi : \Theta \rightarrow \mathbb{R}^k : \varphi \text{ is } \mathbb{F}\text{-adapted, and takes values in } A \right\};$
- $\mathcal{A}_h := \left\{ \varphi \in \mathcal{A} : \varphi \text{ is constant on } [t_i^h, t_{i+1}^h) \text{ for } i, \text{ and takes values in } N^h \right\};$
- $\mathcal{U} := \left\{ \varphi : \Theta \rightarrow \mathbb{R}^d : \varphi \text{ is bounded and adapted} \right\};$
- $\mathcal{D}_h := \left\{ f : [0, T] \rightarrow \mathbb{R}^k : f \text{ is constant on } [t_i^h, t_{i+1}^h) \text{ for } i, \text{ and takes values in } N^h \right\}.$

For the following it is important to note that \mathcal{D}_h is a finite set of piecewise constant functions.

1.2 The Markovian case

We consider stochastic control problems of the form :

$$u_0 = \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}_0} \left[\int_0^T e^{-\int_0^t r(s, \alpha_s, X_s^\alpha) ds} f(t, \alpha_t, X_t^\alpha) dt + e^{-\int_0^T r(s, \alpha_s, X_s^\alpha) ds} g(X_T^\alpha) \right], \quad (\text{VIII.1.1})$$

where X^α is a d -dimensional controlled diffusion defined as

$$X^\alpha := \int_0^\cdot \mu(t, \alpha_t, X_t^\alpha) dt + \int_0^\cdot \sigma(t, \alpha_t, X_t^\alpha) dB_t,$$

and the functions μ, σ, f, r satisfy the following assumption.

Assumption VIII.1.1. *The functions μ, σ, f, r defined on $\mathbb{R}^+ \times A \times \mathbb{R}^d$ takes values in $\mathbb{R}^d, \mathbb{R}^{d \times m}, \mathbb{R}, \mathbb{R}$ respectively. Assume that μ, σ, f, r are uniformly bounded Hölder continuous in t , continuous in α and Lipschitz in x , uniformly in (α, x) . Also assume that $g : \mathbb{R}^d \rightarrow \mathbb{R}$ is continuous.*

Our main result is a duality in the spirit of [24] that allows us to replace the stochastic control problem by a family of suitably discretised deterministic control problems. The key analytic ingredient in our estimate is the following lemma which is a direct consequence of Theorem 2.3 in Krylov [76].

Define the function

$$u_0^h := \sup_{\alpha \in \mathcal{A}_h} \mathbb{E}^{\mathbb{P}_0} \left[\int_0^T e^{-\int_0^t r(s, \alpha_s, X_s^\alpha) ds} f(t, \alpha_t, X_t^\alpha) dt + e^{-\int_0^T r(s, \alpha_s, X_s^\alpha) ds} g(X_T^\alpha) \right].$$

Lemma VIII.1.2. *Suppose Assumption VIII.1.1 holds and g is bounded. We have*

$$u_0 = \lim_{h \rightarrow \infty} u_0^h. \quad (\text{VIII.1.2})$$

Remark VIII.1.3. *Theorem 2.3 in [76] also gives a rate of convergence for the discretisation in Lemma VIII.1.2 : There exists a constant $C > 0$ such that*

$$|u_0 - u_0^h| \leq Ch^{\frac{1}{3}}$$

for all $0 < h \leq 1$.

For the following statement, we introduce :

$$v^h := \inf_{\varphi \in \mathcal{U}} \mathbb{E}^{\mathbb{P}_0} \left[\sup_{a \in \mathcal{D}_h} \left\{ e^{-\int_0^T r(s, a_s, X_s^a) ds} g(X_T^a) + \int_0^T e^{-\int_0^t r(s, a_s, X_s^a) ds} f(t, a_t, X_t^a) dt - \int_0^T e^{-\int_0^t r(s, a_s, X_s^a) ds} \varphi_t^\top(X^a) \sigma(t, a_t, X_t^a) dB_t \right\} \right]. \quad (\text{VIII.1.3})$$

Remark VIII.1.4. It is noteworthy that stochastic integrals are defined in L^2 -space, so it is in general meaningless to take the pathwise supremum of a family of stochastic integrals. However, as we mentioned before, the set \mathcal{D}_h is of finite elements. So there is a unique random variable in L^2 equal to the maximum value of the finite number of stochastic integrals, \mathbb{P}_0 -a.s.

The following theorem allows to recover the stochastic optimal control problem as a limit of discretised deterministic control problems.

Theorem VIII.1.5. *Suppose Assumption VIII.1.1 holds and g is bounded. Then we have*

$$u_0 = \lim_{h \rightarrow 0} v^h.$$

Proof We first prove that $u_0 \leq \lim_{h \rightarrow 0} v^h$. Recall u_0^h defined in (VIII.1.2). For all $\varphi \in \mathcal{U}$, the process $\int_0^\cdot e^{-\int_0^t r(s, \alpha_s, X_s^\alpha) ds} \varphi_t^\top(X^\alpha) \sigma(t, \alpha_t, X_t^\alpha) dB_t$ is a martingale and we have

$$\begin{aligned} u_0^h &= \sup_{\alpha \in \mathcal{A}_h} \mathbb{E}^{\mathbb{P}_0} \left[e^{-\int_0^T r(s, \alpha_s, X_s^\alpha) ds} g(X_T^\alpha) + \int_0^T e^{-\int_0^t r(s, \alpha_s, X_s^\alpha) ds} f(t, \alpha_t, X_t^\alpha) dt - \int_0^T e^{-\int_0^t r(s, \alpha_s, X_s^\alpha) ds} \varphi_t^\top(X^\alpha) \sigma(t, \alpha_t, X_t^\alpha) dB_t \right] \\ &\leq \mathbb{E}^{\mathbb{P}_0} \left[\sup_{a \in \mathcal{D}_h} \left\{ e^{-\int_0^T r(s, a_s, X_s^a) ds} g(X_T^a) + \int_0^T e^{-\int_0^t r(s, a_s, X_s^a) ds} f(t, a_t, X_t^a) dt - \int_0^T e^{-\int_0^t r(s, a_s, X_s^a) ds} \varphi_t^\top(X^a) \sigma(t, a_t, X_t^a) dB_t \right\} \right]. \end{aligned}$$

The desired result follows.

To show $u_0 \geq \lim_{h \rightarrow 0} v^h$ we construct an explicit minimiser φ^* . First note that under Assumption VIII.1.1, it is easy to verify that u_t defined as

$$u_t(x) := \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}_0} \left[\int_t^T e^{-\int_t^s r(\ell, \alpha_\ell, X_\ell^\alpha) d\ell} f(s, \alpha_s, X_s^\alpha) ds + e^{-\int_t^T r(s, \alpha_s, X_s^\alpha) ds} g(X_T^\alpha) \Big| X_t^\alpha = x \right],$$

is a viscosity solution to the Dirichlet problem of the HJB equation :

$$\begin{aligned}
 & -\partial_t u - \sup_{a \in A} \mathcal{L}^a u = 0, \quad u_T = g, \\
 & \text{where } \mathcal{L}^a u := \mu(t, a, x) \cdot \partial_x u + \frac{1}{2} \text{Tr} \left((\sigma \sigma^T)(t, a, x) \partial_{xx}^2 u \right) - r(t, a, x) u + f(t, a, x). \tag{VIII.1.4}
 \end{aligned}$$

We next define the mollification $u^{(\varepsilon)} := u * K^{(\varepsilon)}$ of u , where K is a smooth function with compact support in $(-1, 0) \times O_1$ (O_1 is the unit ball in \mathbb{R}^d), and $K^{(\varepsilon)}(x) := \varepsilon^{-n-2} K(t/\varepsilon^2, x/\varepsilon)$. Clearly, $u^{(\varepsilon)} \in C_b^\infty$ and $u^{(\varepsilon)}$ converges uniformly to u . Further, by a convexity argument as in Krylov [75, proof of Theorem 2.1], we obtain that $u^{(\varepsilon)}$ is a classical supersolution to the HJB equation (VIII.1.4). Consequently for all $\alpha \in \mathcal{A}$, we have

$$\begin{aligned}
 I_\varepsilon^\alpha & := e^{-\int_0^T r(s, \alpha_s, X_s^\alpha) ds} g(X_T^\alpha) + \int_0^T e^{-\int_0^t r(s, \alpha_s, X_s^\alpha) ds} f(t, \alpha_t, X_t^\alpha) dt \\
 & \quad - e^{-\int_0^T r(s, \alpha_s, X_s^\alpha) ds} u_T^{(\varepsilon)}(X_T^\alpha) + u_0^{(\varepsilon)} + \int_0^T e^{-\int_0^t r(s, \alpha_s, X_s^\alpha) ds} \mathcal{L}^{\alpha_t} u^{(\varepsilon)}(t, X_t^\alpha) dt \\
 & \leq e^{-\int_0^T r(s, \alpha_s, X_s^\alpha) ds} \left(g(X_T^\alpha) - u_T^{(\varepsilon)}(X_T^\alpha) \right) + u_0^{(\varepsilon)}
 \end{aligned}$$

By Assumption VIII.1.1, it is clear that I_ε^α is uniformly bounded from above. It is easy to verify that the function u is continuous and therefore uniformly continuous on $S_{Lyons} := [0, T] \times \{|x| \leq L\}$ for any $L > 0$ and $u^{(\varepsilon)}$ converges uniformly to u on S_{Lyons} . Letting

$$\rho_{Lyons}(\varepsilon) := \max_{|x| \leq L} \left| g(x) - u_T^{(\varepsilon)}(x) \right|$$

we note that $\lim_{\varepsilon \rightarrow 0} \rho_{Lyons}(\varepsilon) = 0$. Therefore,

$$\begin{aligned}
 w_\varepsilon^h & := \mathbb{E}^{\mathbb{P}_0} \left[\sup_{a \in \mathcal{D}_h} I_\varepsilon^a \right] = \mathbb{E}^{\mathbb{P}_0} \left[\sup_{a \in \mathcal{D}_h} I_\varepsilon^a; \sup_{a \in \mathcal{D}_h} |X_T^a| \leq L \right] + \mathbb{E}^{\mathbb{P}_0} \left[\sup_{a \in \mathcal{D}_h} I_\varepsilon^a; \sup_{a \in \mathcal{D}_h} |X_T^a| > L \right] \\
 & \leq C \rho_{Lyons}(\varepsilon) + u_0^{(\varepsilon)} + C \mathbb{P}_0 \left[\sup_{a \in \mathcal{D}_h} |X_T^a| > L \right],
 \end{aligned}$$

where C is a constant independent of L and ε . Letting ε tend to zero we deduce that

$$\overline{\lim}_{\varepsilon \rightarrow 0} w_\varepsilon^h \leq u_0 + C \mathbb{P}_0 \left[\sup_{a \in \mathcal{D}_h} |X_T^a| > L \right]$$

for any $L > 0$. Further, since $\mathbb{P}_0 \left[\sup_{a \in \mathcal{D}_h} |X_T^a| > L \right] \rightarrow 0$ as $L \rightarrow \infty$, we conclude that

$$\overline{\lim}_{\varepsilon \rightarrow 0} w_\varepsilon^h \leq u_0. \tag{VIII.1.5}$$

It follows from the Itô formula that

$$e^{-\int_0^T r(s, a_s, X_s^a) ds} u_T^{(\varepsilon)}(X_T^a) - u_0^{(\varepsilon)} = \int_0^T e^{-\int_0^t r(s, a_s, X_s^a) ds} \mathcal{L}^{a_t} u^{(\varepsilon)}(t, X_t^a) dt + \int_0^T e^{-\int_0^t r(s, a_s, X_s^a) ds} \partial_x u_t^{(\varepsilon)}(X_t^a)^\top \sigma(t, a_t, X_t^a) dB_t, \quad \text{for all } a \in \mathcal{D}_h, \quad \mathbb{P}_0\text{-a.s.}$$

and, therefore,

$$w_\varepsilon^h = \mathbb{E}^{\mathbb{P}_0} \left[\sup_{a \in \mathcal{D}_h} \left\{ e^{-\int_0^T r(s, a_s, X_s^a) ds} g(X_T^a) + \int_0^T e^{-\int_0^t r(s, a_s, X_s^a) ds} f(t, a_t, X_t^a) dt - \int_0^T e^{-\int_0^t r(s, a_s, X_s^a) ds} \partial_x u_t^{(\varepsilon)}(X_t^a)^\top \sigma(t, a_t, X_t^a) dB_t \right\} \right] \geq v^h. \quad (\text{VIII.1.6})$$

Combining (VIII.1.5) and (VIII.1.6), we conclude that $v^h \leq u_0$ for all $1 \geq h > 0$. \square

The boundedness assumption on g may be relaxed by means of a simple cut off argument :

Corollary VIII.1.6. *Assume that g is of polynomial growth. Let $M > 0$, g^M a continuous compactly supported function that agrees with g on $O_M \subseteq \mathbb{R}^d$ and satisfies $|g^M| \leq |g|$. Let $v^{h,M}$ denote the approximations defined in (VIII.1.3), with respect to g^M in place of g . Then we have*

$$\lim_{M \rightarrow 0} \left| u_0 - \lim_{h \rightarrow 0} v^{h,M} \right| = 0.$$

Proof Define u_0^M as in (VIII.1.1) by using the approximation g^M . By Theorem VIII.1.5, we know that $u_0^M = \lim_{h \rightarrow 0} v^{h,M}$. Further, we have

$$\begin{aligned} |u_0 - u_0^M| &\leq C \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}_0} \left[|g(X_T^\alpha) - g^M(X_T^\alpha)| \right] \\ &\leq C \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}_0} \left[|X_T^\alpha|^p + 1; |X_T^\alpha| \geq M \right] \\ &\leq \frac{C'}{M}. \end{aligned}$$

The last estimate is due the Chebyshev inequality and the moment estimate in Krylov [77] (Lemma 2 on page 78). The proof is completed. \square

We conclude the section with two remarks, both relevant to the numerical simulation of the approximation derived in Theorem VIII.1.5.

Remark VIII.1.7. *To approximate v^h in our numerical examples we will as in the proof of Theorem VIII.1.5 use fixed functions φ_h^* for the minimisation. The calculations in (VIII.1.5) and (VIII.1.6) make it clear that the natural choice for these minimisers are the (numerical*

approximations) of the function $\partial_x u_t$. Note that these approximations are readily available from the numerical scheme [60] that is used to compute the complementary lower bounds.

Remark VIII.1.8. In the proof of Theorem VIII.1.5 we showed that $u_0^h \leq v^h \leq u_0$. It therefore follows from Remark VIII.1.3 that there exists a constant $C > 0$ such that

$$|u_0 - v^h| \leq Ch^{\frac{1}{3}}$$

for all $0 < h \leq 1 \wedge T$.

2 Some extensions

2.1 The non-Markovian case

In our first extension we consider stochastic control problems of the form

$$u_0 = \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}_0} \left[g(X_{T \wedge \cdot}^\alpha) \right],$$

where X^α is d -dimensional diffusion defined by $X^\alpha := \int_0^\cdot \mu(t, \alpha_t) dt + \int_0^\cdot \sigma(t, \alpha_t) dB_t$. Note that in this setting μ and σ only depend on α and t , but the payoff function g is path dependent.

Remark VIII.2.1. The arguments in this subsection are based on the "path-freezing" approach developed in Ekren, Touzi and Zhang [38]. In order to be able to apply their approach we have restricted the class of diffusions X^α we consider compared to the Markovian control problem.

Writing $\mathbb{P}_\alpha := \mathbb{P}_0 \circ (X^\alpha)^{-1}$, we have

$$u_0 = \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}_\alpha} \left[g(B_{T \wedge \cdot}) \right].$$

Throughout this subsection we will impose the following regularity assumptions.

Assumption VIII.2.2. The functions $\mu, \sigma : \mathbb{R}^+ \times A \rightarrow E$ (E is the respective metric space) and $g : \Omega^d \rightarrow \mathbb{R}$ are uniformly bounded such that

- (i) μ, σ are Hölder continuous in t , continuous in α ;
- (ii) g is uniformly continuous.

Example VIII.2.3. *Arguing as in Corollary VIII.1.6 we may also consider unbounded payoffs. Hence, possible path-dependent payoffs that fit our framework include e.g. the maximum $\max_{s \in [0, T]} \omega_s$ and Asian options $\int_0^T \omega_s ds$.*

Let $\mathbb{L}_\varepsilon := \{t_0 = 0, t_1, t_2, \dots, t_n = T\}$ be a partition of $[0, T]$ with mesh bounded above by ε . For $k \leq n$ and $\pi_k = (x_1 = 0, x_2, \dots, x_k) \in \mathbb{R}^{d \times k}$, denote by $\Gamma_\varepsilon^{\mathbb{L}_\varepsilon, k}(\pi_k)$ the path generated by the linear interpolation of the points $\{(t_i, x_k)\}_{0 \leq i \leq k}$. Where no confusion arises with regards to the underlying partition we will in the following drop the superscript \mathbb{L}_ε and write $\Gamma_\varepsilon^k(\pi_k)$ in place of $\Gamma_\varepsilon^{\mathbb{L}_\varepsilon, k}(\pi_k)$, but it must be emphasised that the entire analysis in this subsection is carried out with a fixed but arbitrary partition \mathbb{L}_ε in mind. Define the interpolation approximation of g by

$$g^\varepsilon(\pi_n) := g\left(\Gamma_\varepsilon^n(\pi_n)\right)$$

and define an approximation of the value function by letting

$$\theta_0^\varepsilon := \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}^\alpha} \left[g^\varepsilon\left((B_{t_i})_{0 \leq i \leq n}\right) \right].$$

The following lemma justifies the use of linear interpolation for approximating dependent payoff.

Lemma VIII.2.4. *Under Assumption 2.1, we have*

$$\lim_{\varepsilon \rightarrow 0} \theta_0^\varepsilon = u_0.$$

Proof Recall that g is uniformly continuous. Let ρ be a modulus of continuity of g . If necessary, we may choose ρ to be concave. Further, we define

$$\mathbf{w}_B(\varepsilon, T) := \sup_{s, t \leq T; |s-t| \leq \varepsilon} |B_s - B_t|.$$

Clearly, we have

$$\left| \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}^\alpha} \left[g^\varepsilon\left((B_{t_i})_{0 \leq i \leq n}\right) \right] - \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}^\alpha} \left[g(B_{T \wedge \cdot}) \right] \right| \leq \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}^\alpha} \left[\rho\left(\mathbf{w}_B(\varepsilon, T)\right) \right] \leq \rho\left(\sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}^\alpha} \left[\mathbf{w}_B(\varepsilon, T) \right] \right),$$

It follows from Theorem 1 in Fisher and Nappo [50] that $\mathbb{E}^{\mathbb{P}^\alpha} \left[\mathbf{w}_B(\varepsilon, T) \right]$ converges to 0 uniformly in α , as $\varepsilon \rightarrow 0$. \square

We next define the controlled diffusion with time-shifted coefficients by setting

$$X^{\alpha, t} := \int_0^s \mu(t+r, \alpha_r) dr + \int_0^s \sigma(t+r, \alpha_r) dB_r, \quad s \in [0, T-t], \quad \mathbb{P}_0\text{-a.s.},$$

and the corresponding law :

$$\mathbb{P}_\alpha^t := \mathbb{P}_0 \circ (X^{\alpha,t})^{-1}.$$

Further, for $1 \leq k \leq n-2$ let $\eta_k := t_{k+1} - t_k$, and define recursively a family of stochastic control problems :

$$\begin{aligned} \theta^\varepsilon(\pi_{n-1}; t, x) &:= \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}_\alpha^{t_{n-1}+t}} \left[g^\varepsilon \left((\pi_{n-1}, x_{n-1} + x + B_{\eta_{n-1}-t}) \right) \right], \quad t \in [0, \eta_{n-1}), \quad x \in \mathbb{R}^d \\ \theta^\varepsilon(\pi_k; t, x) &:= \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}_\alpha^{t_k+t}} \left[\theta^\varepsilon \left((\pi_k, x_k + x + B_{\eta_k-t}), 0, 0 \right) \right], \quad t \in [0, \eta_k), \quad x \in \mathbb{R}^d. \end{aligned} \tag{VIII.2.1}$$

Clearly, $\theta^\varepsilon(0; 0, 0) = \theta_0^\varepsilon$.

Lemma VIII.2.5. *Fix $\varepsilon > 0$. The function $\theta^\varepsilon(\pi; t, x)$ is Borel-measurable in all the arguments and uniformly continuous in (t, x) uniformly in π .*

Proof It follows from the uniform continuity of g and the fact that interpolation with respect to a partition \mathbb{L}_ε is a Lipschitz function (in this case from $\mathbb{R}^{n \times d}$ into the continuous functions), that g^ε is also uniformly continuous. Denote by ρ^ε a modulus of continuity of g^ε , chosen to be increasing and concave if necessary. For any $\pi_{n-1}, \pi'_{n-1} \in \mathbb{R}^{(n-1) \times d}$, given $t \in [0, \eta_{n-1}]$, $x, x' \in \mathbb{R}^d$, we have

$$\begin{aligned} &|\theta^\varepsilon(\pi_{n-1}; t, x) - \theta^\varepsilon(\pi'_{n-1}; t, x')| \\ &\leq \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}_\alpha^{t_{n-1}+t}} \left[\left| g^\varepsilon \left((\pi_{n-1}, x_{n-1} + x + B_{\eta_{n-1}-t}) \right) - g^\varepsilon \left((\pi'_{n-1}, x_{n-1} + x' + B_{\eta_{n-1}-t}) \right) \right| \right] \\ &\leq \rho^\varepsilon(|(\pi_{n-1}, x) - (\pi'_{n-1}, x')|). \end{aligned}$$

Similarly, for any $k < n-1$ and $\pi_k, \pi'_k \in \mathbb{R}^{k \times d}$, given $t \in [0, \eta_k]$, $x, x' \in \mathbb{R}^d$, we have

$$\begin{aligned} &|\theta^\varepsilon(\pi_k; t, x) - \theta^\varepsilon(\pi'_k; t, x')| \\ &\leq \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}_\alpha^{t_k+t}} \left[\left| \theta^\varepsilon \left((\pi_k, x_k + x + B_{\eta_k-t}), 0, 0 \right) - \theta^\varepsilon \left((\pi'_k, x_k + x' + B_{\eta_k-t}), 0, 0 \right) \right| \right] \\ &\leq \rho^\varepsilon(|(\pi_k, x) - (\pi'_k, x')|). \end{aligned} \tag{VIII.2.2}$$

For $0 \leq t^0 < t^1 \leq \eta_k$, it follows from the dynamic programming principle that

$$\theta^\varepsilon(\pi_k; t^0, x) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}_\alpha^{t_k+t^0}} \left[\theta^\varepsilon(\pi_k; t^1, x + B_{t^1-t^0}) \right] \tag{VIII.2.3}$$

and (VIII.2.3) and (VIII.2.2) we deduce that

$$\begin{aligned}
 |\theta^\varepsilon(\pi_k; t^0, x) - \theta^\varepsilon(\pi_k; t^1, x)| &\leq \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}_\alpha^{t_k+t^0}} \left[\left| \theta^\varepsilon(\pi_k; t^1, x + B_{t^1-t^0}) - \theta^\varepsilon(\pi_k; t^1, x) \right| \right] \\
 &\leq \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}_\alpha^{t_k+t^0}} \left[\rho^\varepsilon(|B_{t^1-t^0}|) \right] \\
 &\leq \rho^\varepsilon \left(\sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}_\alpha^{t_k+t^0}} [|B_{t^1-t^0}|] \right).
 \end{aligned} \tag{VIII.2.4}$$

For any $\hat{\mu}$ and $\hat{\sigma}$ satisfying Assumption 2.1 define the controlled diffusion and the corresponding law :

$$X_t^{\hat{\mu}, \hat{\sigma}} = \int_0^t \hat{\mu}_s ds + \int_0^t \hat{\sigma}_s dB_s, \quad \mathbb{P}^{\hat{\mu}, \hat{\sigma}} := \mathbb{P}_0 \circ (X^{\hat{\mu}, \hat{\sigma}})^{-1}.$$

Note that the bound

$$\sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}_\alpha^{t_k+t^0}} [|B_{t^1-t^0}|] \leq \sup_{|\hat{\mu}|, |\hat{\sigma}| \leq C} \mathbb{E}^{\mathbb{P}^{\hat{\mu}, \hat{\sigma}}} [|B_{t^1-t^0}|] \tag{VIII.2.5}$$

does not depend on π_k and t_0 . It follows from (VIII.2.4) that

$$|\theta^\varepsilon(\pi_k; t^0, x) - \theta^\varepsilon(\pi_k; t^1, x)| \leq \rho^\varepsilon \left(\sup_{|\hat{\mu}|, |\hat{\sigma}| \leq C} \mathbb{E}^{\mathbb{P}^{\hat{\mu}, \hat{\sigma}}} [|B_{t^1-t^0}|] \right). \tag{VIII.2.6}$$

Hence, combining (VIII.2.2) and (VIII.2.5) we conclude that $\theta^\varepsilon(\pi_k; t, x)$ is uniformly continuous in (t, x) uniformly in π_k . \square

The functions $\theta^\varepsilon(\pi_k; \cdot, \cdot)$ are defined as the value functions of stochastic control problems, and one can easily check that they are viscosity solutions to the corresponding Hamilton-Jacobi-Bellman equations. For $j = 1, \dots, n-1$, we define a family of PDEs by letting

$$\begin{aligned}
 &\mathbf{Lyons}^j \theta = 0, \quad \text{on } [0, \eta_j) \otimes \mathbb{R}^d, \\
 &\text{where } \mathbf{Lyons}^j \theta := -\partial_t \theta - \sup_{\alpha \in \mathcal{A}} \left\{ \mu(t_j + \cdot, \alpha) \cdot \partial_x \theta + \frac{1}{2} \text{Tr} \left((\sigma \sigma^\top)(t_j + \cdot, \alpha) \partial_{xx}^2 \theta \right) \right\}.
 \end{aligned} \tag{VIII.2.7}$$

The following proposition links the stochastic control problems with the PDE and applies, analogous to the Markovian case, a mollification argument.

Proposition VIII.2.6. *There exists a function $u^{(\varepsilon)} : (\pi, t, x) \mapsto \mathbb{R}$ such that $u^{(\varepsilon)}(0, 0, 0) = \theta_0^\varepsilon + \varepsilon$ and for all $\pi_k \in \Pi_\varepsilon$, $u^{(\varepsilon)}(\pi_k; \cdot, \cdot)$ is a classical supersolution to the PDE (VIII.2.7) with $j = k$ and the boundary condition :*

$$\begin{aligned}
 u^{(\varepsilon)}(\pi_k; \eta_k, x) &= u^{(\varepsilon)}((\pi_k, x); 0, 0), & \text{if } k < n-1; \\
 u^{(\varepsilon)}(\pi_k; \eta_k, x) &\geq g^\varepsilon((\pi_k, x)), & \text{if } k = n-1.
 \end{aligned}$$

Proof Define $\theta^{\varepsilon,\delta}(\pi_k; \cdot, \cdot) := \theta^\varepsilon(\pi_k; \cdot, \cdot) * K^\delta$ for all $\pi_k \in \mathbb{R}^{k \times d}$, $k \leq n$, where K is a smooth function with compact support in $(-1, 0) \times O_1$ (O_1 is the unit ball in \mathbb{R}^d), and $K^\delta(t, x) := \delta^{-d-2} K(t/\delta^2, x/\delta)$. By Lemma VIII.2.5, $\theta^{\varepsilon,\delta}(\pi_k; \cdot, \cdot)$ converges uniformly to $\theta^\varepsilon(\pi_k; \cdot, \cdot)$ uniformly in π_k , as $\delta \rightarrow 0$. Take δ small enough so that $\|\theta^{\varepsilon,\delta} - \theta^\varepsilon\| \leq \frac{\varepsilon}{2n}$. As in the Markovian case (compare the proof of Theorem VIII.1.5) using a convexity argument analogous Krylov [75], we can prove that $\theta^{\varepsilon,\delta}(\pi_k; \cdot, \cdot)$ is a classical supersolution for (VIII.2.7). Note that $\theta^{\varepsilon,\delta}(\pi_k; \cdot, \cdot) + c$ is still a supersolution for any constant c . So there exists a smooth function $v^\varepsilon(0; \cdot, \cdot)$ on $[0, t_1] \times \mathbb{R}^d$ such that

$$v^\varepsilon(0; 0, 0) = \theta^\varepsilon(0; 0, 0) + \frac{\varepsilon}{n}, \quad v^\varepsilon(0; \cdot, \cdot) \geq \theta^\varepsilon(0; \cdot, \cdot)$$

and smooth functions $v^\varepsilon(\pi_k; \cdot, \cdot)$ on $[0, \eta_k] \times \mathbb{R}^d$ for $1 \leq k \leq n-1$ such that

$$v^\varepsilon(\pi_k; 0, 0) = v^\varepsilon(\pi_{k-1}; \eta_{k-1}, x_k - x_{k-1}) + \frac{\varepsilon}{n}, \quad v^\varepsilon(\pi_k; \cdot, \cdot) \geq \theta^\varepsilon(\pi_k; \cdot, \cdot).$$

Finally, we define for $\pi_k \in \mathbb{R}^{k \times d}$ and $(t, x) \in [0, \eta_k] \times \mathbb{R}^d$

$$u^{(\varepsilon)}(\pi_k; t, x) := v^\varepsilon(\pi_k; t, x) + \frac{n-k+1}{n}\varepsilon.$$

It is now straightforward to check that $u^{(\varepsilon)}$ satisfies the requirements. \square

The discrete framework we just developed may be linked to pathspace by means of linear interpolation along the partition \mathbb{L}_ε . Recall that Θ was defined to be $[0, T] \times \Omega$.

Corollary VIII.2.7. Define $\bar{u}^{(\varepsilon)} : \Theta \rightarrow \mathbb{R}$ by

$$\bar{u}^{(\varepsilon)}(t, \omega) := u^{(\varepsilon)}\left((\omega_{t_i})_{0 \leq i \leq k}; t - t_k, \omega_t - \omega_{t_k}\right), \quad \text{for } t \in [t_k, t_{k+1}).$$

There exist adapted processes $\lambda(t, \alpha_t)$, $\mu(t, \alpha_t)$, $\varphi_t(x)$, $\eta(t, x)$ such that for all $\alpha \in \mathcal{A}$

$$\bar{u}^{(\varepsilon)}(T, X^\alpha) = \bar{u}_0^{(\varepsilon)} + \int_0^T \left(\lambda + \mu(t, \alpha_t)\varphi + \frac{1}{2} \text{Tr}\left((\sigma\sigma^\top)(t, \alpha_t)\eta\right)(t, X^\alpha) \right) dt + \int_0^T \varphi_t^\top(X^\alpha)\sigma(t, \alpha_t)dB_t,$$

\mathbb{P}_0 -a.s., and

$$\left(\lambda + \mu(t, \alpha)\varphi + \frac{1}{2} \text{Tr}\left((\sigma\sigma^\top)(t, \alpha)\eta\right)(t, \omega) \right) \leq 0, \quad \text{for all } \alpha \in \mathcal{A}, (t, \omega) \in \Theta.$$

Proof The result follows by applying Itô's formula on each interval $[t_k, t_{k+1})$ and using the supersolution property of $u^{(\varepsilon)}$ in Proposition VIII.2.6. \square

Finally, we prove an approximation analogous to Theorem VIII.1.5 in our non-Markovian

setting.

Theorem VIII.2.8. *Suppose Assumption VIII.2.2 holds. Then we have*

$$u_0 = \lim_{h \rightarrow 0} v^h, \text{ where } v^h := \inf_{\varphi \in \mathcal{U}} \mathbb{E}^{\mathbb{P}_0} \left[\sup_{a \in \mathcal{D}_h} \left\{ g(X_{T \wedge \cdot}^a) - \int_0^T \varphi_t^\top(X^a) \sigma(t, a_t) dB_t \right\} \right].$$

Proof Arguing as in the proof of Theorem VIII.1.5, one can easily deduce using the Ito formula that $u_0 \leq \lim_{h \rightarrow 0} v^h$.

Consider the function $\bar{u}^{(\varepsilon)}$ and let φ be the process defined in Corollary VIII.2.7. We have

$$\begin{aligned} v_h &\leq \mathbb{E}^{\mathbb{P}_0} \left[\sup_{a \in \mathcal{D}_h} \left\{ g(X_{T \wedge \cdot}^a) - \int_0^T \varphi_t^\top(X^a) \sigma(t, a_t) dB_t \right\} \right] \\ &\leq \mathbb{E}^{\mathbb{P}_0} \left[\sup_{a \in \mathcal{D}_h} \left\{ g(X_{T \wedge \cdot}^a) - \bar{u}_T^{(\varepsilon)}(X^a) + \bar{u}_0^{(\varepsilon)} \right\} \right] \\ &\leq \mathbb{E}^{\mathbb{P}_0} \left[\sup_{a \in \mathcal{D}_h} \left\{ g(X_{T \wedge \cdot}^a) - g^\varepsilon((X_{t_i}^a)_{0 \leq i \leq n}) \right\} \right] + \theta_0^\varepsilon + \varepsilon. \end{aligned}$$

For the last inequality, we use the fact that $\bar{u}_0^{(\varepsilon)} = u^{(\varepsilon)}(0; 0, 0) = \theta_0^\varepsilon + \varepsilon$. Note that there are only finite elements in the set \mathcal{D}_h . Therefore, by Lemma VIII.2.4

$$\begin{aligned} &\overline{\lim}_{\varepsilon \rightarrow 0} \left(\mathbb{E}^{\mathbb{P}_0} \left[\sup_{a \in \mathcal{D}_h} \left\{ g(X_{T \wedge \cdot}^a) - g^\varepsilon((X_{t_i}^a)_{0 \leq i \leq n}) \right\} \right] + \theta_0^\varepsilon + \varepsilon \right) \\ &\leq \overline{\lim}_{\varepsilon \rightarrow 0} \left(\sum_{a \in \mathcal{D}_h} \mathbb{E}^{\mathbb{P}_0} \left[\left| g(X_{T \wedge \cdot}^a) - g^\varepsilon((X_{t_i}^a)_{0 \leq i \leq n}) \right| \right] + \theta_0^\varepsilon + \varepsilon \right) \\ &= u_0. \end{aligned}$$

We conclude that $v^h \leq u_0$ for all $h \in (0, 1 \wedge T]$. □

2.2 Example of a duality result for an American option

In this subsection we give an indication how our approach may be extended to American options. To this end we consider a toy model, in which the d -dimensional controlled diffusion X^α takes the particular form $X^\alpha := \int_0^\cdot \alpha_t^0 dt + \int_0^\cdot \alpha_t^1 dB_t$ and carry out the analysis in this elementary setting. The stochastic control problem is now

$$u_0 = \sup_{\alpha \in \mathcal{A}, \tau \in \mathcal{T}_T} \mathbb{E}^{\mathbb{P}_0} \left[g(X_\tau^\alpha) \right],$$

where \mathcal{T}_T is the set of all stopping times smaller than T . Throughout this subsection we will make the following assumption :

Assumption VIII.2.9. *Suppose $g : \mathbb{R}^d \rightarrow \mathbb{R}$ to be bounded and uniformly continuous.*

For $\alpha \in \mathcal{A}$ define probability measures $\mathbb{P}_\alpha := \mathbb{P}_0 \circ (X^\alpha)^{-1}$, let $\mathcal{P} := \{\mathbb{P}_\alpha : \alpha \in \mathcal{A}\}$ and define the nonlinear expectation $\mathcal{E}[\cdot] := \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^\mathbb{P}[\cdot]$. It will be convenient to use the shorthand $\alpha^1 \cdot B$ for the stochastic integral $\int_0^1 \alpha_s^1 dB_s$. We have

$$u_0 = \sup_{\tau \in \mathcal{T}_T} \mathcal{E} \left[g(B_\tau) \right].$$

Further, we define the dynamic version of the control problem :

$$u_t(x) := \sup_{\tau \in \mathcal{T}_{T-t}} \mathcal{E} \left[g(x + B_\tau) \right], \quad \text{for } (t, x) \in [-1, T] \times \mathbb{R}^d.$$

The following lemma shows that the function u satisfies a dynamic programming principle (see for example Lemma 4.1 of [36] for a proof).

Lemma VIII.2.10. *The value function u is continuous in both arguments, and we have*

$$u_{t_1}(x) = \sup_{\tau \in \mathcal{T}_{T-t_1}} \mathcal{E} \left[g(x + B_\tau) 1_{\{\tau < t_2\}} + u_{t_2} 1_{\{\tau \geq t_2\}} \right].$$

In particular, u is a \mathbb{P} -supermartingale for all $\mathbb{P} \in \mathcal{P}$.

Next we apply the familiar mollification technique already employed in Section 1.2. Define $u^{(\varepsilon)} := u * K^{(\varepsilon)}$.

Lemma VIII.2.11. *$\{u^{(\varepsilon)}(t, B_t)\}_t$ is a \mathbb{P} -supermartingale for all $\mathbb{P} \in \mathcal{P}$, and $u^{(\varepsilon)} \geq g^{(\varepsilon)} := g * K^{(\varepsilon)}$.*

Proof For any $s \leq t \leq T$ and $x \in \mathbb{R}$, we have by Lemma VIII.2.10

$$\begin{aligned} \mathcal{E} \left[u^{(\varepsilon)}(t, x + B_{t-s}) \right] &= \mathcal{E} \left[\int u(t-r, x-y + B_{t-s}) K^{(\varepsilon)}(r, y) dy dr \right] \\ &\leq \int \mathcal{E} \left[u(t-r, x-y + B_{t-s}) \right] K^{(\varepsilon)}(r, y) dy dr \\ &\leq \int u(s-r, x-y) K^{(\varepsilon)}(r, y) dy dr = u^{(\varepsilon)}(s, x). \end{aligned}$$

This implies that for all $\mathbb{P} \in \mathcal{P}$ we have

$$\mathbb{E}^\mathbb{P} \left[u^{(\varepsilon)}(t, x + B_{t-s}) \right] \leq u^{(\varepsilon)}(s, x).$$

Therefore, $\{u^{(\varepsilon)}(t, B_t)\}_t$ is a \mathbb{P} -supermartingale for all $\mathbb{P} \in \mathcal{P}$. On the other hand, it is clear from the definition of u that $u \geq g$ and, hence, $u^{(\varepsilon)} \geq g^{(\varepsilon)}$. \square

Again, the stochastic control problem can be discretised.

Lemma VIII.2.12. *Under Assumption VIII.2.9, it holds*

$$u_0 = \lim_{h \rightarrow 0} u_0^h, \quad \text{where } u_0^h := \sup_{\alpha \in \mathcal{A}_h, \tau \in \mathcal{J}_T} \mathbb{E}^{\mathbb{P}_0} \left[g(X_\tau^\alpha) \right]. \quad (\text{VIII.2.8})$$

Proof We only prove the case $\alpha^0 = 0$ and $\alpha = \alpha^1 \in \mathbb{R}$, i.e. $X^\alpha = (\alpha \cdot B)$. The general case follows by a straightforward generalisation of the same arguments. Note that it is sufficient to show that $u_0 \leq \underline{\lim}_{h \rightarrow 0} u_0^h$. Fix $\varepsilon > 0$. There exists $\alpha^\varepsilon \in \mathcal{A}$ such that

$$u_0 < \sup_{\tau \in \mathcal{J}_T} \mathbb{E}^{\mathbb{P}_0} \left[g((\alpha^\varepsilon \cdot B)_\tau) \right] + \varepsilon. \quad (\text{VIII.2.9})$$

For any h sufficiently small define a process $\tilde{\alpha}^h$ by letting

$$\tilde{\alpha}_t^h := \sum_i \frac{1}{t_{i+1}^h - t_i^h} \int_{t_i^h}^{t_{i+1}^h} \mathbb{E}_{t_i^h}^{\mathbb{P}_0} (\alpha_s^\varepsilon) ds \mathbf{1}_{[t_i^h, t_{i+1}^h)}(t).$$

Clearly, $\tilde{\alpha}^h$ is piecewise constant on each interval $[t_i^h, t_{i+1}^h)$. Further, define $\hat{\alpha}^h := h \left\lfloor \frac{\tilde{\alpha}^h}{h} \right\rfloor$ and note that we have $\hat{\alpha}^h \in \mathcal{A}_h$. A standard argument using the martingale convergence theorem yields

$$\lim_{h \rightarrow 0} \mathbb{E}^{\mathbb{P}_0} \int_0^T (\alpha_s^\varepsilon - \tilde{\alpha}_s^h)^2 ds = 0 \quad (\text{VIII.2.10})$$

and, hence,

$$\lim_{h \rightarrow 0} \mathbb{E}^{\mathbb{P}_0} \int_0^T (\alpha_s^\varepsilon - \hat{\alpha}_s^h)^2 ds = 0.$$

With ρ an increasing and concave modulus of continuity of g we have

$$\begin{aligned} \sup_{\tau \in \mathcal{J}_T} \mathbb{E}^{\mathbb{P}_0} \left[g((\alpha^\varepsilon \cdot B)_\tau) \right] - \sup_{\tau \in \mathcal{J}_T} \mathbb{E}^{\mathbb{P}_0} \left[g((\hat{\alpha}^h \cdot B)_\tau) \right] &\leq \mathbb{E}^{\mathbb{P}_0} \left[\rho(\|(\alpha^\varepsilon \cdot B) - (\hat{\alpha}^h \cdot B)\|) \right] \\ &\leq \rho \left(\mathbb{E}^{\mathbb{P}_0} \left[\|(\alpha^\varepsilon \cdot B) - (\hat{\alpha}^h \cdot B)\|^2 \right]^{\frac{1}{2}} \right) \\ &= \rho \left(\mathbb{E}^{\mathbb{P}_0} \left[\int_0^T (\alpha_s^\varepsilon - \hat{\alpha}_s^h)^2 ds \right] \right) \end{aligned} \quad (\text{VIII.2.11})$$

Combining (VIII.2.9), (VIII.2.11) we have

$$\begin{aligned} u_0 &< \sup_{\tau \in \mathcal{T}_T} \mathbb{E}^{\mathbb{P}^0} \left[g \left((\hat{\alpha}^h \cdot B)_\tau \right) \right] + \rho \left(\mathbb{E}^{\mathbb{P}^0} \left[\int_0^T (\alpha_s^\varepsilon - \hat{\alpha}_s^h)^2 ds \right] \right) + \varepsilon \\ &\leq u_0^h + \rho \left(\mathbb{E}^{\mathbb{P}^0} \left[\int_0^T (\alpha_s^\varepsilon - \hat{\alpha}_s^h)^2 ds \right] \right) + \varepsilon. \end{aligned}$$

Letting $h \rightarrow 0$ we deduce

$$u_0 \leq \underline{\lim}_{h \rightarrow 0} u_0^h + \varepsilon.$$

for all $\varepsilon > 0$. □

We conclude the section by proving the analogous approximation result for American options.

Theorem VIII.2.13. *Suppose Assumption VIII.2.9 holds. Then we have*

$$u_0 = \lim_{h \rightarrow 0} v^h, \text{ where } v^h := \inf_{\varphi \in \mathcal{U}} \mathbb{E}^{\mathbb{P}^0} \left[\sup_{\alpha \in \mathcal{D}_h, t \in [0, T]} \left\{ g(X_t^\alpha) - \int_0^t \varphi_s^\top(X^\alpha) \alpha_s dB_s \right\} \right].$$

Proof We first prove that the left hand side is smaller. Recall u_0^h defined in (VIII.2.8). For all $\varphi \in \mathcal{U}$, the process $\int_0^\cdot \varphi_t^\top(X^\alpha) \alpha_t^1 dB_t$ is a martingale, and we have

$$\begin{aligned} u_0^h &\leq \sup_{\alpha \in \mathcal{A}_h, \tau \in \mathcal{T}_T} \mathbb{E}^{\mathbb{P}^0} \left[g(X_\tau^\alpha) - \int_0^\tau \varphi_t^\top(X^\alpha) \alpha_t^1 dB_t \right] \\ &\leq \mathbb{E}^{\mathbb{P}^0} \left[\sup_{a \in \mathcal{D}_h, t \in [0, T]} \left\{ g(X_t^a) - \int_0^t \varphi_s^\top(X^a) a_s^1 dB_s \right\} \right], \text{ for all } \varphi \in \mathcal{U}. \end{aligned}$$

The desired result follows by Lemma VIII.2.12. For the converse note that since $u^{(\varepsilon)} \in C^{1,2}$ and $u^{(\varepsilon)}(t, B_t)$ is a \mathbb{P} -supermartingale for all $\mathbb{P} \in \mathcal{P}$, we have

$$\partial_t u^{(\varepsilon)} + \sup_{a \in A} \left\{ a^0 \partial_x u^{(\varepsilon)} + \frac{1}{2} \text{Tr} \left(a_s^1 (a_s^1)^T \partial_{xx}^2 u^{(\varepsilon)} \right) \right\} \leq 0.$$

Hence, for all $h > 0$

$$\begin{aligned}
 v_h &\leq \mathbb{E}^{\mathbb{P}_0} \left[\sup_{a \in \mathcal{D}_h, t \in [0, T]} \left\{ g(X_t^a) - \int_0^t (\partial_x u_s^{(\varepsilon)})^T (X_s^a) a_s^1 dB_s \right\} \right] \\
 &\leq \mathbb{E}^{\mathbb{P}_0} \left[\sup_{a \in \mathcal{D}_h, t \in [0, T]} \left\{ g(X_t^a) - u_t^{(\varepsilon)}(X_t^a) + u_0^{(\varepsilon)} \right. \right. \\
 &\quad \left. \left. + \int_0^t \left(\partial_t u_s^{(\varepsilon)}(X_s^a) + a_s^0 \cdot \partial_x u_s^{(\varepsilon)}(X_s^a) + \frac{1}{2} \text{Tr} \left(a_s^1 (a_s^1)^T \partial_{xx}^2 u_s^{(\varepsilon)}(X_s^a) \right) \right) ds \right\} \right] \\
 &\leq \mathbb{E}^{\mathbb{P}_0} \left[\sup_{a \in \mathcal{D}_h, t \in [0, T]} \left\{ g(X_t^a) - g^{(\varepsilon)}(X_t^a) \right\} \right] + u_0^{(\varepsilon)},
 \end{aligned}$$

where we have used Ito's formula and the inequality $u^{(\varepsilon)} \geq g^{(\varepsilon)}$ proved in Lemma VIII.2.11. It is straightforward to check that

$$\lim_{\varepsilon \rightarrow 0} \left[\mathbb{E}^{\mathbb{P}_0} \left[\sup_{a \in \mathcal{D}_h, t \in [0, T]} \left\{ g(X_t^a) - g^{(\varepsilon)}(X_t^a) \right\} \right] + u_0^{(\varepsilon)} \right] = u_0.$$

□

3 Examples :

3.1 Uncertain volatility model

As a first example, we consider an uncertain volatility model (UVM), first considered in [1] and [83]. We will consider a range of options with payoff F_T at a maturity T . Let $D \subseteq \mathbb{R}^d \times \mathbb{R}^{d \times d}$ be a compact domain such that for all $\xi := (\sigma^i, \rho^{ij})_{1 \leq i, j \leq d} \in D$ the matrix

$$\left(\rho^{ij} \sigma^i \sigma^j \right)_{1 \leq i, j \leq d}$$

is positive semi-definite, $\rho^{ij} = \rho^{ji} \in [-1, 1]$ and $\rho^{ii} = 1$. If $d = 2$ an example of such a domain is obtained by setting

$$D = \left(\prod_{i=1}^2 [\underline{\sigma}^i, \bar{\sigma}^i] \right) \times \left\{ \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} : \rho \in [\underline{\rho}, \bar{\rho}] \right\},$$

where $0 \leq \underline{\sigma}^i \leq \bar{\sigma}^i$ and $-1 \leq \underline{\rho} \leq \bar{\rho} \leq 1$. An adapted process $(\sigma, \rho) = (\sigma_t, \rho_t)_{0 \leq t \leq T}$ is in the set of admissible volatility processes Ξ_D if it takes values in D .

In the UVM the stock prices follow the dynamics

$$dX_t^i = \sigma_t^i X_t^i dW_t^i, \quad d\langle W^i, W^j \rangle_t = \rho^{ij} dt, \quad 1 \leq i < j \leq d,$$

where $(\sigma, \rho) \in \Xi_D$ is the unknown volatility process and correlation. The time- t value of the option in the UVM, interpreted as a super-replication price under uncertain volatilities, is given by

$$u_t = \sup_{(\sigma, \rho) \in \Xi_D} \mathbb{E}^{\mathbb{Q}}[F_T | \mathcal{F}_t].$$

For European style payoffs $F_T = g(X_T)$, the value $u(t, x)$ is then the unique viscosity solution (under suitable growth condition on g) of the nonlinear PDE :

$$\partial_t u(t, x) + H(x, D_x^2 u(t, x)) = 0, \quad u(T, x) = g(x)$$

with the Hamiltonian

$$H(X, \Gamma) = \frac{1}{2} \max_{(\sigma^i, \rho^{ij})_{1 \leq i \leq j \leq d}} \sum_{i, j=1}^d \rho^{ij} \sigma^i \sigma^j X^i X^j \Gamma^{ij}.$$

Denote by S^d the space of symmetric $d \times d$ matrices. The 2-BSDE associated to this PDE has driver

$$f(x) = \frac{1}{2} \max_{(\sigma^i, \rho^{ij})_{1 \leq i \leq j \leq d} \in D} \sum_{i, j=1}^d \rho^{ij} \sigma^i \sigma^j x^i x^j$$

and dynamics

$$\begin{aligned} d\widehat{X}_t^i &= \widehat{\sigma}^i \widehat{X}_t^i dW_t^i, \quad dW_t^i dW_t^j = \widehat{\rho}^{ij} dt, \quad 1 \leq i \leq j \leq d \\ dY_t &= -f(\widehat{X}_t) dt + \sum_{i=1}^d Z_t^i \circ \widehat{\sigma}^i \widehat{X}_t^i dW_t^i \\ dZ_t^i &= A_t^i dt + \sum_{j=1}^d \Gamma_t^{ij} \widehat{\sigma}^j \widehat{X}_t^j dW_t^j \\ Y_T &= g(\widehat{X}_T). \end{aligned}$$

Note that \widehat{X}_t follows a log normal diffusion with some admissible variance $\widehat{\sigma}$ (we are free to choose e.g. $\widehat{\sigma} = (\bar{\sigma} + \underline{\sigma})/2$) and some constant correlation $\widehat{\rho}$. Let $(Y_t, Z_t, \Gamma_t, A_t)$ be the quadruple of adapted processes taking values in \mathbb{R} , \mathbb{R}^d , S^d and \mathbb{R}^d respectively, solving the 2-BSDE (in the sense of [16]). For details regarding existence and uniqueness of this 2-BSDE solution we refer to [60, p.52f] and the references therein.

We recall a numerical scheme based on this second-order backward stochastic differential equation that has been proposed in [48] and [60].

Partition $[0, T]$ into sub-intervals (t_{i-1}, t_i) , $1 \leq i \leq n$ and set $\Delta t_i = t_i - t_{i-1}$, $\Delta W_{t_i} = W_{t_i} - W_{t_{i-1}}$. In [60], the (backward) scheme reads

$$\begin{aligned} \widehat{X}_{t_i}^j &= \widehat{X}_0^j e^{-(\hat{\sigma}^j)^2 \frac{t_i}{2} + \hat{\sigma}^j W_{t_i}^j}, \quad \Delta W_{t_i}^j \Delta W_{t_i}^k = \widehat{\rho}_{jk} \Delta t_i \\ Y_{t_n} &= g(\widehat{X}_{t_n}) \\ \hat{\sigma}^j \hat{\sigma}^k \widehat{X}_0^j \widehat{X}_0^k \Gamma_{t_{i-1}}^{jk} &= \mathbb{E}_{i-1}^{\mathbb{P}}[Y_{t_i} (U_{t_i}^j U_{t_i}^j - (\Delta t_i)^{-1} \widehat{\rho}_{jk}^{-1} - \hat{\sigma}^j U_{t_i}^j \delta_{jk})] \\ Y_{t_{i-1}} &= \mathbb{E}_{i-1}^{\mathbb{P}}[Y_{t_i}] + \left(H(\widehat{X}_{t_{i-1}}, \Gamma_{t_{i-1}}) - \frac{1}{2} \sum_{j,k=1}^n \widehat{X}_{t_{i-1}}^j \widehat{X}_{t_{i-1}}^k \Gamma_{t_{i-1}}^{jk} \widehat{\rho}_{pk} \hat{\sigma}^j \hat{\sigma}^k \right) \Delta t_i \end{aligned} \quad (\text{VIII.3.1})$$

with

$$U_{t_i}^j \equiv \sum_{k=1}^d \widehat{\rho}_{jk}^{-1} \Delta W_{t_i}^k / \Delta t_i.$$

The scheme requires us to compute conditional expectations at some discretization dates t_i . Once the $\Gamma_{t_{i-1}}$ are computed during the backward induction, one gets a (sub-optimal) estimation of the volatilities (σ^{back}). Performing a second independent (forward) Monte-Carlo using this sub optimal control, we obtain a lower bound $u_0^{\text{LS}} \leq u_0$. So far the primal algorithm developed in [60]. For the dual bounds derived in this paper we next determine the numerical approximation for $\nabla_x u$, which will serve as the minimiser φ^* , which may be computed using the relation

$$\hat{\sigma}^j \widehat{X}_0^j (\varphi_{t_{i-1}}^*)^j = \mathbb{E}_{i-1}^{\mathbb{P}}[Y_{t_i} U_{t_i}^j]$$

and letting

$$\varphi_s^* = \sum_{i=1}^N \varphi_{t_{i-1}}^* (\widehat{X}_{t_{i-1}}) 1_{s \in [t_{i-1}, t_i)}$$

Below, we denote $Y_0 = u_0^{\text{BSDE}}$. Using our candidate φ^* in the minimisation, we get an upper bound

$$u_0^{\text{LS}} \leq u_0 \leq u_0^{\text{dual}} \equiv \lim_{N \rightarrow \infty} \mathbb{E} \left[\sup_{(\sigma, \rho) \in \mathcal{D}_N} \left\{ g(\widehat{X}_{t_N}) - \sum_{i=1}^N \varphi_{t_{i-1}}^* (\widehat{X}_{t_{i-1}}) (\widehat{X}_{t_i} - \widehat{X}_{t_{i-1}}) \right\} \right]$$

where

$$\widehat{X}_{t_i}^k = \widehat{X}_0^k e^{-(\sigma^k)^2 \frac{t_i}{2} + \sigma^k W_{t_i}^k}, \quad \Delta W_{t_i}^k \Delta W_{t_i}^j = \widehat{\rho}_{kj} \Delta t_i.$$

3.1.1 The algorithm

The algorithm can be summarized by the following four steps :

1. Simulate N_1 replications of X with a lognormal diffusion (we choose $\hat{\sigma} = (\bar{\sigma} + \underline{\sigma})/2$).
2. Apply the backward algorithm using a regression approximation. Basis coefficients for the Delta at each discretization time are stored. Compute $Y_0 = u_0^{\text{BSDE}}$.
3. Simulate N_2 independent replication of X using the sub-optimal controls. Give a low-biased estimate u_0^{LS} .
4. Simulate independent increment ΔW_{t_i} and optimize $g(X_{t_N}) - \sum_{i=1}^N \varphi_{t_{i-1}}^*(X_{t_{i-1}})(X_{t_i} - X_{t_{i-1}})$ over (σ) . In our numerical experiments, as the payoff may be non-smooth, we have used a direct search polytope algorithm. Then average.

3.1.2 Numerical experiments

In our experiments, we take $T = 1$ year and for each asset α , $X_0^\alpha = 100$, $\underline{\sigma}^\alpha = 0.1$, $\bar{\sigma}^\alpha = 0.2$ and we use the constant mid-volatility $\hat{\sigma}^\alpha = 0.15$ to generate the first N_1 replication of X . For the second independent Monte-Carlo using our sub-optimal control, we take $N_{\text{LS}} = 2^{15}$ replications of X and a time step $\Delta_{\text{LS}} = 1/400$. In the backward and dual algorithms, we pick $N_1 = 2^{15}$ and choose the $\Delta = (1/2, 1/4, 1/8, 1/12)$ that give the higher u_0^{LS} and the lower u_0^{dual} . The conditional expectations at t_i are computed using parametric regressions. The regression basis consists in some polynomial basis with the Black-Scholes price/delta/gamma with mid-volatilities. The exact price is obtained by solving the (one or two-dimensional) HJB equation with a finite-difference scheme.

1. 90 – 110 call spread $(X_T - 90)^+ - (X_T - 110)^+$, basis= 5-order polynomial :

$$u_0^{\text{LS}} = 11.07 < u_0^{\text{PDE}} = 11.20 < u_0^{\text{dual}} = 11.70, u_0^{\text{BSDE}} = 10.30$$

2. Digital option $1_{X_T \geq 100}$, basis= 5-order polynomial :

$$u_0^{\text{LS}} = 62.75 < u_0^{\text{PDE}} = 63.33 < u_0^{\text{dual}} = 66.54, u_0^{\text{BSDE}} = 52.03$$

3. Outperformer option $(X_T^2 - X_T^1)^+$ with 2 uncorrelated assets,

$$u_0^{\text{LS}} = 11.15 < u_0^{\text{PDE}} = 11.25 < u_0^{\text{dual}} = 11.84, u_0^{\text{BSDE}} = 11.48$$

4. Outperformer option with 2 correlated assets $\rho = -0.5$

$$u_0^{\text{LS}} = 13.66 < u_0^{\text{PDE}} = 13.75 < u_0^{\text{dual}} = 14.05, u_0^{\text{BSDE}} = 14.14$$

5. Outperformer spread option $(X_T^2 - 0.9X_T^1)^+ - (X_T^2 - 1.1X_T^1)^+$ with 2 correlated assets $\rho = -0.5$,

$$u_0^{\text{LS}} = 11.11 < u_0^{\text{PDE}} = 11.41 < u_0^{\text{dual}} = 12.35, u_0^{\text{BSDE}} = 9.94$$

Note that in examples 3.-5. the regression basis we used consisted of

$$\{1, X^1, X^2, (X^1)^2, (X^2)^2, X^1 X^2\}.$$

The dual bounds we have derived complement the lower bounds derived in [60]. They allow us to access the quality of the regressors used in computing the conditional expectations.

3.2 Credit valuation adjustment

Our second example arises in credit valuation adjustment. We will show that for this particular example, we can solve the deterministic optimisation problems arising in the dual algorithm efficiently by recursively solving ODEs. More specifically, we consider the stochastic control problem

$$u^{\text{HJB}}(t, X_t) = \sup_{\lambda_t \in [0, c], \text{adapted}} \mathbb{E}_t[e^{-\int_t^T \lambda_s ds} g(X_T)], \quad \text{where } dX_t = \sigma(t, X_t) dW_t,$$

for which the HJB equation reads

$$\partial_t u^{\text{HJB}} + \frac{1}{2} \sigma^2(t, x) \partial_{xx}^2 u^{\text{HJB}} + c(u^{\text{HJB}})^- = 0.$$

The nonlinear PDE corresponds to the pricing equation in the case of counterparty value adjustment.

3.2.1 CVA interpretation

This nonlinear PDE corresponds to the pricing equation in the case of unilateral counterparty value adjustment (see [61] for more details). We have one counterparty, denoted with C, that may default and another, B, that cannot. We assume that B is allowed to dynamically trade in the underlying X - that is described by a local martingale $dX_t = \sigma(t, X_t) dW_t$ under a

risk-neutral measure \mathbb{Q} . The default of C is modeled by a Poisson jump process with a constant intensity c . We denote by u the value of B's long position in a single derivative contracted by C, given that C has not defaulted so far. For simplicity, we assume zero rate. The no-arbitrage condition gives that $u(t, X_t)$ is a \mathbb{Q} -martingale, characterized by

$$\partial_t u + \frac{1}{2} \sigma^2(t, x) \partial_{xx}^2 u + c(\tilde{u} - u) = 0.$$

where \tilde{u} is the derivative value just after the counterparty has defaulted. At the default event, in the case of zero recovery, \tilde{u} is given by

$$\tilde{u} = (-u)^+$$

Indeed, if the value of u is positive, meaning that u should be paid by the counterparty, nothing will be received by B after the default. If the value of u is negative, meaning that u should be received by the counterparty, B will pay u in the case of default of C. Finally, we obtain the relation

$$u(t, x) = e^{-c(T-t)} u^{\text{HJB}}(t, x).$$

3.2.2 Dual Bound

We are interested in deriving an efficient upper bound for $u^{\text{HJB}}(0, X_0)$. Writing $\Lambda_t = \int_0^t \lambda_s ds$ and letting $\hat{\mathcal{D}}_n := \left\{ \int_0^T \lambda_s ds : \lambda \in \mathcal{D}_{\frac{1}{n}} \right\}$ our dual expression is

$$\begin{aligned} u^{\text{HJB}}(0, X_0) &= \lim_{k \rightarrow \infty} \inf_{\varphi \in \mathcal{U}} \mathbb{E} \left[\sup_{\Lambda_t \in \hat{\mathcal{D}}_k} \left\{ e^{-\Lambda_T} g(X_T) - \int_0^T e^{-\Lambda_t} \varphi(t, X_t) dX_t \right\} \right] \\ &\leq \lim_{k \rightarrow \infty} \mathbb{E} \left[\sup_{\Lambda_t \in \hat{\mathcal{D}}_k} \left\{ e^{-\Lambda_T} g(X_T) - \int_0^T e^{-\Lambda_t} \varphi^*(t, X_t) dX_t \right\} \right], \end{aligned}$$

where φ^* is a fixed strategy. Rewriting the integral in Stratonovich form we have

$$\begin{aligned} &\int_0^T e^{-\Lambda_t} \varphi^*(t, X_t) dX_t \\ &= \int_0^T e^{-\Lambda_t} \varphi^*(t, X_t) \circ dX_t - \frac{1}{2} \int_0^T e^{-\Lambda_t} \left(\frac{\partial}{\partial x} \varphi^* \right) (t, X_t) \sigma^2(t, X_t) dt \end{aligned}$$

Therefore, using the classical Zakai approximation of the Stratonovich integral, it follows that

$$\begin{aligned}
 & \mathbb{E} \left[\sup_{\Lambda_t \in \mathcal{D}_k} \left\{ e^{-\Lambda T} g(X_T) - \int_0^T e^{-\Lambda t} \varphi^*(t, X_t) dX_t \right\} \right] \\
 = & \lim_{n \rightarrow \infty} \left[\mathbb{E} \sup_{\Lambda_t \in \mathcal{D}_k} \left\{ e^{-\Lambda T} g(X_T^n) \right. \right. \\
 & \left. \left. - \int_0^T e^{-\Lambda t} \varphi^*(t, X_t^n) \circ dX_t^n + \frac{1}{2} \int_0^T e^{-\Lambda t} \left(\frac{\partial}{\partial x} \varphi^* \right) (t, X_t^n) \sigma^2(t, X_t^n) dt \right\} \right] \\
 = & \lim_{n \rightarrow \infty} \mathbb{E} \left[\sup_{\Lambda_t \in \tilde{\mathcal{D}}_k} \left\{ e^{-\Lambda T} g(X_T^n) \right. \right. \\
 & \left. \left. - \int_0^T e^{-\Lambda t} \left(\varphi^*(t, X_t^n) \sigma(t, X_t^n) \dot{W}_t^n - \frac{1}{2} \left(\frac{\partial}{\partial x} \varphi^* \right) (t, X_t^n) \sigma^2(t, X_t^n) \right) dt \right\} \right] \\
 \leq & \lim_{n \rightarrow \infty} \mathbb{E} \left[\sup_{\Lambda_t \in \tilde{\mathcal{D}}} \left\{ e^{-\Lambda T} g(X_T^n) \right. \right. \\
 & \left. \left. - \int_0^T e^{-\Lambda t} \left(\varphi^*(t, X_t^n) \sigma(t, X_t^n) \dot{W}_t^n - \frac{1}{2} \left(\frac{\partial}{\partial x} \varphi^* \right) (t, X_t^n) \sigma^2(t, X_t^n) \right) dt \right\} \right],
 \end{aligned}$$

where $\tilde{\mathcal{D}}$ denotes the set of all absolutely continuous controls. For almost every ω we may consider for all n the following deterministic optimisation problem. Set

$$\begin{aligned}
 g_{\omega,n} &= g(X_T^n(\omega)), \quad \alpha_{\omega,n}(t) = -\varphi^*(t, X_t^n(\omega)) \sigma(t, X_t^n(\omega)) \dot{W}_t^n(\omega), \\
 \beta_{\omega,n}(t) &= \frac{1}{2} \left(\frac{\partial}{\partial x} \varphi^* \right) (t, X_t^n(\omega)) \sigma^2(t, X_t^n(\omega)),
 \end{aligned}$$

and consider the function :

$$u_{\omega,n}^{\text{HJ}}(t) = \sup_{\lambda_s \in [0,c]} \left\{ e^{-\mathbb{L}_T + \mathbb{L}_t} g_{\omega,n} + \int_t^T e^{-\mathbb{L}_s + \mathbb{L}_t} \left(\alpha_{\omega,n}(s) + \beta_{\omega,n}(s) \right) ds \right\}.$$

Note that u^{HJ} is the solution of the (path-wise) Hamilton-Jacobi equation :

$$(u_{\omega,n}^{\text{HJ}})'(t) + c \left(-u_{\omega,n}^{\text{HJ}}(t) \right)^+ + \alpha_{\omega,n}(t) + \beta_{\omega,n}(t) = 0, \quad u_{\omega,n}^{\text{HJ}}(T) = g_{\omega,n}.$$

The ODE for $u_{\omega,n}^{\text{HJ}}$ can be solved analytically. Fix a $t^0 \in [0, T]$, and let

$$t^* = \sup \left\{ s < t^0 : u_{\omega,n}^{\text{HJ}}(t^0) u_{\omega,n}^{\text{HJ}}(s) < 0 \right\} \vee 0.$$

$c, (1 - e^{-cT})$	PDE	1/2	1/4	1/8	1/12	1/24	1/50	1/100	1/200
0.01 (1%)	0.26	0.23	0.25	0.26	0.26	0.26	0.26	0.26	0.26
0.05 (4.9%)	1.29	1.14	1.22	1.26	1.27	1.28	1.29	1.29	1.29
0.1 (9.5%)	2.52	2.24	2.39	2.46	2.48	2.51	2.52	2.52	2.52
0.7 (50.3%)	13.60	12.63	13.25	13.53	13.61	13.71	13.75	13.77	13.77

TABLE VIII.1 – $\mathbb{E}[u_{\omega,n}^{\text{HJ}}(0)]$ as a function of the time discretization when $\varphi^*(t, x) = e^{-c(T-t)}$.

$c, (1 - e^{-cT})$	PDE	$E[u_{\omega,n}^{\text{HJ}}(0)]$
0.01 (1%)	0.26	0.40
0.05 (4.9%)	1.30	1.95
0.1 (9.5%)	2.53	3.80
0.7 (50.3%)	13.60	20.08

TABLE VIII.2 – $\mathbb{E}[u_{\omega,n}^{\text{HJ}}(0)]$ when $\varphi^*(t, x) = 0$.

For all $t \in [t^*, t_0]$ we get the following recurrence equation :

$$\begin{aligned}
 u_{\omega,n}^{\text{HJ}}(t) &= \begin{cases} -\int_t^{t^0} e^{-c(s-t)} (\alpha_{\omega,n}(s) + \beta_{\omega,n}(s)) ds + u_{\omega,n}^{\text{HJ}}(t^0) e^{c(t^0-t)}, & u_{\omega,n}^{\text{HJ}}(t^0) < 0 \\ -\int_t^{t^0} (\alpha_{\omega,n}(s) + \beta_{\omega,n}(s)) ds + u_{\omega,n}^{\text{HJ}}(t^0), & u_{\omega,n}^{\text{HJ}}(t^0) > 0 \end{cases}, \\
 u_{\omega,n}^{\text{HJ}}(T) &= g_{\omega,n}.
 \end{aligned}$$

Finally, we observe that,

$$u^{\text{HJB}}(0, X_0) \leq \lim_{n \rightarrow \infty} \mathbb{E}[u_{\omega,n}^{\text{HJ}}(0)].$$

We illustrate the quality of our bounds by the following numerical example.

Remark VIII.3.1. *This example falls into the framework of [24], [32]. By virtue of their (continuous) pathwise analysis the upper bounds derived above could in the limit be replaced with equalities. Only the error introduced by the choice of φ^* remains.*

Numerical example

We take $\sigma(t, x) = 1$, $T = 1$ year, $X_0 = 0$. $g(x) = x$. We use two choices : $\varphi^*(t, x) = e^{-c(T-t)}$ (which corresponds to $\partial_x u^{\text{HJB}}$ at the first-order near $c = 0$) and $\varphi^*(t, x) = 0$. We have computed $\mathbb{E}[u_{\omega,n}^{\text{HJ}}(0)]$ as a function of the time discretization (see Table VIII.1 and VIII.2). The exact value has been computed using a one-dimensional PDE solver (see column PDE). We have used different values of c corresponding to a probability of default at T equal to $(1 - e^{-cT})$.

The approximation has two separate sources of error. First, there is the suboptimal choice

of the minimiser φ^* for the discretised optimisation implying an upper bias. The second error arises from the discretisation of the deterministic optimisation problems, which in this example underestimates the true value of the optimisation. The choice $\varphi^* = e^{-c(T-t)}$ in our example - as expected - close to optimal, so for small values of n in the discretisation of the deterministic optimisation problems the optimisation error dominates converging at a rate $n^{-1/2}$ to the upper bound. The case $\varphi^* = 0$ demonstrates the effect of the gain of information, when the stochastic optimisation problem is replaced by the deterministic problems without a Lagrange multiplier to compensate.

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